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Investigation of the Incompressible Viscous Newtonian Fluids Flow using Three-Dimensions Linear Navier-Stokes Equations

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Abstract: In this paper, the unsteady and nonlinear Navier-Stokes equations in three Cartesian coordinates are converted to the linear diffusion equations based on the concept of linear velocity operator ($\underline{\hat{v}}$. $\underline{\nabla}$). The stream function Ψ(x, y, z, t) represents the analytical solutions of dimensional continuity and linear Navier-Stokes equations. As a physical application, the viscous Newtonian fluid flow in a 3D peristaltic horizontal tube is described by non-dimensional continuity and linear Navier-Stokes's equations. The analytical solution in terms of stream function is obtained for different values of time, wavelengths, and Reynolds numbers for a first time. Moreover, the streamlines change from laminar, to transit, and then to turbulent flow with increasing time interval. Authors introduced the 3D analytical solutions of linear and nonlinear Navier-Stokes equations as a millennium problem.

Keywords: Linear velocity operator. Linear Navier-Stokes equations in 3D. Peristaltic flow. Linear convective acceleration. Stream function in 3D. wave lengths. Laminar, transit, and turbulent flow.

1 Introduction

The Navier-Stokes equations represent the soul of fluid mechanics for describing the fluid flow under the effect of different forces. The Navier–Stokes equations may be used to model the weather, ocean currents, petroleum, biomedical engineering, water flow in a pipes and problems of air flow around a wing. Moreover, in their full and simplified forms, help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other problems [1- 11]. In fluid mechanics, the physical problems are formulated by continuity, Navier-Stokes, energy, and state equations [12]. There are many problems that can be formulated by the nonlinear partial differential equations, which face some difficulties in the way of analytical solutions [13-14]. The analytical and numerical method for solving the dissipative, nonlinear, and non-stationary partial differential equations is obtained by Mohammadein [15-22], Nugroho [21], Obada [22], Schöffel [23], and Shalaby [24]. The development of the basis of quantum mechanics is illustrated in detail by Obada et al [22]. Moreover, the Navier-stokes equation is transformed to Schrödinger equation [25] based on assumption that the fluid velocity is equal to the gradient of potential by using Hopf-Cole transformation [4, 25]. The numerical technique in two-phase bubbly flow is treated by Bilicki [26].

On the existence, Regularity and decay of Navier-Stokes solutions, the mathematics scientists tried to make proof that the solution of Navier-Stokes equations exists, and the solution is unique [8]. Here there are very important question for specialist mathematician scientists if the solution is existing and unique, then where the efforts to find the analytical solution? The solution exists because Navier-Stokes equation is a well posed problem and there are many treatments for solving the nonlinear Navier-Stokes equations in different cases [6-7]. The treatments involve numerical, approximate, or analytical methods. The numerical methods like Rung-Kutta, finite difference, and volume element methods [3, 9, 10-11, 15-17] ;which are applied for solving many problems.

Approximate methods like similarity parameters, homotopy, perturbation and iteration methods are used as a special solution [4, 10, 22]. Moreover, there are many analytical methods like integration, integrodifferential, Hopf-Cole transformation, Green's function, and special physical assumptions are used for nonlinear problems [4, 25]. In the following, we mention very brief history for some proposals. Some scientists go to the special cases of fluid and flow kinds like taking very small velocity of incompressible fluid, or to ignore the nonlinear term in the Navier-Stokes equations [15-17, 24]. New approximate analytical solutions are obtained for two- and three-dimensional unsteady viscous incompressible flows by using the kinetically reduced local Navier-Stokes equations as Harfash [9]. In the previous treatments, without taking special cases in consideration, the nonlinear term in acceleration still represents an obstacle for analytical solution.

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The new treatment of fluid mechanics by Mohammadein [18] solved the obstacle of Navier-Stokes equation, which exists in the nonlinear term of acceleration. The transformation of linear velocity operator based on linear momentum operator in quantum mechanics [5, 12] is derived. The nonlinear Navier-Stokes, Burger and Korteweg-de Vries (KDV) are transformed to linear diffusion equations based on this simplest physical idea [18]. Moreover, the continuity and linear Navier-Stokes equations are solved in analytical way by Mohammadein et al [19-20] in two dimensions cartesian coordinates (x, y) and applied to describe the viscous incompressible Newtonian fluid with peristaltic flow. The analytical solution of dimensional and non-dimensional system of Navier-Stokes equations is obtained. The solution in terms of stream function is valid for all values of wavelengths and Reynolds numbers. Based on the pressure gradient as surface force, the Navier-Stokes equations are reformulated. Moreover, both linear and original nonlinear Navier-Stokes equations are satisfied by the obtained solutions.

Mohammadein theory for the nonlinear term $(\underline{\hat{v}}.\underline{\nabla})\underline{v}$.

