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On Congruences of Principal GK_2-Algebras

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Abstract: We investigate some features of principal GK_2-algebras (PGK_2-algebras). Necessary and sufficient conditions for a principal GK_2- algebra to have 2-permutable congruences are obtained. Furthermore, it is established how 2-permutable congruences are characterized using pairs of principal congruences. Also, a generalization of the 2-permutability of the primary congruences of the GK_2-algebras concept to the concept of the n-permutable congruences is provided. We round off with strong extensions of principal GK_2-algebras.

Keywords: *MS*-algebra, *GMS*-algebra, GK_2-algebra, principal GK_2-algebra, congruence pair, 2-permutability of congruences, *n*-permutability of congruences, strong extension

1 Introduction

T.S. Blyth and J.C. Varlet [1] introduced the variety **MS** of *MS*-algebras. In [2], they determined the subvarieties of **MS**. Many properties of *MS*-algebras, principal *MS*-algebras, principal *p*-algebras and decomposable *MS*-algebras are investigated in [3,4,5,6,7,8]. The variety **GMS** was defined and characterized by D. Ševčovič in [9]. Certain modular generalized MS-algebras with distributive skeletons, called K_2-algebras, were introduced by A. Badawy [10]. Each K_2-algebra was built using quadruples. A. Badawy [11] considered the subclass $\mathbf{GK_2}$ of GK_2-algebras. He constructed any PGK_2-algebra by means of triple. Also, he deduced that each congruence α on a GK_2-algebra L can be constructed by a congruence pair (α_1, α_2) in a unique way, where $\alpha_1 \in Con(L^{\circ\circ})$ and α_2 is a congruence of lattices on the bounded lattice $D(L)$. Many authors considered the concepts of permutable congruences, strong extensions and related properties (see [12], [13] and [14]).
This paper applies the concepts of 2-permutability of congruences and *n*-permutability of congruences to PGK_2-algebras. We characterize such concepts by using congruence pairs (α_1, α_2) of a principal GK_2-algebra L, where α_1 is a congruence on GK-algebra $L^{\circ\circ}$ of all closed elements of L, and α_2 is a lattice congruence on a lattice bounded $D(L)$. Also, we introduce and characterize the notion of strong extensions of PGK_2-algebras. We proved that a GK_2-algebra L is a strong extension of a subalgebra L_1 if and only if $L^{\circ\circ}$ is a strong extension of $L_1^{\circ\circ}$ and $D(L)$ is a strong extension of $D(L_1)$.

2 Preliminaries

This section contains the basic background and results. We refer to [9,11,15,16,17,18] for details. An *MS*-algebra is an algebra $(;\vee,\wedge,^\circ,0,1)$ such that $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and $^\circ$ is a unary operation satisfying:

(1) $r \leq r^{\circ\circ}$,
(2) $(r\wedge s)^\circ = r^\circ \vee s^\circ$,
(3) $1^\circ = 0$.

The subvariety **M** (De Morgan algebras) of **MS** is defined by

$$r = r^{\circ\circ} \tag{1}$$

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The subvariety **K** (Kleene algebras) of **M** is characterized by :

$$r \wedge r^\circ \leq s \vee s^\circ \tag{2}$$

The class **S**(Stone algebras) of **MS** is the subvariety which is defined by:

$$r \wedge r^\circ = 0 \tag{3}$$

The subvariety **B** (Boolean algebras) of **MS** is defined by the identity

$$r \vee r^\circ = 1 \tag{4}$$

A generalized De Morgan algebra (simply GM-algebra) $(L;\vee,\wedge,^\circ,0,1)$, where $(L;\vee,\wedge,0,1)$ is a bounded lattice with

(1) $r = r^{\circ\circ}$,
(2) $(r\wedge s)^\circ = r^\circ \vee s^\circ$,
(3) $1^\circ = 0$.

If a GM-algebra satisfies:

$$r \wedge r^\circ \leq s \vee s^\circ \tag{5}$$

it becomes a generalized Kleene algebra.
If we drop the distributivity condition of MS-algebra, we obtain GMS-algebra.

Lemma 2.1.[9] *For any two elements r,s of a GMS-algebra L, we have*

(1) $0^\circ = 1$,
(2) $r \leq s \Longrightarrow r^\circ \geq s^\circ$,
(3) $r^\circ = r^{\circ\circ\circ}$,
(4) $(r\vee s)^\circ = r^\circ \wedge s^\circ$,
(5) $(r\vee s)^{\circ\circ} = r^{\circ\circ} \vee s^{\circ\circ}$,
(6) $(r\wedge s)^{\circ\circ} = r^{\circ\circ} \wedge s^{\circ\circ}$.

