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Analytical Methods for the Solution of Linear Fractional Order Systems

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Abstract: This paper introduces fractional-order systems, which are modeled by differential equations involving real number orders for their derivatives. Various numerical and approximation methods are commonly used to derive solutions for equations that lack exact analytical solutions. The primary goal of this study is to develop analytical approaches and acquire a clear commensurate-order fractional linear system state equation solution expression.

Keywords: Fractional order calculus, functions of matrices, fractional order differential equation, state-space representation.

1 Introduction

The applications of FODEs have gained much attention and importance in recent decades due to their extensive applications in various fields, including control systems, porous media, electrochemistry, viscoelasticity, and electromagnetism theory. [1,2,3,4]. Fractional calculus extends integer-order calculus to a real or complex order [2].

The fundamental operator of fractional differentiation and integration is a cornerstone in the analysis and solution of FDEs and their applications, and it is a key concept in the field of FC is defined as (1). Specifically, the real-order generalization of this operator can be introduced as:

$$ {}_aD_t^m = \begin{cases} \frac{d^m}{dt^m}, & m > 0 \\ 1, & m = 0 \\ \int_a^t (d\tau)^m, & m < 0 \end{cases} \tag{1} $$

where $[m \in \mathbb{R}]$.

There are multiple fractional operators definitions that have been proposed in the literature. However, the most widely used ones are the Riemann-Liouville, Grünwald-Letnikov and Caputo which under certain conditions, these expressions can be considered represent for a wide range of functions [2].

In this article, we introduced the fundamentals of fractional calculus to researchers new to the field. The main definitions, properties as well as the Laplace transform of these operators are discussed. Important functions used in fractional calculus as well as systems of fractional order described in fractional differential equations are presented and briefly examined.

2 Fractional operators

Fractional operators can be seen as a natural extension of integer-order operators, where differentiation of a function f is considered for successive integer orders.

$$ \frac{d^n}{dt^n} f(t) \equiv f^{(n)}(t) = \lim_{h \to 0} \frac{1}{h^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(t - jh) \tag{2} $$

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where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$.

2.1 *Basic definitions of fractional operators*

2.1.1 Riemann-Liouville definitions

The fractional order m integral in fractional calculus for the $f(t)$ is defined as [4]:

$$ {}_{a}I_{t}^{m} f(t) = {}_{a}D_{t}^{-m} f(t) = \frac{1}{\Gamma(m)} \int_{a}^{t} (t-\tau)^{m+1} f(\tau)\, d\tau \tag{3} $$

where $a < t$, $m > 0$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the Gamma function.
The Riemann-Liouville fractional derivative is foundational for much of the theory and applications of fractional calculus and is often used in the context of mathematical analysis and modeling is as follows [4]:

$$ {}_{RL}D_{t_0}^{m} f(t) = \frac{1}{\Gamma(n-m)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-\tau)^{n-m-1} f(\tau)\, d\tau \tag{4} $$

where $t_0 < t$ and the integer n is such that $(n-1) < m < n$.

2.1.2 Definition of Grünwald-Letnikov

Extending the expression in (2) to fractional orders leads to the definition of Grünwald-Letnikov , which defines the integral or derivative of fractional m of $f(t)$ as follows [4]:

$$ {}_{GL}D_{t}^{m} f(t) = \lim_{h \to 0} \frac{1}{h^m} \sum_{j=0}^{\infty} (-1)^j \binom{m}{j} f(t - jh) \tag{5} $$

where $m \in R$ with $m < 0$ for the derivative and $m > 0$ for the integral and $\binom{m}{j} = \frac{\Gamma(m+1)}{\Gamma(j+1)\,\Gamma(m-j+1)}$. This expression is widely used in the literature for numerical calculation of fractional differentiation and integration.

2.1.3 Caputo definition

Caputo introduced another formulation of the fractional derivative defined by [4]:

$$ {}_{C}D_{t_0}^{m} f(t) = \frac{1}{\Gamma(n-m)} \int_{t_0}^{t} (t-\tau)^{n-m-1} f^{(n)}(\tau)\, d\tau \tag{6} $$

where the integer n is such that $(n-1) < m < n$ and $f^{(n)}(t)$ is the integer order derivative n of the $f(t)$. The definitions of Riemann-Liouville ${}_{RL}\mathrm{D}_t{}^m f(t)$ and Caputo ${}_{C}\mathrm{D}_t{}^m f(t)$ are equivalent and are related by the following relation [4]:

$$ {}_{RL}D_{t}^{m} f(t) = {}_{C}D_{t}^{m} f(t) + \sum_{k=0}^{n-1} \frac{(t)^{(k-m)}}{\Gamma(k-m+1)} f^{(k)}(0^+) \tag{7} $$

for $f^{(k)}(a) = 0$, where $(k = 0, 1, ..., n-1)$.