Based on the Mohammadein new treatment theory [18], the total operator for any physical function f called $\frac{Df}{Dt}$ as proposed by Landau [12] takes the form

$$\frac{D\ldots}{Dt} = \frac{\partial\ldots}{\partial t} + (\underline{\hat{v}}.\underline{\nabla})\ldots \qquad (1)$$

Based on quantum mechanics, the linear momentum operator term $\underline{\hat{P}} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ play a real role in proofing Schrodinger equation and then linear velocity operator becomes $\underline{\hat{v}} = \frac{\hbar}{im}\frac{\partial}{\partial x}$. Then, the general operator $\frac{D\ldots}{Dt}$ becomes

$$\frac{D\ldots}{Dt} = \frac{\partial\ldots}{\partial t} + (-M^{*}\underline{\nabla}.\underline{\nabla})\ldots \qquad (2)$$

where $\underline{\hat{v}} = -M^{*}\underline{\nabla}$, is the physical proposed and M^{*} called Mohammadein parameter. The total differentiation of a function f in the point of view of Mohammadein description [18] has a simplest form

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} - M^{*}\underline{\nabla}^2 f \qquad (3)$$

where f is valid for any physical function represents the velocity, pressure, temperature, mass diffusion or blood concentration in the bio tissues, plasma, thermal, binary thermal, and other fields of heat and mass transfer in fluid mechanics. For example, the total differentiations for fluid velocity and its temperature in the diffusion form become

$$\underbrace{\frac{D\underline{v}}{Dt}}_{total\ acceleration} = \underbrace{\frac{\partial \underline{v}}{\partial t}}_{local\ acceleration} - \underbrace{\nu\,\underline{\nabla}^2\underline{v}}_{diffusion\ acceleration} \qquad (4)$$

and

$$\underbrace{\frac{DT}{Dt}}_{total\ heat} = \underbrace{\frac{\partial T}{\partial t}}_{local\ heat} - \underbrace{a_l\,\underline{\nabla}^2 T}_{diffusion\ heat}, \qquad (5)$$

where $\underline{v}(u, \mathrm{v}, w)$ and T (x, y, z, .t) are the velocity and temperature of fluid in 3D respectively. We note that M^{*}equal to kinematic viscosity ν and thermal diffusivity a_l for a linear vector velocity $\underline{v}$ and scalar temperature T of any fluid respectively.

In this paper, the continuity and unsteady dimensional and non-dimensional linear Navier-Stokes equations are formulated in three dimensions cartesian coordinates (x, y, z) based on Mohammadein et al [18-20] model. This system is used to describe the fluid flow of viscous incompressible Newtonian fluid in 3D a peristaltic horizontal Tube. The simplest analytical solution is obtained in terms of stream function $\Psi(x, y, z, t)$ and fluid velocity components (u, v, w); which are valid for all different values of wave lengths. The change of laminar to transit and then to turbulent flows are observed through the increasing of time.

2 The physical and Mathematical Model in 3D

consider the incompressible viscous fluid flow in a 3D peristaltic horizontal tube (see Fig. 1) in x-axis. The flow is caused by infinite sinusoidal wave train moving ahead with constant velocity c along the walls of the tube. The gravity force is ignored in our case. The peristaltic of four walls around x-axis in y and z directions. The peristaltic boundary conditions have the form

$$y = \mp H_1 \mathrm{a} + a_1 \sin\left(\frac{2\pi}{\lambda_1}(x - ct)\right)$$
$$z = \mp H_2 = \mathrm{b} + b_1 \sin\left(\frac{2\pi}{\lambda_2}(x - ct)\right) \qquad (6)$$

where a and b are the tube half-length of width and height in the direction y and z with wave amplitudes $a_1$and b_1 respectively. Moreover, the wave lengths are represented by λ_1 and λ_2 in the x- direction as in Fig.1.

The mathematical model is described by continuity and three-dimensional Navier-Stokes equations for the viscous incompressible fluid flow in x-direction under the effect of surface and body forces in the following vector form

$$\underline{\nabla}\cdot\underline{v} = 0 \qquad (7)$$

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{1}{\rho}\underline{\nabla}P + \frac{1}{\rho}\underline{\nabla}\cdot\tau_{ij} \qquad (8)$$

where $\underline{\nabla}P$ is the gradient of pressure field of fluid flow. The shearing stress between layers of fluid τ_{ij} takes

$$\underline{\nabla}\cdot\tau_{ij} = \begin{cases} \eta\,\underline{\nabla}^2\underline{v} & for\ \text{Newtonian fluids} \\ \underline{\nabla}\cdot\tau_{ij} & for\ \text{Non Newtonian fluids} \end{cases} \qquad (9)$$

The vector Navier-Stokes equation (8) for a Newtonian fluid and based on Mohammadein theory [18], becomes

$$\frac{\partial \underline{v}}{\partial t} = -\frac{1}{\rho}\underline{\nabla}P + 2\nu\underline{\nabla}^2\underline{v} \qquad (10)$$

The continuity and dimensional linear Navier-Stokes equations (7) and (10) respectively in 3D Cartesian coordinates become

$$u_x + v_y + w_z = 0, \qquad (11)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\,\nu(u_{xx} + u_{yy} + u_{zz}), \qquad (12)$$

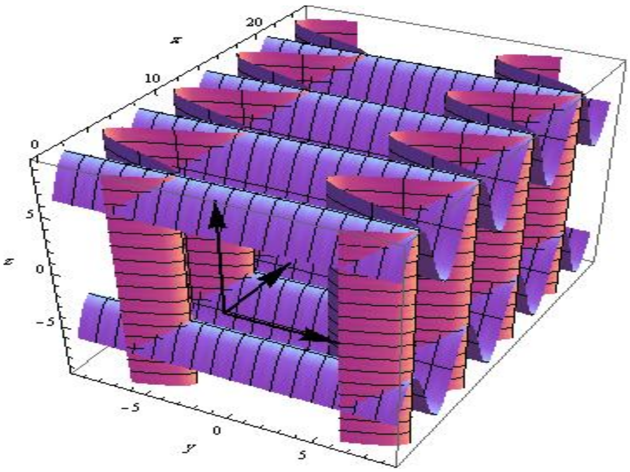

Fig.1. The sketch of problem.