Definition 2.1.[11] *A GK_2-algebra L is a GMS-algebra satisfying:*

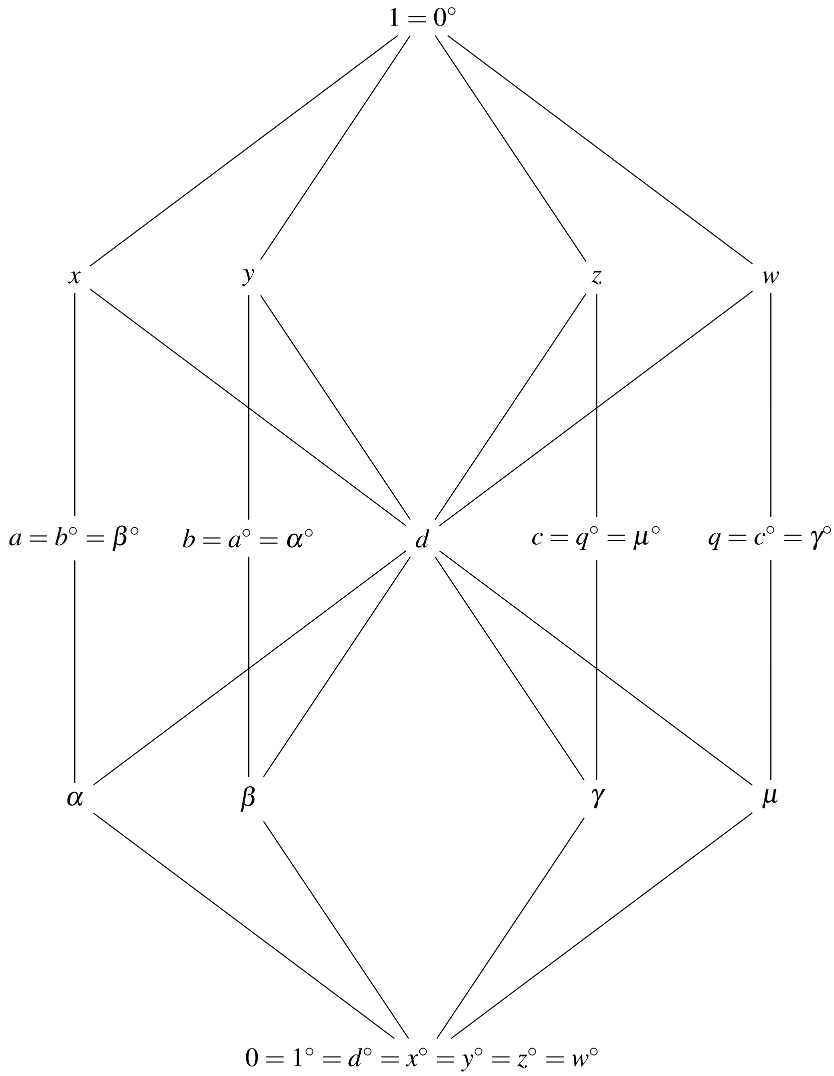
(1) $r\wedge r^\circ = r^{\circ\circ}\wedge r^\circ \ \forall\, r \in L$,
(2) $r\wedge r^\circ \leq s\vee s^\circ \ \forall\, r,s \in L$.

Let L be a GK_2-algebra. An element r of L is called closed if $r^{\circ\circ} = r$ and an element $d \in L$ is called dense if $d^\circ = 0$. Set $L^{\circ\circ}$ to denote the set of all closed elements of L and $D(L)$ for the set of all dense elements of L.

Lemma 2.2.[11] *Let $L \in GK_2$-algebra. Then*

(1) *$L^{\circ\circ}$ is a GK-subalgebra of L,*
(2) *$D(L)$ is a filter of L.*

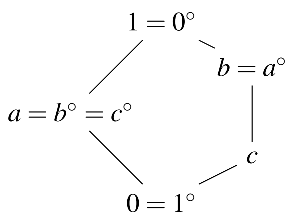
Example 2.1. (1) Every MS-algebra is a GMS-algebra.
(2) Every S-algebra (pseudo-complement lattice satisfying the Stone identity, $r^* \vee r^{**} = 1$,where $r^* = max\{s : s\wedge r = 0\}$ is the pseudo-complement of r) is a GMS-algebra.
(3) The following is a GMS-algebra $(L_1,\ ^\circ)$ satisfying the Stone identity $r^*\vee r^{**} = 1$. We observe that it is not an S-algebra; for example, the element μ has not pseudo-complement.



Also, we have
$L_1^{\circ\circ} = \{0, a, b, c, q, 1\}$ is a modular GK-algebra, and $D(L_1) = \{d, x, y, z, w, 1\}$ is a modular lattice.

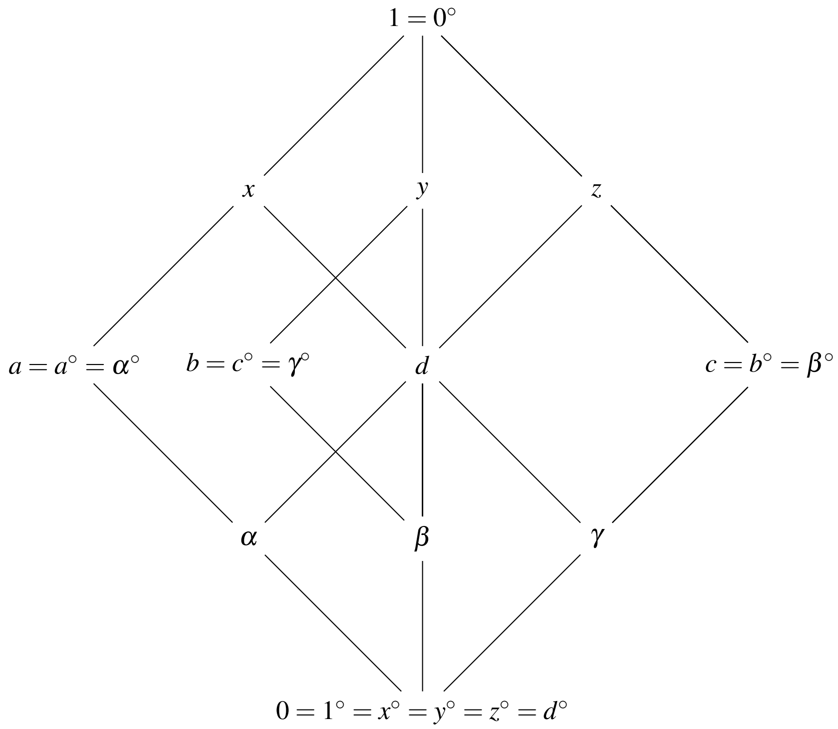
Definition 2.2.[11] *A GK_2-algebra L is a PGK_2-algebra if:*
(1) $D(L) = [d)$ for some $d \in L$,
(2) The generator d is distributive, that is, $(r \wedge s) \vee d = (r \vee d) \wedge (s \vee d)$ for all $r, s \in L$,
(3) $r = r^{\circ\circ} \wedge (r \vee d)$ for all $r \in L$.