3 Linear fractional order systems

Over the past few decades, FC has been increasingly linked to multiple applications in science and engineering, where fractional derivatives and integrals have been found to provide accurate models for a wide range of processes. This has led to the development of fractional FDEs, which provide a precise demonstrate of the properties of these processes, thereby enabling a deeper understanding of complex phenomena [1,2,3], [5,6,7]. As a result, much emphasis has been placed on fractional systems, with the goal of developing trustworthy and efficient methods for their representation, analysis, and solution. This section is dedicated to exploring the representation, stability analysis, observability and controllability of Linear fractional systems with commensurate orders, as well as techniques for solving these systems [4], [1,8,9,10].

3.1 Fractional linear systems with commensurate orders

An important class of LIT fractional systems is the set of fractional systems of commensurate order.

3.1.1 Commensurate-order fractional differential equation

A time-invariant linear (TIL) fractional system whose behavior is described by a differential equation where the derivatives are of fractional order (FDEq) is said to be commensurable if all exponents are integer numbers resulting from the repeated multiplication of the same real number m, i.e. $\alpha_k = km$ and $\beta_k = km$, with 0< m <1. The ensuing expression represents the linear fractional differential equation with a commensurate order:

$$\sum_{k=0}^{N} a_k D^{km} y(t) = \sum_{k=0}^{M} b_k D^{km} u(t) \tag{8}$$

The fractional equation of state space is also defined by:

$$\begin{aligned} D_t^m x(t) &= \frac{d^m x(t)}{dt^m} = Ax(t) + Bu(t) \quad , \quad 0 < m < 1 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{9}$$

with the fractional order being m, the state vector being $x(t)$, the input vector being $u(t)$, and the output vector being $y(t)$. The state-space matrices of the system are A, B, C, and D. The system is of appropriate dimensions.

3.1.2 Transfer function from state-space

The equation of fractional state-space of (9) can be transformed into the frequency domain by using the LT with non-zero initial conditions, and the Caputo definition of differentiation, we obtain [4] :

$$\begin{aligned} & s^m X(s) - s^{m-1} x(0) = AX(s) + BU(s) \\ & Y(s) = CX(s) + DU(s) \end{aligned} \tag{10}$$

Then, we will have:

$$\begin{aligned} & X(s) = (s^m I - A)^{-1} BU(s) + s^{m-1} (s^m I - A)^{-1} x(0) \\ & Y(s) = CX(s) + DU(s) \end{aligned} \tag{11}$$

In the case of zero initial conditions, equation (11) becomes:

$$\begin{aligned} & X(s) = (s^m I - A)^{-1} BU(s) \\ & Y(s) = CX(s) + DU(s) \end{aligned} \tag{12}$$

So, the function $G(s)$ is given as:

$$G(s) = \frac{Y(s)}{U(s)} = C\ (s^m I - A)^{-1} B + D \tag{13}$$

The $G(s)$ can be represented as a ratio of two polynomials, where the numerator and denominator are expressed in terms of the integer power of the LT operator s^m.

3.2 Solution of fractional linear systems of commensurate order [4], [11]

In this section, we will present the solution of the commensurate fractional linear systems represented by the state-space equation form:

$$\mathrm{D}^m \mathrm{x}(\mathrm{t}) = \mathrm{A}\,\mathrm{x}(\mathrm{t}) + Be(t) \tag{14}$$

The system has a fractional derivative m of Caputo (for $0 < m < 1$), state vector $x(t) \in \mathbb{R}^N$, input $e(t)$, and state matrix $A \in \mathbb{R}^{N \times N}$. By using the LT of (14) and rearranging, we obtain:

$$X(s) = (s^m I - A)^{-1} \left[s^{(m-1)} x(0) \right] + (s^m I - A)^{-1} BE(s) \tag{15}$$

By performing the inverse Laplace transform on equation (15), the state vector $x(t)$ can be derived, with $x(0)$ representing the initial state.

$$x(t) = L^{-1}\{X(s)\} = L^{-1}\left\{ s^{(m-1)}\left(s^{m}I - A\right)^{-1}\right\} x(0) + L^{-1}\left\{\left(s^{m}I - A\right)^{-1}\right\} * B e(t) \tag{16}$$