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + 2\,\nu\left(v_{xx} + v_{yy} + v_{zz}\right), \quad (13)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + 2\,\nu\left(w_{xx} + w_{yy} + w_{zz}\right). \quad (14)$$

The above 3 D linear system can be solved by analytical way under the proposed of initial and boundary conditions.

Pressure field

The pressure field performs an important parameter in Navier-Stokes equations. Based on Bernoulli and Mohammadein theory [18], the gradient pressure becomes

$$\frac{1}{\rho}\underline{\nabla}P = \nu\,\underline{\nabla}^2\underline{v} - g\,\underline{\hat{n}}. \quad (15)$$

The final form of Navier-Stokes equation (10) for Newtonian fluid flow by using equation (15) has the form

$$\frac{\partial \underline{v}}{\partial t} = \nu\underline{\nabla}^2\underline{v} + g\,\underline{\hat{n}} \quad (16)$$

The continuity and Navier-Stokes equations in 3D tube for fluid flow in the x-direction ($g_x = g_y = 0, g_z = -\,g\,\underline{\hat{n}}$), are in the form

$$u_x + v_y + w_x = 0, \quad (17)$$

$$\frac{\partial u}{\partial t} = \nu(u_{xx} + u_{yy} + u_{zz}), \quad (18)$$

$$\frac{\partial v}{\partial t} = \nu\left(v_{xx} + v_{yy} + v_{zz}\right), \quad (19)$$

$$\frac{\partial w}{\partial t} = \nu\left(w_{xx} + w_{yy} + w_{zz}\right) + g\,\underline{\hat{n}}, \quad (20)$$

Where $\underline{\hat{n}}$ is the unit normal vector.

Stream function $\Psi(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t})$

In a 3D incompressible fluid flow, the relation between stream function $\Psi(x,y,z,t)$ and fluid velocity components u, v, and w as in Appendix I has the form.

$$u = 2\frac{\partial^2\Psi}{\partial y\,\partial z}, \quad v = -\frac{\partial^2\Psi}{\partial z\,\partial x}, and \;\; w = -\frac{\partial^2\Psi}{\partial x\,\partial y} \quad (21)$$

The simplest analytical solution for the above continuity and linear Navier-Stokes equations (17-20) by using Picard method [19] become

$$u(x,y,z,t) = 2\,c_2\,c_3 A_1 e^{\left({c_1}^2+{c_2}^2+{c_3}^2\right)t-\frac{c_1x+c_2y+c_3z}{\sqrt{\nu}}}$$

$$v(x,y,z,t) = -c_3c_1A_1e^{\left({c_1}^2+{c_2}^2+{c_3}^2\right)t-\frac{c_1x+c_2y+c_3z}{\sqrt{\nu}}}$$

$$w(x,y,z,t) = -c_1c_2A_1e^{\left({c_1}^2+{c_2}^2+{c_3}^2\right)t-\frac{c_1x+c_2y+c_3z}{\sqrt{\nu}}} - gt \quad (22)$$

where c_1,c_2, c_3 and $A_1$are constants, which are obtained based on initial and boundary conditions of the proposed physical problem. Moreover, the 3D fluid velocity components (22) are satisfied the continuity and linear Navier-Stokes equations (17-20). Moreover, the 3D stream function $\Psi(x,y,z,t)$ becomes

$$\Psi(x,y,z,t) = A_1e^{\left({c_1}^2+{c_2}^2+{c_3}^2\right)t-\frac{(c_1x+c_2y+c_3z)}{\sqrt{\nu}}} \quad (23)$$

The unknowns A_1, c_1, c_2, and c_3 under the following initial and boundary conditions

$$\psi(x,y,z,0) = A_1e^{-(c_1\,x+\,c_2\,y+\,c_3z)} \quad (24)$$

and six boundary conditions

$$\psi(L_1,y,z,t) = q_1, \quad \psi(L_2,y,z,t) = q_2$$
$$\psi(x,h_1,z,t) = q_3, \quad \psi(x,h_2,z,t) = q_4,$$
$$\psi(x,y,h_3,t) = q_5, \quad \psi(x,y,h_4,t) = q_6, \quad (25)$$

are obtained in the form

$$A_1 = 1,\; c_1 = \frac{1}{L_2-L_1}\ln\frac{q_1}{q_2}, \quad c_2 = \frac{1}{h_2-h_1}\ln\frac{q_3}{q_4}, \text{ and}$$

$$c_3 = \frac{1}{h_4-h_3}\ln\frac{q_5}{q_6}, \quad (26)$$

where $q_1, q_2, q_3, q_4, q_5, q_6$ are given constants or functions.