Example 2.2. (1) Every K_2-algebra is a GK_2-algebra.
(2) Every S-algebra is a GK_2-algebra.
(3) The GK_2-algebra L_1 of Example 2.4(3) is a PGK_2-algebra which is not an S-algebra.
(4) The following GK_2-algebra represents an S-algebra L_2, where $L_2^{\circ\circ} = \{0, a, b, 1\}$ is a Boolean subalgebra and $D(L_2) = \{1\}$. It is clear that it is not a principal S-algebra as $c^{\circ\circ} \wedge (c \vee 1) \neq c$.



From this example, it is not true that every finite GK_2-algebra is principal.

(5) The following GMS-algebra is a PGK_2- algebra.



Definition 2.3.[6] *A binary relation α defined on a lattice L is said to be a lattice congruence if :*
(1) α is an equivalence relation on L,
(2) $(r,s),(u,v)\in\alpha$ implies $(r\wedge u, s\wedge v),(a\vee c, b\vee d)\in\alpha$.
For a congruence relation α on a lattice L, $[r]\alpha$ is given

$$[r]\alpha=\{t\in L:(t,r)\in\alpha\}. \tag{6}$$

It can be prove that $(L/\alpha,\vee,\wedge)$ forms a lattice, where

$$L/\alpha=\{[r]\alpha: r\in L\} \tag{7}$$

is the quotient lattice of L modulo α and

$$[r]_\alpha\vee[s]_\theta=[r\vee s]_\alpha \text{ and } [r]_\alpha\wedge[s]_\alpha=[r\wedge s]_\alpha \tag{8}$$

A lattice congruence α on a GK_2-algebra $(L;\,^\circ)$ is called a congruence on L if $r\equiv s(\alpha)$ implies $r^\circ\equiv s^\circ(\alpha)$.
For a GK_2-algebra L, Con(L) is used to denote the set of all congruence on L and $\alpha_{L^{\circ\circ}},\alpha_{D(L)}$ are used for α restricted to $L^{\circ\circ}$ and $D(L)$, respectively. Obviously, $(\alpha_{L^{\circ\circ}},\alpha_{D(L)})\in$Con$(L^{\circ\circ})\times$Con$(D(L))$. Also, we use $\bigtriangledown_L=L\times L$ and $\triangle_L=\{(r,r): r\in L\}$ for the universal and the identity congruences on L, respectively.
A congruence relation α on a lattice L is called principal if there exist $r,s\in L$ such that α is the smallest congruence relation for which $r\equiv s(\alpha)$. Indeed,

$$\alpha(r,s)=\bigwedge\{\alpha\in Con(L)\mid r\equiv s(\alpha)\} \tag{9}$$

Definition 2.4.[11] *Let d be the smallest dense element of a PGK_2-algebra L . Then a pair $(\alpha_1,\alpha_2)\in Con(L^{\circ\circ})\times Con(D(L))$ is called a congruence pair of L if $r\equiv s(\alpha_1)$ implies $r\vee d\equiv s\vee d(\alpha_2)$.*

A characterization of a congruence relation on PGK_2-algebras is given as follows:

Theorem 2.1.[11] *Let d be the smallest dense element of a PGK_2-algebra L. Then any $\alpha\in Con(L)$ determines a congruence pair $(\alpha_{L^{\circ\circ}},\alpha_{D(L)})$. Conversely, any congruences pair (α_1,α_2) uniquely determines an $\alpha\in Con(L)$satisfies $\alpha_{L^{\circ\circ}}=\alpha_1$ and $\alpha_{D(L)}=\alpha_2$, by the rule: $r\equiv s(\alpha)\Leftrightarrow r^{\circ\circ}\equiv s^{\circ\circ}(\alpha_1)$ and $r\vee d\equiv s\vee d(\alpha_2)$.*

Lemma 2.3.[11] *Let L be a PGK_2-algebra and let $A(L)$ be the set of all congruence pairs of L. Then :*
(1) $(\forall\beta\in Con(D(L)))(\triangle_{L^{\circ\circ}},\beta)\in A(L)$,
(2) $(\forall\eta\in Con(L^{\circ\circ}))(\eta,\nabla_{D(L)})\in A(L)$.

3 2-Permutability of PGK_2-algebras

We extend the concept of 2-permutability of congruences to PGK_2-algebras. Some basic properties are proved, and necessary and sufficient conditions for a principal GK_2-algebra to have 2-permutable congruences are provided. Moreover, it is established how to characterise 2-permutable congruences in terms of pairs of main congruences.