Let's define the matrices of dimensions $(NxN)\,\hat{\psi}(t)$ and $\psi(t)$ as follows:

$$\hat{\psi}(t) = L^{-1}\left\{\left(s^{m}I - A\right)^{-1}\right\} \; and \; \psi(t) = L^{-1}\left\{ s^{(m-1)}\left(s^{m}I - A\right)^{-1}\right\} \tag{17}$$

The state vector $x(t)$ can be obtained as a function of time t, which represents the state of the system at that time.

$$x(t) = L^{-1}\{X(s)\} = \psi(t)\,x(0) + \hat{\psi}(t) * B\,e(t) \tag{18}$$

From equation (17), we can write that $\psi(t) = \hat{\psi}(t) * \chi_{(m-1)}(t)$; where $\chi_{(m-1)}(t)$ can be expressed as:

$$\chi_{(m-1)}(t) = L^{-1}\left(s^{(m-1)}\right) = \begin{cases} \frac{t^{-m}}{\Gamma(1-m)} & , \; m < 1 \\ \delta(t) & , \; m = 1 \end{cases} \tag{19}$$

The following expression is obtained by taking the ILT of the state-space equation in the s-domain, and applying the property of the LT related to the convolution product, where $\delta(t)$ is the function of Dirac delta .

$$x(t) = \psi(t)\,x_0 + \hat{\psi}(t) * [B u(t)] \quad \Rightarrow \quad x(t) = \psi(t)\,x_0 + \int_0^t \hat{\psi}(t-\tau)\,B u(\tau)\,d\tau \tag{20}$$

Note that the solution $x(t)$ of the state equation (14) consists of two terms. The terms represent the free and forced responses, respectively.
To solve this linear fractional system of the state-space equation of (14), we need to find the matrices $\psi(t)$ and $\hat{\psi}(t)$. The goal of this study is to formulate a methodology for computing these matrices expressions, as indicated by the following equations:

$$\hat{\psi}(t) = L^{-1}\left\{\left(s^{m}I - A\right)^{-1}\right\} \; and \; \psi(t) = L^{-1}\left\{ s^{(m-1)}\left(s^{m}I - A\right)^{-1}\right\} \tag{21}$$

3.2.1 Calculation of matrices $\psi(t)$ and $\hat{\psi}(t)$

In this section, three methods are presented for calculating the expressions of $\psi(t)$ and $\hat{\psi}(t)$ matrices. The proposed methods extend the well-known techniques employed for solving linear LTI systems with integer-order derivatives. It is assumed that the matrix A with N eigenvalues propres λ_i $(1 \le i \le N)$ distincts.

3.2.1.1. Inverse Laplace transform (ILT) method [4]

The calculation procedure is given as follows:

1.Calculate the matrix $\widehat{\Psi}(s^m) = (s^m I - A)^{-1}$ which a matrix of rational functions in s^m as follows:

$$\widehat{\Psi}(s^m) = (s^m I - A)^{-1} = \frac{1}{\det\{(s^m I - A)\}}\{Adjointe\,(s^m I - A)\}^T \tag{22}$$

with $\det(s^m I - A) = \prod_{i_3=1}^{N}\left(s^m - \lambda_{i_3}\right)$ is the polynomial associated with the system's characteristics. So, for $1 \le i_1, i_2 \le N$, each element of the matrix $\widehat{\Psi}(s^m)$ is given by:

$$\widehat{\Psi}_{i_1,i_2}(s) = \frac{\alpha_{i_1,i_2}(s)}{\prod_{i_3=1}^{N}\left(s^m - \lambda_{i_3}\right)} \tag{23}$$

where the $\alpha_{i1,i2}(s)$ are the cofactors of the matrix elements $(s^m I - A)$.

2.By decomposing into simple elements of equation (23) in terms of s^m, we get:

$$\widehat{\Psi}_{i_1,i_2}(s) = \frac{\alpha_{i_1,i_2}(s)}{\prod_{i_3=1}^{N}(s^m - \lambda_{i_3})} = \sum_{i_3=1}^{N} \frac{r(i_1, i_2, i_3)}{s^m - \lambda_{i_3}} \tag{24}$$

where $r(i_1, i_2, i_3)$, for $1 \le i_1, i_2 \le N$, are the residues.

$$\hat{\psi}_{i_1,i_2}(t) = L^{-1}\left\{ \widehat{\Psi}_{i_1,i_2}(s^m) \right\} \tag{25}$$

