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Designing Advanced Reliability Testing Mathematical Model for Modern Products

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Abstract: The modern era is the age of science, technology and at the same time it is the age of competition. The advancement of new technology and increased global competition have emphasized the importance of product strength and reliability estimation. As a result, producers and manufacturers must now verify the strength and reliability of their products prior to releasing them to the market. In the past, reliability data analysis was a critical tool for this purpose. Traditionally, reliability data analysis entails quantifying these life characteristics through the examination of failure data. However, in many situations, obtaining such failure data has been extremely difficult, if not impossible, due to the length of time between designing and releasing a product, and the difficulty of designing a product that will last a long period due to its continuous use and operation. Faced with this challenge, reliability statisticians developed a technique called Accelerated Reliability Testing to rapidly determine the reliability and life characteristics of products. This technique increases product reliability and identifies when and how a product will fail in its intended environment. In the present work, we plan to investigate these mathematical reliability models to determine the costs associated with the various product guarantees. If component lifetimes follow the power-function distribution, the problem is examined under increasing stress using percent failure censoring. The method is referred as a process that applies accelerated testing to estimate the cost of age-replacement for goods sold under warranty. Additionally, a mathematical illustration is presented to illustrate the results.

Keywords: Reliability, Life Testing, Warranty Policy, Statistical Model, Application Example.

1 Introduction

Each manufacturing product is distinct, with its own workspace and marketing requirements. The entire manufacturing process is dependent on several constraints that restrict the start or progression of field operations, which can have a negative impact on the production's reliability and overall performance. Constraints are defined as any condition, such as time/space, safety/quality concerns, or warranty/guaranty, which can prevent a product from achieving its marketing objectives if not managed well, particularly when the product is sold electronically. Also, due to competitive business, the requirements for the improvement of reliability and quality of products and systems have also increased. Simultaneously, products are becoming more complex, and an increasing number of components have become vital for the function of a product and system. Therefore, there is a need to put severe reliability and quality requirements on components and parts of such products and systems. The product can only survive in the market if it has been tested in the manufacturing process. Therefore, the long-term survival of the product is dependent on its better-expected life, quality/reliability, and a decent warranty/guaranty plan.

Offering a strong warranty policy is always a wise choice for businesses, even more so for items sold online. It was recognized early on that a warranty could provide potential buyers with insurance against product failure. For electronic goods, newcomers offer extended warranties to reassure consumers that they are dealing with a reputable brand and to help customers

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identify their brand. As a result, the product's reliability must be sufficient for the manufacturer to extend the warranty and thus increase the product's value.

To achieve all these measures, reliability statisticians have developed a method called accelerated life testing (ALT) to improve the reliability of products and systems. ALT models have mostly been used to estimate or measure reliability with reduced time and cost. It helps to identify weak points in a design and determines when and how a product will fail in its intended environment.

In manufacturing companies, quality engineers have been using accelerated life testing (ALT) processes for several years now in order to fast collect data that is reliable. Within the confines of an adequate testing period, ALTs function as a technique for estimating the lifetime of highly reliable objects. The components are worked beyond their normal functioning limitations to hasten the onset of early failures. To calculate the life distribution, the results of the accelerated test are entered into a model and compared to the stress levels that were specified in the design.

The current study is very hybrid in nature. It connects **advanced statistics** with reliability engineering, E-marketing, and computer science. There are so many studies on accelerated life testing approaches. To begin with the introduction and terminology of accelerated life testing, one can refer to Nelson [1]. Accelerated Life Testing with a Rebate Warranty Application: A Comprehensive Study and Design by Lone and Ahmed [2]. Using data from accelerated life tests, YAng and Wang [3] analysed the features of insulation deterioration. To create a step-stress partially accelerated life testing strategy for competing risk, Lone and Rahman [4] described the adaptive type-I progressive hybrid censoring. To deal with UH censored data, Lone and Panahi [5] trained a constant-stress model with the Gompertz distribution. The recent ones include Pascual et al. [6], Asadi et al. [7], Lone et al. [8], Ismail [9], Sindhu et al. [10] and Alam et al. [11]. Lone et al. [12] discussed the statistical analysis under geometric processes in accelerated life testing plans for a generalized exponential distribution. Lone [13] recently presented a simulation analysis of the Fréchet distribution under a partially accelerated, multiply censored life testing plan. To help predict results from a two-dimensional warranty, as those provided by US automobile manufacturers, Manna et al. [14] demonstrate and explain the application of a use-rate ALT model.