3 The Incompressible and Viscous Newtonian Fluid Flow in a Horizontal 3D peristaltic Tube for different wave lengths ($\boldsymbol{\delta \neq 0}$) described by continuity and non-dimensional linear Navier-Stokes equations

The most previous problems are described by the nonlinear Navier-Stokes equations, which are approximately solved for long wavelength $\delta = 0$ and low Reynolds numbers [6,7,13-14, 23-24]. The simplest analytical solution of Navier-Stokes equation in a 2D fluid flow is already solved and discussed by Mohammadein et al. [18-20] In this application, the proposed 3 D problem is solved analytically. Moreover, the stream function $\Psi(x,y,z,t)$ and fluid velocity components u (x, y, z, t), v (x, y, z, t), and w (x, y, z, t) are obtained for different values of time, wave lengths λ and Reynolds number.

3.1 The physical and mathematical description

The motion of flow of an incompressible Newtonian viscous fluid in a peristaltic horizontal tube (see Fig. 2) in x-axis is considered.

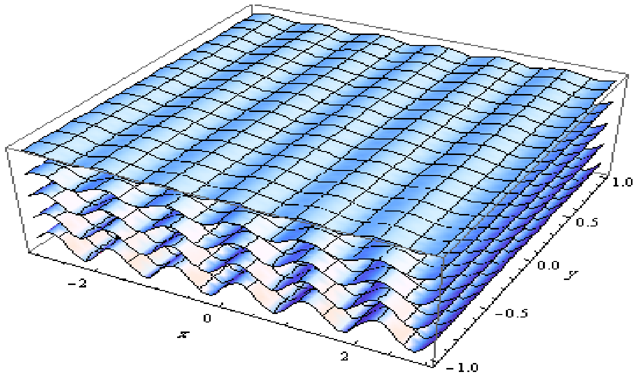

Fig.2. The sketch of problem

The fluid flows in x-direction. The two surfaces 0xy and 0xz move as a peristaltic motion around x-axis. The gravity force is ignored in a horizontal tube. The peristaltic boundary conditions of two walls (h_1, h_2) in y-axis and other two walls (h_3, h_4) in z-axis has the form

$$h_1 = 1 + e_1 \sin(2\pi(x - ct)),$$
$$h_2 = -1 - e_1 \sin(2\pi(x - ct)),$$
$$h_3 = 1 + e_2 \sin(2\pi(x - ct)),$$
$$h_4 = -1 - e_2 \sin(2\pi(x - ct)). \quad (27)$$

3.2 Transformation of dimensional system to the non-dimensional form

In section 2, on the basis of the linear equations (17-20), the fluid velocities and stream function are obtained in the dimensional form. In this section, we need to find the solutions in terms of non-dimensional numbers. The above system (17-20), can be rewritten in the frame $(\bar{x}, \bar{y}, \bar{z})$ as follows

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \quad (28)$$
$$\frac{\partial \bar{u}}{\partial \bar{t}} = \nu\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right), \quad (29)$$
$$\frac{\partial \bar{v}}{\partial \bar{t}} = \nu\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2}\right), \quad (30)$$
$$\frac{\partial \bar{w}}{\partial \bar{t}} = \nu\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{z}^2}\right), \quad (31)$$

where

$$\bar{u} = 2\frac{\partial^2 \bar{\Psi}}{\partial \bar{y}\,\partial \bar{z}}, \quad \bar{v} = -\frac{\partial^2 \bar{\Psi}}{\partial \bar{z}\,\partial \bar{x}}, and\ \bar{w} = -\frac{\partial^2 \bar{\Psi}}{\partial \bar{x}\,\partial \bar{y}} \quad (32)$$

The non-dimensional parameters in terms of dimensional ones have the form

$\bar{x} = \lambda x,\ \bar{y} = ay,\ \bar{z} = b\,z,\ \bar{u} = cu,\ \bar{v} = cv\delta,,\ \bar{w} = c\,w\delta,\ \delta = \frac{\sqrt{a^2+b^2}}{\lambda},$

$, \delta_1 = \frac{a}{\lambda_1},\ \delta_2 = \frac{b}{\lambda_2}, e_1 = \frac{a_1}{a},\ e_2 = \frac{b_1}{b},\ ,\ \bar{t} = \frac{\lambda}{c}t,\ \bar{\psi} = c\sqrt{a^2+b^2}\,\psi.$

$$h_1 = \frac{H_1}{a},\ h_2 = \frac{H_2}{b} \quad (33)$$

The above equations (28-31) by using the above transformations (33) in frame (x, y, z) introduces a linear partial differential equation in terms of stream function ψ in the form

$$\frac{\partial \Psi}{\partial t} = \frac{1}{R_e\,\delta_1}\left(\delta_1{}^2\Psi_{xx} + \Psi_{yy} + \frac{\delta_1{}^2}{\delta_2{}^2}\Psi_{zz}\right), \quad (34)$$

where gravity is ignored in this study. The analytical solution by using Picard method [10] of the above linear partial differential equation (34) has the form become

$$\psi(x,y,z,t) = A_1 e^{\frac{1}{R_e\,\delta_1}\left(c_1^2\,\delta_1^2 + c_2^2 + \frac{\delta_1^2}{\delta_2^2}c_3^2\right)t - (c_1x + c_2y + c_3z)}$$
$$, \delta_1 \neq 0, \delta_2 \neq 0\,. \quad (35)$$