Definition 3.1. *Let L be a PGK_2-algebra. Then $\alpha, \delta \in Con(L)$ are 2-permutable congruences (briefly 2-permutable) if $\alpha \circ \delta = \delta \circ \alpha$, that is, $r \equiv s(\alpha)$ and $s \equiv p(\delta)$ imply the existence of an element $u \in L$ such that $r \equiv u(\delta)$ and $u \equiv p(\alpha)$.*

A PGK_2-algebra L is called 2-permutable congruences if any pair of congruences permute. Let L be a principal GK_2-algebra. Define a relation Γ on L as follows:

$$\begin{aligned}(r,s) \in \Gamma &\Longleftrightarrow r^{\circ\circ} = s^{\circ\circ}\\ &\Longleftrightarrow r^{\circ} = s^{\circ}.\end{aligned}$$

Lemma 3.1. *Let L be a PGK_2-algebras. Then*

(1) $\Gamma \in Con(L)$ with $Ker\,\Gamma = \{0\}$ and $Coker\,\Gamma = D(L)$,
(2) $r^{\circ\circ}$ is the maximum element of the $[r]\Gamma$, where $[r]\Gamma = \{s \in L : s^{\circ\circ} = r^{\circ\circ}\}$,
(3) $[r]\Gamma = [r^{\circ\circ}]\Gamma$ for any $r \in L$,
(4) L/Γ is a GK-algebra,
(5) $L/\Gamma \cong L^{\circ\circ}$.

Proof. (1) It is straightforward to show that Γ is an equivalent relation on L. Let $(r,s),(u,v) \in \Gamma$. Then $r^{\circ\circ} = s^{\circ\circ}$ and $u^{\circ\circ} = v^{\circ\circ}$. Now we have

$$\begin{aligned}(r\wedge u)^{\circ\circ} &= r^{\circ\circ} \wedge u^{\circ\circ}\\ &= s^{\circ\circ} \wedge v^{\circ\circ}\\ &= (s\wedge v)^{\circ\circ}.\end{aligned}$$

Then $(r\wedge u, s\wedge v) \in \Gamma$. Also, we have

$$\begin{aligned}(r\vee u)^{\circ\circ} &= r^{\circ\circ} \vee u^{\circ\circ}\\ &= s^{\circ\circ} \vee v^{\circ\circ}\\ &= (s\vee v)^{\circ\circ}.\end{aligned}$$

Then $(r\vee u, s\vee v) \in \Gamma$. Now, let $(r,s) \in \Gamma$. Then we have

$$\begin{aligned}(r,s) \in \Gamma &\Longrightarrow r^{\circ\circ} = s^{\circ\circ}\\ &\Longrightarrow r^{\circ\circ\circ} = s^{\circ\circ\circ}\\ &\Longrightarrow (r^{\circ}, s^{\circ}) \in \Gamma.\end{aligned}$$

Then $\Gamma \in Con(L)$. We observe that

$$\begin{aligned}Ker\,\Gamma &= \{r \in L : (r,0) \in \Gamma\}\\ &= \{r \in L : r^{\circ\circ} = 0^{\circ\circ} = 0\}\\ &= \{r \in L : r^{\circ} = 1\}\\ &= \{0\}.\end{aligned}$$

Moreover,

$$\begin{aligned}Coker\,\Gamma &= \{r \in L : (r,1) \in \Gamma\}\\ &= \{r \in L : r^{\circ\circ} = 1^{\circ\circ} = 1\}\\ &= \{r \in L : r^{\circ} = 0\}\\ &= D(L).\end{aligned}$$

(2) Since $(r^{\circ\circ})^{\circ\circ} = r^{\circ\circ}$, then $r^{\circ\circ} \in [x]\Gamma$. Let $s \in [r]\Gamma$. Then $s \leq s^{\circ\circ} = r^{\circ\circ}$. So, $r^{\circ\circ} \geq s$. Hence, $r^{\circ\circ}$ is the greatest element of $[r]\Gamma$.
(3) Since $r^{\circ\circ\circ} = r^{\circ}$. Then $r^{\circ\circ\circ\circ} = r^{\circ\circ}$ implies $(r^{\circ\circ}, r) \in \Gamma$, and thereby $[r^{\circ\circ}]\Gamma = [r]\Gamma$, $\forall r \in L$.
(4) We have $(L/\Gamma; \vee, \wedge, [0]\Gamma, [1]\Gamma)$ is a bounded lattice with bounds $[0]\Gamma$ and $[1]\Gamma$, where $[r]\Gamma \wedge [s]\Gamma = [r \wedge s]\Gamma$ and $[r]\Gamma \vee [s]\Gamma = [r \vee s]\Gamma$. Define $^{\square}$ on L/Γ by $([r]\Gamma)^{\square} = [r^{\circ}]\Gamma$. Now, we have the following equalities

$$([0]\Gamma)^{\square} = [1]\Gamma \text{ and } ([1]\Gamma)^{\square} = [0]\Gamma,$$

$$([r]\Gamma)^{\square\square} = [r^{\circ\circ}]\Gamma = [x]\Gamma,$$

$$\begin{aligned}([r]\Gamma \wedge [s]\Gamma)^{\square} &= ([r \wedge s]\Gamma)^{\square} \\ &= [(r \wedge s)^{\circ}]\Gamma \\ &= [r^{\circ} \vee s^{\circ}]\Gamma \\ &= [r^{\circ}]\Gamma \vee [s^{\circ}]\Gamma \\ &= ([r]\Gamma)^{\square} \vee ([s]\Gamma)^{\square}.\end{aligned}$$

Then L/Γ is a GM-algebra. Since $r \wedge r^{\circ} \leq s \vee s^{\circ}$, then $[r \wedge r^{\circ}]\Gamma \leq [s \vee s^{\circ}]\Gamma$. Hence,

$$\begin{aligned}[r]\Gamma \wedge ([r]\Gamma)^{\square} &= [r]\Gamma \wedge [r^{\circ}]\Gamma \\ &= [r \wedge r^{\circ}]\Gamma \\ &\leq [s \vee s^{\circ}]\Gamma \\ &= [s]\Gamma \vee [s^{\circ}]\Gamma \\ &= [s]\Gamma \vee ([s]\Gamma)^{\square}.\end{aligned}$$