The objective of the study is to arrive at an estimate of both the total cost and the cost rate for the age-related warranty plan's provision of unit replacements. Assume that the failure times of items follow the power function distribution when increasing stress is applied to the items. In addition, we will presume that the stress imposed on the components is distributed evenly and independently. To determine the model parameters, the maximum likelihood estimation (MLE) method is utilized. With the pro-rata rebate warranty, we can also get an estimate of the expected total cost and an expected cost rate for age-replacement.

2 Modologies and Models

The structured approach will be used to review various types of marketing products, their manufacturing, construction environment, specifications, and their characteristics. A suitable stress mechanism will be applied to the products to generate their failure-time data. Based on the generated failure-time data along with different life-stress mathematical models, an advanced mathematical reliability model can be designed. To test the performance of the developed mathematical model, the simulation techniques will be performed on statistical software using Newton-Raphson techniques.

The Model

Here, the assumed lifetime model for the product life is introduced. The lifetime distribution of the test items is assumed to be a two-parameter Power function. This distribution was first presented by Mukherjee and Islam [15] as a new model to access failure time distribution. Numerous studies have shown interest in modeling lifetime or survival data with power function distribution as of late. The density function (pdf) is.

$$f(t,\alpha,\zeta)=\frac{\alpha}{\zeta^{\alpha}}(t)^{(\alpha-1)},\alpha>0,\zeta>0,0<t<\zeta. \quad (1)$$

The survival function is calculated as

$$\mathrm{S}(t)=1-\left(\frac{t}{\zeta}\right)^{\alpha}, \quad (2)$$

where, α,ζ are shape and scale parameters, respectively.
In recent years, researchers have discovered that the two-parameter Power-Function distribution is an excellent option for analyzing numerous lifetime data sets. Lone [16] has elegantly substituted it for the other lifetime distributions.

3 Estimation Procedures

This study is conducted under conditions of increasing stress S_j,j=1,2,…,k such that, S_u 〖<S〗_1<S_2<⋯<S_k. Where S_uis testing at use conditions. The procedure starts by subjecting units to the test at stress level, and the process is stopped once some fixed percentage of n_j (In_j) units come with failure. The expression t_ij,i=1,2,...In_j;j=1,2,...,k denotes the lifetime of each unit at each at each stress level.

The scale parameter ζ_j is the function of stress S_j modelled by the following equation called the power rule model:

$$\zeta_j = \omega S_j^{-p};\ p,\ \omega > 0, j = 1,2,\dots,k, \quad (3)$$

where, ω is the proportionality constant.

Since n_j items are tested for each level of stress S_j, the total unit count in the experiment is N=$\sum_{i=1}^{k} n_j$. When failure censoring is used, the experiment is terminated as soon as the r_j failure takes place among the n_j units tested at each stress level. Using the failure censoring scheme, the likelihood function can be defined as given below.

$$L(\alpha,\omega,p) = \prod_{j=1}^{k} \frac{n_j}{(n_j-In_j)!}\left[\prod_{i=1}^{In_j} f(t_{ij};\alpha,\omega,p)\right]\left[1-F\left(t_{(In_j j)}\right)\right]^{n_j-In_j} \quad (4)$$

In accordance with this framework, the log-likelihood function is stated as

$$InL = C + \sum_{j=1}^{k} In_j(In\alpha + In\omega) - p\sum_{j=1}^{k} In_j InS_j + \sum_{j=1}^{k}\sum_{i=1}^{In_j}(\alpha-1)Int_{ij} + \sum_{j=1}^{In_j}(n_j-In_j)In\left\{1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}\right\}, \quad (5)$$