where, $c_1,\ c_2,\ c_3$ and A_1 are unknows and can be estimated from the following initial and boundary conditions:

$$\psi(x,y,z,0) = A_1 e^{-(c_1\,x + c_2\,y + c_3 z)}$$
$$\psi(L_1,y,z,t) = 1, \quad \psi(L_2,y,z,t) = 0.3$$
$$\psi(x,h_1,z,t) = 2, \quad \psi(x,h_2,z,t) = 1,$$
$$\psi(x,y,h_3,t) = 1, \quad \psi(x,y,h_4,t) = 0.7, \quad (36)$$

in the form

$$A_1 = 1,\ c_1 = \frac{1}{L_2 - L_1}\ln\frac{10}{3},\ c_2 = \frac{1}{h_2 - h_1}Ln(2),\ \text{and}\ c_3 = \frac{1}{h_4 - h_3}\ln\frac{10}{7}. \quad (37)$$

4 Discussion of analytical solution

In this section, the Navier-Stokes equations are studied for Newtonian incompressible fluid flow inside a peristaltic horizontal 3D Tube. The nonlinear system of Navier-Stokes equations (8) is transformed to a linear one (16). The dimensional linear equations (17-20) are formulated in three dimensions x, y, and z. which behave as a linear diffusion equation. The fluid velocity components in 3D are obtained by equations (22). Moreover, the stream function $\Psi(x,y,z,t)$ is obtained as shown in equation (23). The mathematical model (28-31) represents an application for viscous Newtonian incompressible fluid flow inside a peristaltic horizontal 3D Tube, which is converted to the stream function equation (34). The solution is obtained in terms of stream function $\Psi(x,y,z,t)$ as shown in equation (35); which is function of wave lengths and Reynolds number.

The stream function $\Psi(x, y, z, t)$ for a 3D fluid flow in the time interval {0→1} unit time is shown in Fig. 3. It is observed that the two surfaces 0zx and 0yx move as a peristaltic motion and behaves as a sine function. The behavior of the xz-plan(y=o) flow shows the streamlines of fluid as a laminar flow with an amplitude range {-25→ 0} unit length in z-axis as in Fig.4.The stream function $\Psi(x,y,z,t)$ for a fluid flow in the time interval $\{1 \rightarrow 2\}$ unit time is shown in Fig.5. The xz-plan (y=0) shows the streamlines of fluid as a transit flow with an amplitude rang $\{-50\ \rightarrow -20\}$ unit length as shown in Fig.6. It is observed

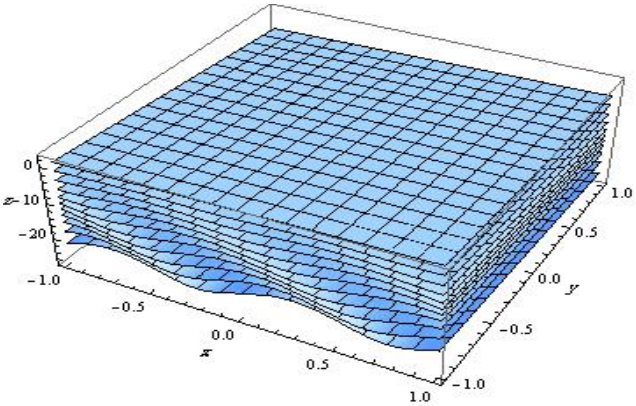

Fig.3. The surface in 3D in time $\{0 - 1\}$.

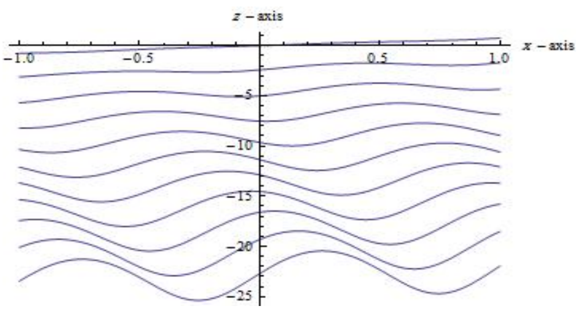

Fig.4. The xz-plan (y=0) in time interval $\{0 - 1\}$unit time

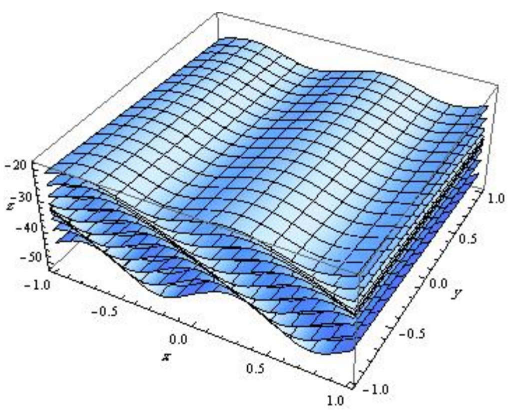

Fig.5. The surface in 3D in time $\{1 - 2\}$unit time.