Thus, L/Γ is a GK-algebra.
(5) Define $f : L^{\circ\circ} \longrightarrow L/\Gamma$ by

$$f(r) = [r]\Gamma \ \forall r \in L^{\circ\circ} \tag{10}$$

It is clear that f is well-defined. Let $f(r) = f(s)$. Then $[r]\Gamma = [s]\Gamma$ implies $r \equiv s(\Gamma)$. Then $r = r^{\circ\circ} = s^{\circ\circ} = s$ as $r, s \in L^{\circ\circ}$. Then f is one-to-one. Let $[s]\Gamma \in L/\Gamma$ for some $s \in L$. Then $[s] = [s^{\circ\circ}]\Gamma$ and so $f(s^{\circ\circ}) = [s^{\circ\circ}]\Gamma = [s]\Gamma$. Then f is onto Also, we need to show that f is a homomorphism. Clearly, $f(r \vee s) = f(r) \vee f(s)$ and $f(r \wedge s) = f(r) \wedge f(s)$. Also,

$$\begin{aligned}f(r^{\circ}) &= [r^{\circ}]\Gamma \\ &= [r^{\circ\circ\circ}]\Gamma \\ &= ([r^{\circ\circ}]\Gamma)^{\square} \\ &= ([r]\Gamma)^{\square} \\ &= (f(r))^{\square}.\end{aligned}$$

Clearly $f(0) = [0]\Gamma$ and $f(1) = [1]\Gamma$. Hence, $L^{\circ\circ} \cong L/\Gamma$.

Lemma 3.2. *Let L be a PGK_2-algebras. Then:*

(1) Γ permutes with any $\alpha \in$Con(L),
(2) $\triangle_L$ permutes with any $\alpha \in$Con(L),
(3) $\bigtriangledown_L$ permutes with any $\alpha \in$Con(L).

Proof. (1) Let $\alpha \in$Con(L). Then we need to show that $\alpha \circ \Gamma = \Gamma \circ \alpha$. Let $r \equiv s(\alpha \circ \Gamma)$. Then $r \equiv p(\alpha)$ and $p \equiv s(\Gamma)$ for some $p \in L$. So, $r \equiv p(\alpha)$ and $p^{\circ\circ} = s^{\circ\circ}$. Now

$$\begin{aligned}r \equiv p(\alpha) &\Longrightarrow r^{\circ\circ} \equiv p^{\circ\circ}(\alpha), s \vee d \equiv s \vee d(\alpha) \\ &\Longrightarrow r^{\circ\circ} \equiv s^{\circ\circ}(\alpha), s \vee d \equiv s \vee d(\alpha) \text{ as } p^{\circ\circ} = s^{\circ\circ} \\ &\Longrightarrow r^{\circ\circ} \wedge (s \vee d) \equiv s^{\circ\circ} \wedge (s \vee d)(\alpha) = s(\alpha) \text{ as } s = s^{\circ\circ} \wedge (s \vee d).\end{aligned}$$

Since $[r^{\circ\circ} \wedge (s \vee d)]^{\circ\circ} = r^{\circ\circ}$, then $r^{\circ\circ} \wedge (s \vee d) \equiv r(\Gamma)$. Since $r \equiv r^{\circ\circ} \wedge (s \vee d)(\Gamma)$ and $r^{\circ\circ} \wedge (s \vee d) \equiv s(\alpha)$, then $r \equiv s(\Gamma \circ \alpha)$.
(2) Let $r \equiv s(\alpha \circ \triangle_L)$. Then $r \equiv p(\alpha), p \equiv s(\triangle_L)$ for some $p \in L$. Hence $r \equiv s(\alpha)$ as $p = s$. Then, $r \equiv r(\triangle_L)$ and $r \equiv s(\alpha)$. Thus, we deduced that $r \equiv s(\triangle_L \circ \alpha)$. Therefore, $\triangle_L$ permutes with any element of Con(L).
(3) Let $r \equiv s(\alpha \circ \bigtriangledown_L)$. Then $r \equiv p(\alpha)$, $p \equiv s(\bigtriangledown_L)$ for some $p \in L$. Then we have $r \equiv s(\bigtriangledown_L)$ and $s \equiv s(\alpha)$. Thus, $r \equiv s(\bigtriangledown_L \circ \alpha)$. Therefore, $\bigtriangledown_L$ permutes with any element of Con(L).

Now, we provide a characterization of 2-permutable congruences.

Theorem 3.1. *Let d be the smallest dense element of a PGK_2-algebra L . Then L has 2-permutable congruences if and only if:*

(1) $L^{\circ\circ}$ has 2-permutable congruences,
(2) $D(L)$ has 2-permutable congruences.

Proof. Suppose that α,δ are 2-permutable on L. First, we prove that $\alpha_{L^{\circ\circ}},\delta_{L^{\circ\circ}}$ are 2-permutable on $L^{\circ\circ}$. Consider that $r,s,p\in L^{\circ\circ}$ be such that $r\equiv s(\alpha_{L^{\circ\circ}})$ and $s\equiv p(\delta_{L^{\circ\circ}})$. Then $r\equiv s(\alpha)$ and $s\equiv p(\delta)$. Since α,δ are 2-permutable, we have $r\equiv q(\delta), q\equiv p(\alpha)$ for some $q\in L$. Now,

$$\begin{aligned} r\equiv q(\delta),q\equiv p(\alpha) &\Longrightarrow r^{\circ\circ}\equiv q^{\circ\circ}(\delta),q^{\circ\circ}\equiv p^{\circ\circ}(\alpha)\\ &\Longrightarrow r\equiv q^{\circ\circ}(\delta),q^{\circ\circ}\equiv p(\alpha) \text{ as } r,p\in L^{\circ\circ}\\ &\Longrightarrow r\equiv q^{\circ\circ}(\delta_{L^{\circ\circ}}),q^{\circ\circ}\equiv p(\alpha_{L^{\circ\circ}}) \text{ as } q^{\circ\circ}\in L^{\circ\circ}.\end{aligned}$$