Where C is a constant and $ln\, L = ln\, L\,(\alpha,\omega,p)$

By differentiating the log-likelihood equation with the given parameters, the maximum likelihood estimators (MLEs) are derived.

$$\frac{\partial\, ln\, L}{\partial\alpha} = \sum_{i=1}^{k}\frac{In_j}{\alpha} + \sum_{j=1}^{k}\sum_{i=1}^{In_j} ln\, t_{ij} - \sum_{j=1}^{k}(n_j-In_j)\alpha\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha-1}\left[1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}\right]^{-1} = 0$$

$$\frac{\partial\, ln\, L}{\partial\omega} = \sum_{i=1}^{k}\frac{In_j}{\omega} + \sum_{j=1}^{k}(n_j-In_j)\alpha S_j^{p\alpha}\omega^{-\alpha-1}t_{In_j,j}^{\alpha}\left[1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}\right]^{-1} = 0$$

$$\frac{\partial\, ln\, L}{\partial p} = -\sum_{i=1}^{k} In_j\, ln\, S_j - \sum_{j=1}^{k}(n_j-In_j)\, ln\, S_j\, \alpha S_j^{p\alpha}\omega^{-\alpha}t_{In_j,j}^{\alpha}\left[1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}\right]^{-1} = 0$$

The MLEs don't seem to possess a solution that can be expressed in closed form. Therefore, the Newton-Raphson method is applied in order to acquire the MLEs.

The following is Fisher-Information matrix.

$$I = \begin{bmatrix} \frac{-\partial^2 InL}{\partial\alpha^2} & \frac{-\partial^2 InL}{\partial\omega\partial\alpha} & \frac{-\partial^2 InL}{\partial p\partial\alpha} \\ \frac{-\partial^2 InL}{\partial\alpha\partial\omega} & \frac{-\partial^2 InL}{\partial\omega^2} & \frac{-\partial^2 InL}{\partial p\partial\omega} \\ \frac{-\partial^2 InL}{\partial\alpha\partial p} & \frac{-\partial^2 InL}{\partial\omega\partial p} & \frac{-\partial^2 InL}{\partial p^2} \end{bmatrix}, \quad (6)$$

where,

$$-\frac{\partial^2\, ln\, L}{\partial\alpha^2} = \sum_{i=1}^{k}\frac{In_j}{\alpha^2} - \sum_{j=1}^{k}(n_j-In_j)\alpha\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha-1}\left[1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}\right]^{-1}\left[\alpha^{-1} + ln\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right) + \frac{\alpha\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha-1}}{1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}}\right]$$

$$-\frac{\partial^2 \ln L}{\partial \omega^2} = \sum_{i=1}^{k} \frac{In_j}{\omega^2} - \sum_{j=1}^{k} (n_j - In_j)\alpha S_j^{p\alpha} t_{In_j j}^{\alpha} \frac{\omega^{-\alpha-1}}{1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}} \left[-\frac{(\alpha+1)}{\omega} - \frac{\alpha S_j^{p\alpha} t_{In_j j}^{\alpha} \omega^{-\alpha-1}}{1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}} \right]$$

$$\frac{\partial^2 \ln L}{\partial p^2} = \sum_{j=1}^{k} (n_j - In_j) \ln S_j \, \alpha \omega^{-\alpha} t_{In_j j}^{\alpha} \frac{S_j^{-p\alpha}}{1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}} \left[-\alpha \ln S_j + \frac{p\alpha\omega^{-\alpha} t_{In_j j}^{\alpha} S_j^{-p\alpha-1}}{1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}} \right]$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial \omega} = -\frac{\partial^2 \ln L}{\partial \omega \partial \alpha}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial \omega} = -\sum_{j=1}^{k} (n_j - In_j)\alpha \omega^{-(\alpha-1)} \left(\frac{t_{In_j j}}{S_j^{-p}}\right)^{\alpha-1} \left[1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}\right]^{-1} \left[-\frac{(\alpha-1)}{\omega} - \frac{t_{In_j j}^{\alpha} S_j^{-p\alpha} \omega^{-\alpha-1}}{1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}} \right]$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial p} = -\frac{\partial^2 \ln L}{\partial p \partial \alpha}$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial p} = \sum_{j=1}^{k} (n_j - In_j)\alpha \left(\frac{t_{In_j j}}{\omega}\right)^{\alpha-1} S_j^{p(\alpha-1)} \left[1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}\right]^{-1} \left[(\alpha-1)\ln S_j + p\alpha S_j^{p\alpha-1} \frac{\left(\frac{t_{In_j j}}{\omega}\right)^{\alpha}}{1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}} \right]$$