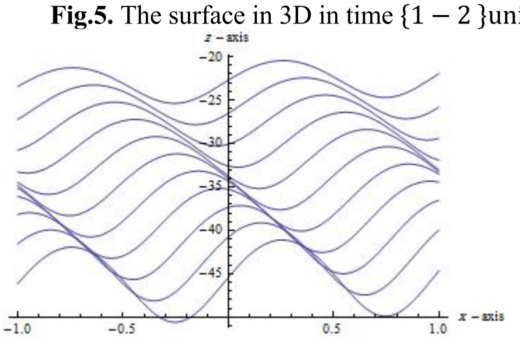

Fig.6. The xz-plan(y=0) in time interval $\{1 - 2\}$unit time

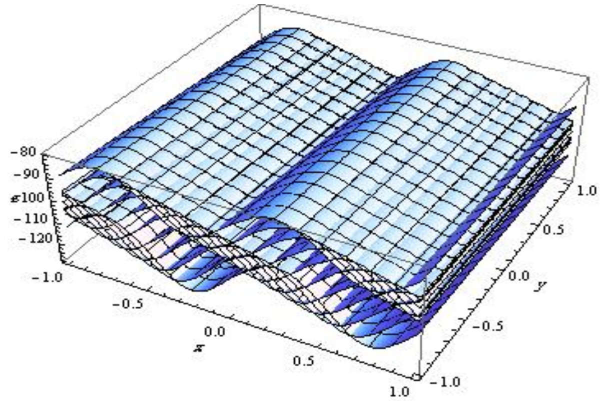

Fig.7. The surface in 3D in time $\{4 - 5\}$.

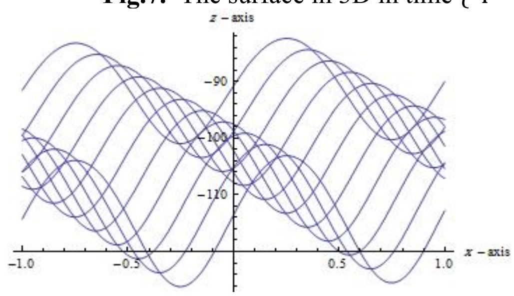

Fig.8. The xz-plan (y=0) in time interval $\{4 - 5\}$unit time

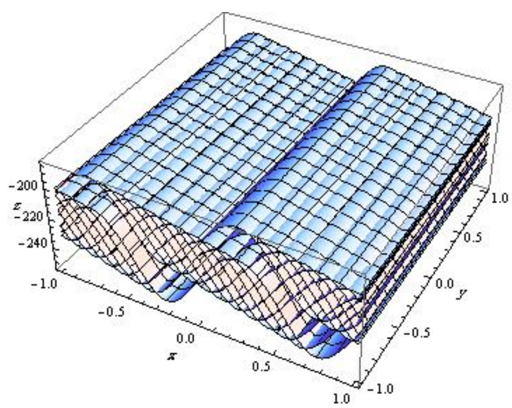

Fig.9. The surface in 3D in time $\{9 - 10\}$unit time .

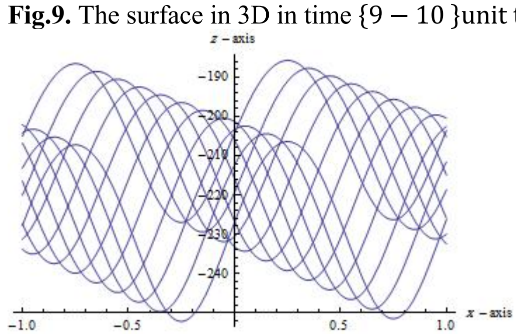

Fig.10. The xz-plan(y=0) in time interval $\{9 - 10\}$unit time

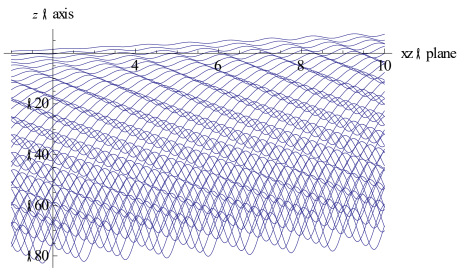

Fig.11. The xz-plan(y=0) in time interval $\{0-10\}$ unit time when y=0, $\delta_1=0.2$, $\delta_2=0.3$, $e_1=e_2=0.1$ *and* R_e=0.5

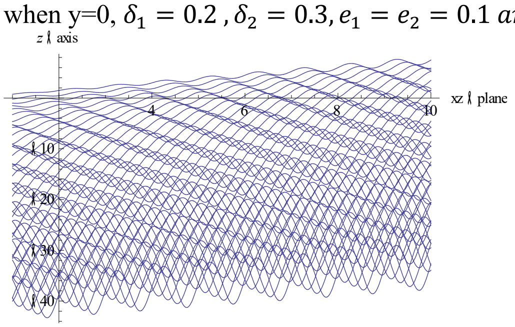

Fig.12. The xz-plan(y=0) in time interval $\{0-10\}$ unit time when y=0, $\delta_1=0.4$, $\delta_2=0.5$, $e_1=e_2=0.1$ *and* R_e=0.5