Therefore $\alpha_{L^{\circ\circ}},\delta_{L^{\circ\circ}}$ are 2-permutable on $L^{\circ\circ}$ and (1) is proved. Secondly, we show that 2-permutability of α,δ implies 2-permutability of $\alpha_{D(L)}$ and $\delta_{D(L)}$. Let $r,s,p\in D(L)$ be such that $r\equiv s(\alpha_{D(L)})$ and $s\equiv p(\delta_{D(L)})$. Then $r\equiv s(\alpha)$, $s\equiv p(\delta)$. Since α,δ are 2-permutable, then $r\equiv u(\delta)$ and $u\equiv p(\alpha)$ for some $u\in L$. Now,

$$\begin{aligned} r\equiv u(\delta),u\equiv p(\alpha) &\Longrightarrow r\vee d\equiv u\vee d(\delta),u\vee d\equiv p\vee d(\alpha)\\ &\Longrightarrow r\equiv u\vee d(\delta),u\vee d\equiv p(\alpha) \text{ as } r,p\geq d\\ &\Longrightarrow r\equiv u\vee d(\delta),u\vee d\equiv p(\alpha) \text{ where } u\vee d\in D(L).\end{aligned}$$

Hence $r\equiv u\vee d(\delta_{D(L)})$ and $u\vee d\equiv p(\alpha_D(L))$. Therefore $\alpha_{D(L)}$ and $\delta_{D(L)}$ are 2-permutable congruences on $D(L)$. For the converse direction, let $\alpha,\delta\in$Con(L) such that $\alpha_{L^{\circ\circ}},\delta_{L^{\circ\circ}}$ and $\alpha_{D(L)},\delta_{D(L)}$ are 2-permutable on $L^{\circ\circ}$ and $D(L)$, respectively. Consider the elements $r,s,p\in L$ with $r\equiv s(\alpha)$ and $s\equiv p(\delta)$. We have, by Theorem 2.9, that $r^{\circ\circ}\equiv s^{\circ\circ}(\alpha_{L^{\circ\circ}})$ and $s^{\circ\circ}\equiv p^{\circ\circ}(\delta_{L^{\circ\circ}})$. Since $\alpha_{L^{\circ\circ}},\delta_{L^{\circ\circ}}$ are 2-permutable congruences on $L^{\circ\circ}$, then $r^{\circ\circ}\equiv u(\delta_{L^{\circ\circ}})$ and $u\equiv p^{\circ\circ}(\alpha_{L^{\circ\circ}})$ with $u\in L^{\circ\circ}$ implies that $r^{\circ\circ}\equiv u(\delta)$ and $u\equiv p^{\circ\circ}(\alpha)$. On the other hand, also by Theorem 2.9, we get $r\vee d\equiv s\vee d(\alpha_{D(L)})$ and $s\vee d\equiv p\vee d(\delta_{D(L)})$. Since $\alpha_{D(L)},\delta_{D(L)}$ are 2-permutable congruences on $D(L)$, then $r\vee d\equiv v(\delta_{D(L)})$ and $v\equiv p\vee d(\alpha_{D(L)})$ for some $v\in D(L)$. It follows that

(11)

Since L is a PGK_2-algebra, then we have $r=r^{\circ\circ}\wedge(r\vee d)$ and $p=p^{\circ\circ}\wedge(p\vee d)$. Then we have

$$r^{\circ\circ}\equiv u(\delta),r\vee d\equiv v(\delta) \text{ imply that } r=r^{\circ\circ}\wedge(r\vee d)\equiv u\wedge v(\delta), \tag{12}$$

and

$$u\equiv p^{\circ\circ}(\alpha),v\equiv p\vee d(\alpha) \text{ imply that } a\wedge v\equiv p^{\circ\circ}\wedge(p\vee d)(\alpha)=p \tag{13}$$

Consequently, we deduce that $r\equiv u\wedge v(\delta)$ and $u\wedge v\equiv p(\alpha)$. Therefore α,δ are 2-permutable congruences.

Theorem 3.2. *A PGK_2-algebra L has 2-permutable congruences if and only if every pair of principal congruences on L permutes.*

Proof. The first statement is obvious. Assume that any pair of principal congruences on L permute. Let $\alpha,\delta\in$Con(L). Consider $r,s,p\in L$ with $r\equiv s(\alpha)$ and $s\equiv p(\delta)$. Then $r\equiv s(\alpha(r,s))$, $s\equiv p(\delta(s,p))$. It is clear that $\alpha(r,s)\subseteq\alpha$ and $\delta(s,p)\subseteq\delta$. Hence, $r\equiv p(\alpha(r,s)\circ\delta(s,p))$. Since $\alpha(r,s),\delta(s,p)$ are 2-permutable, then $r\equiv u(\delta(s,p))$ and $u\equiv p(\alpha(r,s))$ for some $u\in L$. Consequently, $r\equiv u(\delta)$ and $u\equiv p(\alpha)$ and hence $r\equiv p(\delta\circ\alpha)$.