$$-\frac{\partial^2 \ln L}{\partial \omega \partial p} = -\frac{\partial^2 \ln L}{\partial p \partial \omega}$$

$$-\frac{\partial^2 \ln L}{\partial \omega \partial p} = -\sum_{j=1}^{k} (n_j - In_j)\alpha S_j^{p\alpha} \omega^{-\alpha-1} t_{In_j j}^{\alpha} \left[1-\left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha}\right]^{-2} \left[\alpha \left(\frac{t_{In_j j}}{\omega S_j^{-p}}\right)^{\alpha-1} \left(\frac{t_{In_j j}}{\omega}\right) p S_j^{p-1} \right]$$

It is possible to invert the Fisher-Information matrix to obtain the variance-covariance matrix of $(\hat{\alpha}, \hat{\omega} \& \hat{p})$, denoted by Λ is:

$$\Lambda = I^{-1} \quad (7)$$

Hence, the approximate confidence intervals (CIs) for α, ω & p are

$$\hat{\alpha} \pm Z_{\phi/2}\sqrt{var(\hat{\alpha})},\ \hat{\omega} \pm Z_{\phi/2}\sqrt{var(\hat{\omega})} \text{ and } \hat{p} \pm Z_{\phi/2}\sqrt{var(\hat{p})} \quad (8)$$

$Z_{\phi/2}$ is the $100(1-\lambda/2)\%$ **percentile of standard normal variate.**

4 Simulation studies

Computational studies are performed to investigate the performance of the MLEs by finding their mean square error (MSE) and relative absolute bias (RAB). With the help of the invariance property, we can use the following equation to estimate the MLEs of ζ_j.

$$\zeta_j = \omega S_j^{-p}; \omega > 0, p > 0, j = 1,2,\ldots,3$$

The steps involved in the simulation using R- Software are outlined in detail below.

1. Following a Power-Function distribution, we generated 1,000 samples with sizes of 60, 110, 160, and 210. Each parameter set starts out with a different value.

2. We assume the three levels of stress $(k = 3)$, as $(S_1 = 1.1, S_2 = 1.6, S_3 = 2), n_j = \frac{n}{3}$ & $I = 35\%$ $(i.e. In_j = 35\% n_j)$.

3. All the samples are tested under percent Type-II censoring schemes to figure out the model parameters.

4. An iterative method (Newton Raphson technique) is used to solve the nonlinear equations. And the results (RABs) and (MSE) are obtained for all sets of parameters$(\alpha_0, \omega_0, p_0)$.

5. We obtained the parameter $\zeta_u$at $S_u = 0.5$ (normal stress level) using different values of ω_0 and p_0.

6. The following equation can be used to estimate the reliability at various values of mission time t_0.

$$\hat{R}_u(t_0) = 1 - \left(\frac{t}{\zeta_0}\right)^{\alpha_0}$$

Table-1: Simulation results of the parameters $(\alpha, \omega, p, \xi_1, \xi_2, \xi_3)$ under percent failure censoring.