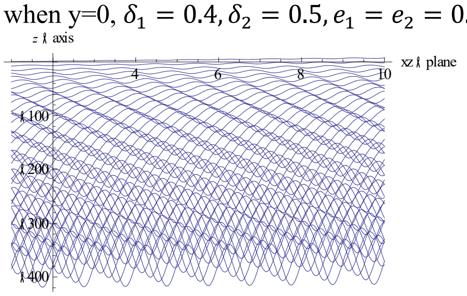

Fig.13. The xz-plan(y=0) in time interval $\{0-10\}$ unit time when y=0, $\delta_1=0.2$, $\delta_2=0.3$, $e_1=e_2=0.1$ *and* R_e=0.1

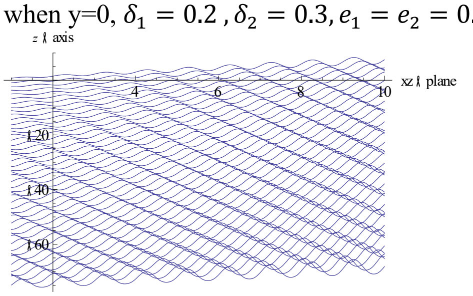

Fig.14. The xz-plan (y=0) in time interval $\{0-10\}$ unit time when y=0, $\delta_1=0.2$, $\delta_2=0.3$, $e_1=0.1$, $e_2=0.2$ *and* R_e=0.5

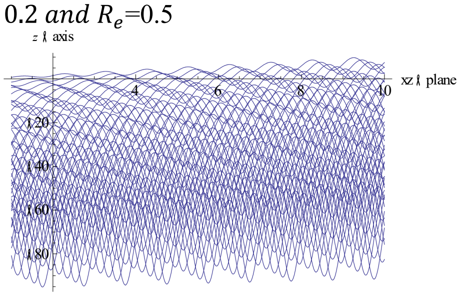

Fig.15. The xz-plan(y=0) in time interval $\{0-10\}$ unit time $\delta_1=0.2$, $\delta_2=0.3$, $e_1=0.3$, $e_2=0.4$ *and* R_e=0.5

that the change of 3D surface and amplitudes increases with time in z-axis. In Fig.7, the stream function $\Psi(x,y,z,t)$ for a fluid flow in the time interval $\{4\to 5\}$ unit time is plotted. The xz-plan(y=0) shows the turbulent flow of streamlines of fluid change with the range of amplitude $\{-130\to -80\}$ unit length in z-axis as shown in Fig.8. The stream function Ψ (x, y, z, t) for a fluid flow in the time interval {9→10} unit time is shown in Fig.9. It is observed that the two surfaces 0zx and 0zy move with peristaltic motion as a sine function. The behavior of the xz-plan(y=0) flow shows the streamlines of fluid as a complicated turbulent flow with an amplitude range {-250 →-195} unit length in z-direction as in Fig.10. The above graphs (Figs.3-10) are plotted when $e_1=e_2=0.1$, $L_1=1$, $L_2=10$, $\delta_1=0.2$, $\delta_2=0.3$ and R_e=0.5. It is noted that the amplitude length of streamlines increases in the negative direction of z-axis with increasing of time. In figures 11-15, the streamlines are plotting in xz-plan (y=0) in the time interval $\{0-10\}$ unit time, $L_1=1$, and $L_2=10$. Streamlines are plotting for different values of wave lengths δ_1 *and* δ_2, Reynolds number R_e, amplitude ratio e_1 and e_2. Streamlines are plotting in xz-plan (y=0) when $\delta_1=0.2$, $\delta_2=0.3$, $e_1=e_2=0.1$ and R_e=0.5 as shown in Fig.11. It observed that the streamlines were affected by wavelengths. The amplitude of streamlines in z-axis the range {-80 →2} unit length. Moreover, Fig.11 is considered as a standard one for comparison with Figs.12-15. In Fig.12, the streamlines for different values of $\delta_1=0.3$, $\delta_2=0.4$ is compared with Fig.11. It observed that, the wavelengths λ_1 and λ_2 decreases with increasing of δ_1 and δ_2 values. The streamlines in z-axes are valid in the range {-42 → 2}. The streamlines for Reynolds number $R_e=0.1$ are plotted as shown in Fig.13. The stream range in z-axes is valid in the interval {-400 → zero}. The viscosity of fluid increases and decays the fluid flow. The streamlines for different values of amplitude ratio $e_1=0.1$ and $e_2=0.2$ are plotted as shown in Fig.14. The stream range in z-axes is valid in the interval {-80→ 2}. It observed that the peristaltic of fluid flow increased with amplitude ratio. The streamlines for different values of amplitude ratio $e_1=0.3$ and $e_2=0.4$ are plotted as shown in Fig.15. The stream range in z-axes is valid in the interval {-100→2}. It observed that the dense peristaltic fluid flow increases with amplitude ratio.

5. Conclusions

The system of continuity and Navier-Stokes equations (7-8) is formulated in dimensional and non-dimensional form (17-20) and (34) respectively. The discussion of results and plotted graphs concluded the following remarks:

1.The nonlinear system of Navier-Stokes equations is converted to a linear one (16) in three dimensions Cartesian coordinates.