3.Then, for $1 \le i1, i2 \le N$, the elements $\widehat{\psi}_{i_1,i_2}(t)$ of the matrix $\hat{\psi}(t) = L^{-1}\left\{ (s^m I - A)^{-1} \right\}$ are given by:

$$\widehat{\psi}_{i_1,i_2}(t) = \left\{ L^{-1}[\widehat{\Psi}_{i_1,i_2}(s^m)] \right\} = L^{-1}\left\{ \sum_{i_3=1}^{N} \frac{r(i_1, i_2, i_3)}{s^m - \lambda_{i_3}} \right\} \tag{26}$$

and, for $1 \le i_1, i_2 \le N$, the elements $\psi_{i_1,i_2}(t)$ of the matrix $\psi(t) = L^{-1}\left\{ s^{(m-1)}(s^m I - A)^{-1} \right\}$ are also given by:

$$\psi_{i_1,i_2}(t) = \left\{ L^{-1}[s^{(m-1)}\widehat{\Psi}_{i_1,i_2}(s^m)] \right\} = L^{-1}\left\{ \sum_{i_3=1}^{N} \frac{r(i_1, i_2, i_3)\ s^{(m-1)}}{s^m - \lambda_{i_3}} \right\} \tag{27}$$

3.2.1.2. Modal decomposition method [4]

The eigenvalues of a matrix A are the scalar values λ that make the matrix $A - \lambda I$ singular, where I is the identity matrix, which are distinct and $v_1, v_2, \ldots$ and v_N represent the eigenvectors of the matrix A that correspond to these eigenvalues. Let $V = \left[v_1 \vdots v_2 \vdots \cdots \vdots v_N \right]$ be the matrix of modes and $J = diag\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ with $J = V^{-1}AV$ and $A = VJV^{-1}$. So we have:

$$\widehat{\Psi}(s^m) = (s^m I - A)^{-1} = \left[s^m I - (V J V^{-1}) \right]^{-1} = V\left[s^m I - J \right]^{-1} V^{-1} \tag{28}$$

The matrix $(s^m I - J)^{-1}$ is given as follows:

$$(s^m I - J)^{-1} = \begin{bmatrix} (s^m - \lambda_1) & 0 & \cdots & 0 \\ 0 & (s^m - \lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (s^m - \lambda_N) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s^m - \lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s^m - \lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s^m - \lambda_N} \end{bmatrix} \tag{29}$$

Therefore, $\widehat{\Psi}(s^m) = (s^m I - A)^{-1}$ is given by the following expression:

$$\widehat{\Psi}(s^m) = (s^m I - A)^{-1} = V\left\{ (s^m I - J)^{-1} \right\} V^{-1} = V \begin{bmatrix} \frac{1}{s^m - \lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s^m - \lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s^m - \lambda_N} \end{bmatrix} V^{-1} \tag{30}$$

Then, $\hat{\psi}(t) = L^{-1}\left\{ \widehat{\Psi}(s^m) \right\} = L^{-1}\left\{ (s^m I - A)^{-1} \right\}$ *and* $\psi(t) = L^{-1}\{ \Psi(s^m) \} = L^{-1}\left\{ s^{(m-1)}(s^m I - A)^{-1} \right\}$ are given as:

$$\hat{\psi}(t) = L^{-1}\left\{ \widehat{\Psi}(s^m) \right\} = V \begin{bmatrix} L^{-1}\left\{\frac{1}{s^m - \lambda_1}\right\} & 0 & \cdots & 0 \\ 0 & L^{-1}\left\{\frac{1}{s^m - \lambda_2}\right\} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L^{-1}\left\{\frac{1}{s^m - \lambda_N}\right\} \end{bmatrix} V^{-1} \tag{31}$$

$$\psi(t)=L^{-1}\left\{\Psi(s^m)\right\}=V\begin{bmatrix} L^{-1}\left\{\frac{s^{(m-1)}}{s^m-\lambda_1}\right\} & 0 & \cdots & 0 \\ 0 & L^{-1}\left\{\frac{s^{(m-1)}}{s^m-\lambda_2}\right\} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L^{-1}\left\{\frac{s^{(m-1)}}{s^m-\lambda_N}\right\} \end{bmatrix}V^{-1} \tag{32}$$