n	Parameter	$(\boldsymbol{\alpha} = 0.3, \omega = 1.5, \boldsymbol{p} = 1.2)$			$(\boldsymbol{\alpha} = 0.5, \boldsymbol{\omega} = 1.5, \boldsymbol{p} = 1.2)$		
		Estimator	RAB	MSE	Estimator	RAB	MSE
60	α	0.319	0.061	0.087	1.120	0.087	0.080
	ω	1.391	0.060	0.056	1.621	0.048	0.059
	p	1.190	0.103	0.059	0.895	0.101	0.081
	ξ_1	1.606	0.086	0.072	1.561	0.100	0.106
	ξ_2	1.093	0.075	0.071	0.781	0.070	0.058
	ξ_3	0.901	0.080	0.069	1.001	0.068	0.073
110	α	0.268	0.071	0.057	1.096	0.060	0.048
	ω	1.589	0.052	0.039	1.601	0.038	0.050
	p	1.093	0.070	0.067	0.986	1.054	0.100
	ξ_1	1.602	0.078	0.068	1.439	0.049	0.048
	ξ_2	1.209	0.050	0.019	1.145	0.039	0.039
	ξ_3	0.903	0.063	0.050	1.091	0.087	0.071
160	α	0.235	0.046	0.038	0.953	0.050	0.048
	ω	1.459	0.029	0.034	1.561	0.040	0.041
	p	0.950	0.054	0.046	1.048	0.052	0.045
	ξ_1	1.557	0.051	0.055	1.532	0.045	0.044
	ξ_2	0.909	0.038	0.039	0.934	0.041	0.037
	ξ_3	0.896	0.050	0.031	0.887	0.061	0.055
210	α	0.249	0.009	0.013	0.980	0.025	0.030
	ω	1.500	0.020	0.003	1.467	0.022	0.023
	p	1.113	0.113	0.010	0.955	0.044	0.031
	ξ_1	1.553	0.017	0.014	1.501	0.022	0.020
	ξ_2	0.910	0.021	0.013	0.811	0.018	0.016
	ξ_3	0.889	0.032	0.030	0.857	0.033	0.025

Table-2: Simulation results of the parameters $(\alpha, \omega, p, \xi_1, \xi_2, \xi_3)$ under percent failure censoring.

n	parameter	$(\alpha = 0.3, \omega = 1, p = 1.2)$			$(\alpha = 0.6, \omega = 1, p = 1.6)$		
		Estimator	RAB	MSE	Estimator	RAB	MSE
60	α	0.228	0.080	0.069	0.931	0.069	0.073
	ω	0.898	0.109	0.050	1.067	0.067	0.066
	p	0.933	0.068	0.060	1.508	0.047	0.101
	ξ_1	0.902	0.069	0.068	1.219	0.065	0.058
	ξ_2	0.768	0.119	0.103	1.099	0.072	0.060
	ξ_3	0.590	0.070	0.071	0.709	0.109	0.080
110	α	0.240	0.048	0.061	0.940	0.055	0.060
	ω	1.100	0.092	0.039	1.056	0.044	0.048
	p	0.937	0.063	0.075	1.442	0.042	0.065
	ξ_1	1.069	0.071	0.073	1.289	0.060	0.046
	ξ_2	0.780	0.091	0.077	1.168	0.049	0.050
	ξ_3	0.587	0.058	0.050	0.786	0.073	0.068
160	α	0.245	0.044	0.045	0.948	0.052	0.038
	ω	1.046	0.049	0.036	1.038	0.041	0.036
	p	0.963	0.042	0.040	1.561	0.040	0.063
	ξ_1	1.051	0.060	0.048	1.230	0.046	0.045
	ξ_2	0.708	0.050	0.040	1.128	0.044	0.036
	ξ_3	0.594	0.049	0.036	0.787	0.057	0.045
210	α	0.250	0.008	0.017	0.981	0.019	0.040
	ω	1.058	0.061	0.023	1.035	0.030	0.031
	p	1.013	0.013	0.021	1.467	0.022	0.025
	ξ_1	1.062	0.010	0.010	1.211	0.019	0.016
	ξ_2	0.718	0.029	0.013	1.111	0.019	0.012
	ξ_3	0.595	0.032	0.016	0.755	0.020	0.018