2. The analytical solutions are obtained for dimensional and non-dimensional equation form (23) and (35) respectively.

3.The solutions are existed for wave lengths $\delta_1$and , δ_2 for

a first time in fluid mechanics.
4. The fluid velocity components in 3D are obtained directly in terms of stream function.
5.The fluid flow patterns change from laminar to transit and then turbulent with increasing of time.
6. During the increase of time, the amplitude of waves and the intersection of lines increases in the negative direction producing a quick transform from laminar to transit and then to turbulent flow.
7.The streamlines in xz-plan(y=0) are proportional inversaly with wave lengths λ_1 *and* λ_2 and amplitude interval through z-axes.
8. The streamlines in xz-plan(y=0) are proportional with Reynolds number R_e values.
9. The dense of peristaltic stream lines in xz-plane is proportional with amplitude ratio values $e_1 and\ e_2$.
10. The fluid velocities and stream function are obtained for the first time in terms of different wavelengths, Reynolds number and any time interval.
11.The output of the physical results proves the validity of the proposed physical and mathematical model of linear Navier-Stokes equations in 3D.
12.The authors derived the stream function for the first time in three different coordinates as in Appendix I.
The solution satisfies all requisites of the Millennium problem definition of the Clay Institute.

Appendix I

1. Derivation of stream function Ψ in three dimensions

Derivation of stream function $\Psi(x, y, z, t)$ in three dimensional Cartesian coordinates (x, y, z)
The continuity equation for an incompressible fluid flow has the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{A1}$$

The above equation can be rewritten in the form

$$\frac{\partial}{\partial x}\left(2\frac{\partial^2\Psi}{\partial y\,\partial z}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial^2\Psi}{\partial z\,\partial x}\right)+\frac{\partial}{\partial z}\left(-\frac{\partial^2\Psi}{\partial x\,\partial y}\right)=0, \tag{A2}$$

Then

$$u=2\frac{\partial^2\Psi}{\partial y\,\partial z},\quad v=-\frac{\partial^2\Psi}{\partial z\,\partial x}, and\ w=-\frac{\partial^2\Psi}{\partial x\,\partial y}. \tag{A3}$$

AS a mathematical trick, the above relations are valid in three dimensions.

2. Derivation of stream function $\Psi(r, \theta, z)$ in three dimensional cylindrical coordinates (x, y, z)

The continuity equation for an incompressible fluid flow has the form

$$\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_\theta}{\partial \theta}+\frac{\partial(ru_z)}{\partial z}=0, \tag{A4}$$

The above equation can be rewritten in the form

$$\frac{\partial}{\partial r}\left(2\frac{\partial^2\Psi}{\partial\theta\,\partial z}\right)+\frac{\partial}{\partial\theta}\left(-\frac{\partial^2\Psi}{\partial z\,\partial r}\right)+\frac{\partial}{\partial z}\left(-\frac{1}{r}\frac{\partial^2\Psi}{\partial r\,\partial\theta}\right)=0, \tag{A5}$$

then

$$u_r=\frac{2}{r}\frac{\partial^2\Psi}{\partial\theta\,\partial z},\quad u_\theta=-\frac{\partial^2\Psi}{\partial z\,\partial r}, and\ u_z=-\frac{\partial^2\Psi}{\partial r\,\partial\theta}. \tag{A6}$$

As a mathematical trick, the above relations are valid in three dimensions.

3. Derivation of stream function $\Psi(r, \theta, \varphi)$ in three dimensional spherical coordinates (r, θ, φ)

The continuity equation for an incompressible fluid flow has the form

$$\frac{\partial(r^2 sin\theta\, u_r)}{\partial r}+\frac{\partial(r\sin\theta\, u_\theta)}{\partial\theta}+\frac{\partial(r\,u_\varphi)}{\partial z}=0, \tag{A7}$$

The above equation can be rewritten in the form

$$\frac{\partial}{\partial r}\left(2\frac{\partial^2\Psi}{\partial\theta\,\partial z}\right)+\frac{\partial}{\partial\theta}\left(-\frac{\partial^2\Psi}{\partial z\,\partial r}\right)+\frac{\partial}{\partial z}\left(-\frac{1}{r}\frac{\partial^2\Psi}{\partial r\,\partial\theta}\right)=0, \tag{A8}$$

then

$$u_r=\frac{2}{r^2 sin\theta}\frac{\partial^2\Psi}{\partial\theta\,\partial\varphi},\quad u_\theta=\frac{-1}{r\ sin\theta}\frac{\partial^2\Psi}{\partial r\,\partial\varphi}, and\ u_\varphi=\frac{-1}{r}\frac{\partial^2\Psi}{\partial r\,\partial\theta} \tag{A9}$$

As a mathematical trick, the above relations are valid in three dimensions.

Appendix II

Picard method

The Picard method is applied for solving the following linear partial differential equations.

$$\frac{\partial\Psi}{\partial t}=\frac{\nu}{R_e\,\delta_1}\left({\delta_1}^2\Psi_{xx}+\Psi_{yy}\right) \tag{A10}$$

$$\Psi_{n+1}=\Psi_0+\frac{\nu}{R_e\,\delta_1}\int_0^t\left(({\delta_1}^2\Psi_n)_{xx}+(\Psi_n)_{yy}\right)dt,\quad n=0,1,\ 2,\ 3,\dots \tag{A11}$$

where $\psi(x, y, z, 0)=\Psi_0 = A_1 e^{-(c_1x+c_2y+c_3z)}$.
The iteration of n values in relation (A11) represents the solution of Eqn. (A10).

Conflict of interest

The authors have no conflicts of interest to disclose.

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