4 n-Permutability of PGK_2-algebras

The results of this section extend the 2-permutability of congruences of PGK_2-algebras to n-permutable congruences. Two congruences α,δ are n-permutable if

$$\alpha\circ\delta\circ\alpha\circ\ldots\ldots\ldots(n-\text{times})=\delta\circ\alpha\circ\delta\circ\ldots\ldots\ldots.(n-\text{times}),\ \text{where } n=1,2,\ldots,n-1 \tag{14}$$

Definition 4.1. *A principal GK_2-algebras L has n-permutable congruences, if every two congruences in L are n-permutable.*

Lemma 4.1. *Let d be the smallest dense element of a PGK_2-algebra L. Let θ,ψ be congruences on L. Then*

(1) $(\alpha\circ\delta\circ\alpha\circ..........)_{L^{\circ\circ}} = \alpha_{L^{\circ\circ}}\circ\delta_{L^{\circ\circ}}\circ\alpha_{L^{\circ\circ}}\circ......(n-times)$,
(2) $(\alpha\circ\delta\circ\alpha\circ......)_{D(L)} = \alpha_{D(L)}\circ\delta_{D(L)}\circ\alpha_{D(L)}\circ.................(n-times)$.

Proof. (1) To show the equality

(15)

Now, let $r,s\in L^{\circ\circ}$ with $r\equiv s(\alpha\circ\delta\circ......)_{L^{\circ\circ}}$. Then $r\equiv s(\alpha\circ\delta\circ......)$. Thus there exist elements $t_1,t_2,..........,t_{n-1}\in L$ be such that $r\equiv t_1(\alpha)$, $t_1\equiv t_2(\delta)$,.......$t_{n-1}\equiv s(\nu)$, where

$$\nu = \begin{cases}\alpha & \text{if n is odd}\\ \delta & \text{if n is even}\end{cases} \tag{16}$$

We have, $r^{\circ\circ}=r\equiv t_1{}^{\circ\circ}(\alpha)$, $t_1{}^{\circ\circ}\equiv t_2{}^{\circ\circ}(\delta)$,..........., $t_{n-1}^{\circ\circ}\equiv s^{\circ\circ}=s(\nu)$,
Then, $r\equiv s(\alpha_{L^{\circ\circ}}\circ\delta_{L^{\circ\circ}}\circ...........)$ because of $t_n{}^{\circ\circ}\in L^{\circ\circ}$ for $n=1,2,....,n-1$.
The reverse inclusion is obvious. Hence,

$$(\alpha\circ\delta\circ.....)_{L^{\circ\circ}} = (\alpha_{L^{\circ\circ}}\circ\delta_{L^{\circ\circ}}.........). \tag{17}$$

(2) Let $r,s\in D(L)$ be such that $r\equiv s((\theta\circ\psi\circ.......)_{D(L)})$, that is $r\equiv s(\alpha\circ\delta\circ....)$. Then there exist $t_1,t_2,......,t_{n-1}\in L$ be such that $r\equiv t_1(\alpha)$, $t_1\equiv t_2(\delta)$, ...$t_{n-1}\equiv b(\nu)$. Then, $r=r\vee d\equiv t_1\vee d(\alpha)$,.........,$t_{n-1}\vee d\equiv b\vee d=s(\nu)$. Therefore, $r\equiv s(\alpha_{D(L)}\circ\delta_{D(L)}\circ.......)$ since $t_n\vee d\in D(L)$ for $n=1,2,.....,n-1$. The reverse inclusion is obvious. Therefore,

$$(\alpha\circ\delta\circ...........)_{D(L)} = (\alpha_{D(L)}\circ\delta_{D(L)}.........(n-\text{time}) \tag{18}$$

Theorem 4.1. *Let d be the smallest dense element of a PGK_2-algebra L. Then L has n-permutable congruences if and only if $L^{\circ\circ}$ and $D(L)$ are n-permutable congruences.*

Proof. ($\Longrightarrow$:) By using Lemma 4.2(1) we have

$$\begin{aligned}\alpha_{L^{\circ\circ}}\circ\delta_{L^{\circ\circ}}\circ.... &= (\alpha\circ\delta\circ....)_{L^{\circ\circ}}\\ &= (\delta\circ\alpha\circ....)_{L^{\circ\circ}}\\ &= \delta_{L^{\circ\circ}}\circ\alpha_{L^{\circ\circ}}\circ....\end{aligned}$$

Again by using Lemma 4.2(2) we have

$$\begin{aligned}\alpha_{D(L)}\circ\delta_{D(L)}\circ.... &= (\alpha\circ\delta\circ....)_{D(L)}\\ &= (\delta\circ\alpha\circ....)_{D(L)}\\ &= \delta_{D(L)}\circ\alpha_{D(L)}\circ....\end{aligned}$$

($\Longleftarrow$:) Let $r\equiv s(\alpha\circ\delta\circ...)$. Then $r^{\circ\circ}\equiv s^{\circ\circ}((\alpha\circ\delta\circ...)_{L^{\circ\circ}})$ and $r\vee d\equiv s\vee d((\alpha\circ\delta\circ....)_{D(L)})$ by Theorem 2.9. Applying Lemma 4.2 we have

(19)

Since

$$\alpha_{L^{\circ\circ}}\circ\delta_{L^{\circ\circ}}\circ.... = \delta_{L^{\circ\circ}}\circ\alpha_{L^{\circ\circ}}\circ....(n-\text{times}) \text{ and } \alpha_{D(L)}\circ\delta_{D(L)}\circ.... = \delta_{D(L)}\circ\alpha_{D(L)}\circ....(n-\text{times}), \tag{20}$$

then we get

$$r^{\circ\circ}\equiv s^{\circ\circ}((\delta\circ\alpha\circ...)_{L^{\circ\circ}}) \text{ and } r\vee d\equiv s\vee d((\delta\circ\alpha\circ...)_{D(L)}). \tag{21}$$

Now, by using Definition 2.5(3) and Theorem 2.9, we get

(22)

Therefore, $r\equiv s(\delta\circ\alpha\circ........)$. Thus, we deduce that δ and α are n permutable.