Table 3: **Simulation results of the parameters $(\alpha, \omega, p, \xi_1, \xi_2, \xi_3)$ under percent failure censoring.**

n	Parameter	$(\alpha = 0.6, \omega = 1.2, p = 1.6)$			$(\alpha = 0.6, \omega = 1.6, p = 1.8)$		
		Estimator	RAB	MSE	Estimator	RAB	MSE
	α	0.540	0.083	0.070	0.582	0.059	0.078
	ω	1.388	0.059	0.044	1.610	0.055	0.058
	p	0.894	0.070	0.065	1.616	0.101	0.065

60	ξ_1	1.608	0.078	0.058	1.404	0.100	0.064
	ξ_2	1.091	0.069	0.044	0.920	0.078	0.058
	ξ_3	1.032	0.067	0.067	1.009	0.060	0.060
110	α	0.519	0.100	0.051	0.600	0.064	0.055
	ω	1.551	0.048	0.038	1.570	0.047	0.052
	p	0.877	0.061	0.081	1.581	0.098	0.101
	ξ_1	1.441	0.084	0.075	1.489	0.076	0.046
	ξ_2	1.127	0.048	0.023	0.899	0.044	0.039
	ξ_3	1.082	0.068	0.044	1.095	0.101	0.060
160	α	0.582	0.052	0.037	0.611	0.060	0.051
	ω	1.432	0.029	0.028	1.561	0.046	0.038
	p	1.306	0.047	0.040	1.682	0.059	0.046
	ξ_1	1.570	0.053	0.048	1.502	0.044	0.060
	ξ_2	1.095	0.033	0.036	1.034	0.040	0.037
	ξ_3	0.902	0.050	0.030	0.904	0.064	0.043
210	α	0.529	0.010	0.022	0.540	0.053	0.028
	ω	1.471	0.020	0.012	1.467	0.033	0.019
	p	1.228	0.131	0.020	1.705	0.051	0.031
	ξ_1	1.530	0.020	0.022	1.576	0.026	0.028
	ξ_2	0.910	0.033	0.022	0.784	0.020	0.020
	ξ_3	0.857	0.035	0.034	0.885	0.104	0.034

Table 4: Simulation results of shape parameter and reliability under use conditions, taking $n = 200$.

α_0	ω_0	p_0	ξ_u	t_0	$R_u(t_0)$
0.25	1.5	1	3.201165	1	0.2523
				1.3	0.2017
				1.5	0.1726
1	1.5	1	2.843896	1	0.6483
				1.3	0.5428
				1.5	0.4725
0.25	1	1	2.139189	1	0.1731
				1.3	0.1170
				1.5	0.0849
1	1	1.5	2.861221	1	0.6504
				1.3	0.5456
				1.5	0.4757

5 The age replacement (AR) policy under pro-rata rebate (PRR) warranty

Under this service agreement, a non-repairable item is replaced at a specified age (τ) or upon failure, whichever comes first. Product is replaced at a cost of downtime $D_c > 0$ and a cost of purchase $P_c > 0$ when an item fails at $t \leq \tau$. If the product does not perform as expected throughout the applicable warranty period (w), the consumer receives a reimbursement of a portion of the purchase cost (P_c). The rebate function of the pro-rata warranty is given by:

$$R(t) = \begin{cases} P_c\left(1 - \frac{t}{w}\right), & 0 \leq t \leq w \\ 0, & t > w \end{cases} \quad (9)$$

The topic of age-replacement policies has been analyzed by numerous authors. In Chien and Chen [17], the authors investigate the effect of a renewed free-replacement warranty (RFRW) on the age-based replacement plan for repairable equipment with a general failure model. Also, Hong-Zhong et al [18] examined the problem of estimating expected warranty costs for items that are rarely and in a variety of ways utilized. More details can be seen in Yang [19] and Dessouky [20].
The procedure involves the following assumptions:

1. The item is changed out either when it sustains damage, which is known as corrective replacement, or when it reaches a certain age, which is known as preventive replacement. Whichever occurs first, it is taken care of.

2. The products are slod under pro-rata rebate (PRR) warranty.

3. Product components that are intended for preventive replacement are assumed to have no value of salvage.

4. The length of the warranty (w) is shorter than the age replacement period (τ).

As a preventive maintenance measure, a product is replaced at a cost of P_c when it reaches the age of τ.