3.2.1.3. Cayley-Hamilton method [4]

The distinct eigenvalues of A are $\lambda_1, \lambda_2, \dots$ and λ_N and let $\Delta(\lambda)$ be its degree N of the polynomial characteristic. Let be a function $f(\lambda)$ that can be represented by its series development as follows:

$$f(\lambda)=\sum_{k=0}^{+\infty}\delta_k\lambda^k \tag{33}$$

It is possible to divide $f(\lambda)$ by $\Delta(\lambda)$, we obtain :

$$f(\lambda)=\left[\Delta(\lambda)\sum_{k=0}^{+\infty}\beta_k\lambda^k\right]+[R(\lambda)] \tag{34}$$

where $R(\lambda)=\alpha_0+\alpha_1\lambda+\dots+\alpha_{N-1}\lambda^{N-1}$ is a degree $(N-1)$ of polynomial because $\Delta(\lambda)$ is a degree N of polynomial. We have $\Delta(\lambda_i)=0$, then we can write that :

$$f(\lambda_i)=R(\lambda_i)=\alpha_0+\alpha_1\lambda_i+\dots+\alpha_{N-1}\lambda_i^{N-1} \tag{35}$$

According to the Cayley–Hamilton theorem, we can therefore have [4] :

$$f(A)=\alpha_0 I+\alpha_1 A+\dots+\alpha_{N-1}A^{N-1}=\sum_{i=0}^{N-1}\alpha_i A^i \tag{36}$$

So, to calculate $f(A)$ just find the coefficients α_i (for $i=1,2,\dots,N-1$). In the case where the eigenvalues of A are distinct, the coefficients α_i $(for i=1,2,\dots,N-1)$ are given by the following expression [4]:

$$\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{N-1}\end{bmatrix}=\begin{bmatrix}1 & \lambda_1 & \cdots & \lambda_1^{N-1}\\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \lambda_N & \cdots & \lambda_N^{N-1}\end{bmatrix}^{-1}\begin{bmatrix}f(\lambda_1)\\ f(\lambda_2)\\ \vdots\\ f(\lambda_N)\end{bmatrix} \tag{37}$$

for $f(\lambda)=(s^m-\lambda)^{-1}$, we can therefore have:

$$f(A)=\widehat{\Psi}(s^m)=(s^m I-A)^{-1}=\sum_{i=0}^{N-1}\alpha_i(s^m)A^i \tag{38}$$

where the coefficients $\alpha_i(s^m)$ $(for i=0,1,\dots,N-1)$ are given by:

$$\begin{bmatrix}\alpha_0(s^m)\\ \alpha_1(s^m)\\ \vdots\\ \alpha_{N-1}(s^m)\end{bmatrix}=\begin{bmatrix}1 & \lambda_1 & \cdots & \lambda_1^{N-1}\\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \lambda_N & \cdots & \lambda_N^{N-1}\end{bmatrix}^{-1}\begin{bmatrix}(s^m-\lambda_1)^{-1}\\ (s^m-\lambda_2)^{-1}\\ \vdots\\ (s^m-\lambda_N)^{-1}\end{bmatrix} \tag{39}$$

From equation (39), each coefficient $\alpha_i(s^m)=\sum_{j=1}^{N}\theta_{i,j}\,(s^m-\lambda_j)^{-1}$ $(for i=0,1,\dots,N-1)$ is a linear combination of functions $(s^m-\lambda_j)^{-1}$ $(for j=1,2,\dots,N-1)$. Then, $\hat{\psi}(t)=L^{-1}\left\{\widehat{\Psi}(s^m)\right\}=L^{-1}\left\{(s^m I-A)^{-1}\right\}$ and

$\psi(t) = L^{-1}\{\ \Psi(s^m)\} = L^{-1}\left\{\ s^{(m-1)}\,(s^m I - A)^{-1}\right\}$ are given as follows :

$$\hat{\psi}(t) = L^{-1}\left\{(s^m I - A)^{-1}\right\} = \sum_{i=0}^{N-1} L^{-1}\{\ \alpha_i(s^m)\}\,A^i = \sum_{i=0}^{N-1}\sum_{j=1}^{N}\theta_{i,j}\,L^{-1}\left\{\ (s^m - \lambda_j)^{-1}\right\}\,A^i \tag{40}$$

$$\psi(t) = L^{-1}\left\{\ s^{(m-1)}\,(s^m I - A)^{-1}\right\} = \sum_{i=0}^{N-1}\sum_{j=1}^{N}\theta_{i,j}\,L^{-1}\left\{\ s^{(m-1)}\,(s^m - \lambda_j)^{-1}\right\}\,A^i \tag{41}$$

4 Conclusion

Fractional calculus derivatives and integrals has been associated with applications across various scientific and engineering disciplines. This has led to the formulation and study of fractional-order differential equations (FODEs). Introduction to fractional operators and systems presented in the paper. The different representations of linear systems of factionary order by differential equations, by transfer function and by equation of state have been given. Finally, Analytical solutions for these fractional linear systems of commensurate order's equation of state were presented.

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