5 Strong extensions of PGK_2-algebras

The concept of strong extensions of PGK_2-algebras is investigated in this section.
An algebra L satisfies the congruence extension property (CEP); if for every subalgebra L_1 of L and every α of L_1, α extends to a congruence of L.(see [19])

Definition 5.1. *Let M_1 and N be a PGK_2-algebras. Then we call the algebra K a strong extension of the algebra K_1 if K_1 is a subalgebra of K and for any $\alpha_{1\in}Con(K_1)$, there exists a unique congruence relation $\alpha \in Con(K)$ such that $\alpha_{K_1} = \alpha_1$.*

Theorem 5.1. *Let K_1 be a subalgebra of a PGK_2-algebra K. Then K is a strong extension of K_1 if and only if*

(1) $D(K)$ is a strong extension of $D(K_1)$,
(2) $K^{\circ\circ}$ is a strong extension of ${K_1}^{\circ\circ}$.

Proof. Let K be a strong extension of K_1. Let $\eta_2 \in$Con$(D(K_1))$. Assume that $\acute{\eta}_2, \bar{\eta}_2 \in$Con$(D(K))$ such that $\acute{\eta}_{2,D(K_1)} = \bar{\eta}_{2,D(K_1)} = \eta_2$ Then, by Lemma 2.10(1), we have

$$(\triangle_{K^{\circ\circ}}, \acute{\eta}_2), (\triangle_{K^{\circ\circ}}, \bar{\eta}_2) \in A(K) \text{ and } (\triangle_{K_1^{\circ\circ}}, \eta_2) \in A(K_1). \tag{23}$$

According to Theorem 2.9, we have $\acute{\eta}, \bar{\eta} \in$ Con(K) and $\eta \in$Con(K_1) corresponding to $(\triangle_{K^{\circ\circ}}, \acute{\eta}_2), (\triangle_{K^{\circ\circ}}, \bar{\eta}_2)$ and $\eta = (\triangle_{K_1^{\circ\circ}}, \eta)$, respectively. We see that $\acute{\eta}_{K_1} = \bar{\eta}_{K_1} = \eta$. We have $\acute{\eta} = \bar{\eta}$. Hence, $\acute{\eta}_2 = \bar{\eta}_2$ proving (1). On the other hand, we need to show that $K^{\circ\circ}$ is a strong extension of $K_1^{\circ\circ}$. Let $\eta_1 \in$Con$({K_1}^{\circ\circ})$ and η_1 extend to a congruence of $K^{\circ\circ}$. Let $\acute{\eta}_1, \bar{\eta}_1 \in$Con $(K^{\circ\circ})$ with $\acute{\eta}_{1,K_1^{\circ\circ}} = \bar{\eta}_{1,K_1^{\circ\circ}} = \eta_1$. Then, by Lemma 2.10(2), we have

$$(\acute{\eta}_1, \triangledown_{D(K)}), (\bar{\eta}_1, \triangledown_{D(K)}) \in A(K) \text{ and } (\eta, \triangledown_{D(K_1)}) \in A(K_1). \tag{24}$$

Again, by Theorem 2.9, we have $\acute{\eta}, \bar{\eta} \in$Con$(K)$ and $\eta \in$Con(K) corresponding to $(\acute{\eta}_1, \triangledown_{D(K)}), (\bar{\eta}_1, \triangledown_{D(K)})$ and $\eta = (\eta, \triangledown_{D(K_1)})$, respectively. We see that $\acute{\eta}_{K_1} = \bar{\eta}_{K_1} = \eta$. Since K is a strong extension of K_1, then $\acute{\eta} = \bar{\eta}$. Therefore $\acute{\eta}_1 = \bar{\eta}_1$, proving that (2). Conversely, suppose that conditions (1) and (2) hold and let $\eta \in$ Con(K_1). Let $\acute{\eta}, \bar{\eta}$ be extensions of η in Con(K). By Theorem 2.9, the congruences $\acute{\eta}, \bar{\eta}$ and η can be represented by the congruence pairs $(\acute{\eta}_1, \acute{\eta}_2), (\bar{\eta}_1, \bar{\eta}_2)$ and (η_1, η_2), respectively. Where

$$\acute{\eta}_{1,K_1^{\circ\circ}} = \bar{\eta}_{1,K_1^{\circ\circ}} = \eta_1 \text{ and } \acute{\eta}_{2,D(K_1)} = \bar{\eta}_{2,D(K_1)} = \eta_2. \tag{25}$$

By (1) and (2) we get

$$\acute{\eta}_1 = \bar{\eta}_1 \text{ and } \acute{\eta}_2 = \bar{\eta}_2. \tag{26}$$

Therefore, $\acute{\eta} = \bar{\eta}$.

Corollary 5.1. *Let K_1 and K be PGK_2-algebras. If K_1 is a strong extension of K, then $Con(K_1) \cong Con(K)$.*

6 Conclusion

The following three key concepts in algebraic structures: 2-Permutability, n- Permutability, and strong extensions were examined for the PGK_2-algebras via congruence pairs. this paper's work could be further developed to study many aspects of GK_2-algebras and related structures. For instance, it can be applied to triple construction of GK_2-algebras, perfect extensions of PGK_2-algebras and substructures of PGK_2-algebras.

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