The following is a breakdown of the overall cost associated with the renewal of this policy:

$$C(d) = \begin{cases} D_c + P_c - R(t) 0 \leq t \leq w \\ D_c + P_c w < t < \tau \\ P_c t \geq \tau \end{cases} \quad (10)$$

Chien and Chen [17] provide useful reviews of the literature on estimated total cost.

$$E\big(C(t)\big) = D_c F(\tau) + P_c \frac{\int_0^w \bar{F}(u)du}{w} \quad (11)$$

The expected cost rate is:

$$E\big(CR(t)\big) = \frac{E\big(C(t)\big)}{\int_0^\tau \bar{F}(u)du} \quad (12)$$

The denominator $\int_0^\tau \bar{F}(u)du$ denotes the expected cycle time. $E\big(T(\tau)\big) = \int_0^\tau \bar{F}(u)du$

Now, applying the above measures using the Power-Function distribution**:**

We have $F(u) = \left(\frac{u}{\zeta}\right)^\alpha, 0 < u < \zeta$ (13)

Therefore $\int_0^w \bar{F}(u)\, du = w - \frac{w^{\alpha+1}}{\zeta^\alpha(\alpha+1)}$ (14)

Also, $\int_0^\tau \bar{F}(u)\, du = \tau - \frac{\tau^{\alpha+1}}{\zeta^\alpha(\alpha+1)}$ (15)

From equations (11), (12), (14), and (15), one can determine the expected total cost $E(C(\tau))$ and the expected cost rate $CR(\tau)$ for the non-repairable component.

For example, if the failure replacement of a product costs $Cd = 40$ because of downtime and the purchase cost $= 990$, we can find the expected total cost $E(C(\tau))$, the expected cycle time $E(T(\tau))$. Also, the expected cost rate for age-replacement $CR(\tau)$ is calculated for different warranty periods (w) under usual conditions.

Table 5: The simulation results of age-replacement warranty using Power-Function distribution.

ζ	α	W	τ	$E(C(\tau))$	$E(T(\tau))$	$CR(\tau)$
2.5	0.30	1	1	428.12	0.422	1102.88
2.5	0.30	1	1.5	429.17	0.530	891.370
2.5	0.30	1	1.75	427.78	0.530	825.791
3	0.50	1	1	648.28	0.867	758.570
3	0.50	1	1.5	782.69	1.082	723.028
3.5	0.75	1.5	175	730.18	1.248	584.675
3.5	0.75	1.5	2	743.79	1.286	578.038
3.5	1	1.5	2	837.52	1.501	558.341
4	1	2	3	775.11	1.875	413.327
4	1	2.5	3	712.50	1.875	380.112
4	1	3	3	650.65	1.905	351.453

6 Results and Conclusions

According to the information presented in Tables (1), (2), and 3, an indication of the overall consistency is that the absolute value of the difference between the actual value of a parameter and its estimator approaches zero. We provide some estimates for the reliability function shown in Table (4) for a range of different values of the parameter ζ_0 and the mission periodt_0. When the duration of the mission is increased, the performance feature of the product becomes less reliable. It is obvious that if a product is tested for a long time, the wear and tear caused will make it less reliable.

From table 5, we can figure out the following.

1) There is a positive relationship between the expected total cost and the expected time cycle. This means that products need to be changed frequently as the rate at which they break is going up very quickly.
2) The parameter α and the predicted cost rate have a positive association. Which means that, if a product that has a higher failure rate would undoubtedly have a higher estimated cost.
3) By extending the duration of the warranty, the expected total cost and rate of cost goes up, but the expected life cycle stays the same. This result shows that the developed warranty model works well when it is used along pro-rata rebate scheme.
4) The correlation between the age of replacement time (τ) and the expected cost rate is negative, but the correlation between the expected total cost and the expected time cycle is positive. This implies that if the product works well, the expected cost will be lower.

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Conflicts of Interest Statement

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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