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A New Extension of Power Hazard Distribution with Applications

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Abstract: A new lifetime distribution is suggested using the Sine function by considering power hazard distribution as baseline distribution. Some mathematical and statistical features are discussed. The Maximum likelihood method is used to estimate the parameters for proposed distribution. Three real data sets are examined to analyze the performance of proposed distribution with some other distributions. The new distribution has been shown better fit to the bladder cancer patients' data and COVID-19 data as compared to some distributions through statistical criterion.

Keywords: Lifetime distribution, power hazard distribution, sine family, data analysis.

1 Introduction

In recent years, many authors have studied several mechanisms for generating many probability distributions. These generalizations enabled the statisticians to have various applications in finance, biology, medicine, physics, engineering and economics. Most generalization methods depend on adding new parameter(s) to a baseline distribution function based on a specific rule. The new parameter(s) can significantly improve the statistical properties of the baseline distribution. Accordingly, new families of distributions have been obtained, namely, the exp-G family, [1], Weibull-G family [2], Topp-Leone generated (TL-G) [3], odd - generalized NH-G [4], a new alpha power transformed-G [5], new power TL-G [6], truncated inverted Kumaraswamy-G [7], a new extended alpha power transformed-G [8], type II general inverse exponential-G [9], exponentiated truncated inverse Weibull-G [10], type II power TL-G [11], and others. In addition, there is a new generalization to obtain new families of continuous probability distributions using the trigonometric transformation. It started with using the sine function to generate the sine-G family by [12, 13].

Let $F(x)$ be the cumulative distribution function (CDF) of a baseline distribution, then the CDF $G(x)$ of sine-G family is

$$G(x) = \sin\left(\frac{\pi}{2}F(x)\right). \tag{1}$$

The corresponding probability density function (PDF) is

$$g(x) = \frac{\pi}{2}f(x)\cos\left(\frac{\pi}{2}F(x)\right), \tag{2}$$

and the hazard rate function (HRF), $h(x)$ is given as

$$h(x) = \frac{\pi}{2}f(x)\left[\frac{\cos\left(\frac{\pi}{2}F(x)\right)}{1-\sin\left(\frac{\pi}{2}F(x)\right)}\right]. \tag{3}$$

The transformation (1) and (2) is called SS transformation, for more details, see [14]. The use of the sine function makes it possible to describe trigonometric distributions via the deformation of any classic distribution. The study of the

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sine-G family has led to the discussion of many trigonometric families with different modeling.

[15] considered the two-parameter power hazard distribution, denoted by PHD (ϑ,γ) and studied its different properties. The PDF and CDF for PHD are given by

$$f(x;\vartheta,\gamma)=\vartheta x^{\gamma}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}},\quad x\geq 0. \tag{4}$$

$$F(x;\vartheta,\gamma)=1-e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}},\quad x\geq 0,\,\vartheta>0,\gamma>-1, \tag{5}$$

where ϑ and γ are the scale and shape parameters, respectively. It is a very flexible model because it has different distributions based on its parameter's values.

PHD is reached to (i) Rayleigh distribution when $\vartheta=1/v^2$ and $\gamma=1$, (ii) Weibull $(\vartheta,1)$ when $\gamma=\vartheta-1$, and (iii) an exponential with mean $1/\vartheta$, when $\gamma=0$. Therefore, the PHD is useful for analyzing and modeling the lifetime data in engineering, biological sciences and medical, and so on.
Many researchers have studied the PHD and estimated its parameters in the case of complete data and different censored samples, see [16,17,18]. The stress-strength reliability model for PHD obtained by [19]. Several applications using the weighted, length biased and transmuted PHD are introduced by [20,21,22] and among others.

In this article, a new distribution referred to as SS-transformation power hazard distribution (SS-PHD) is presented. It can be outlined as follows, the CDF, PDF, and HRF of the SS-PHD are established in Section 2. In Sections 3, some statistical properties are derived. Section 4 consists of the order statistics from SS-PHD. Inequality measures are discussed in Section 5. The parameters of the model are estimated in Section 6. Some numerical applications are shown in Section 7. The Section 8 ends with some conclusion

2 SS Transformation of PHD

Consider the baseline (4), then by SS transformation (2), the PDF of the SS-PHD is obtained by

$$g(x)=\frac{\pi}{2}\vartheta x^{\gamma}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right),\quad \vartheta>0,\gamma>-1,x\geq 0. \tag{6}$$

The CDF and HRF of SS-PHD (ϑ,γ) are given by,

$$G(x)=\cos\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right),\quad \vartheta>0,\gamma>-1,x\geq 0, \tag{7}$$

$$h(x)=\frac{\frac{\pi}{2}\vartheta x^{\gamma}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right)}{1-\cos\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right)}, \tag{8}$$

respectively. For different values of ϑ and γ, one can graph PDF and HRF as shown in the following figures.

3 Distributional Properties

3.1 The moments

The r^{th} moment of SS-PH distribution is

$$\mu_r^{\prime}=\frac{\vartheta}{\gamma+1}\sum_{i=0}^{\infty}\frac{(-1)^i(\pi/2)^{2i+2}}{(2i+1)!}\left(\frac{\gamma+1}{2\vartheta(i+1)}\right)^{\frac{r}{\gamma+1}+1}\Gamma\left(\frac{r}{\gamma+1}+1\right) \tag{9}$$

Proof. We know that

$$\mu_r^{\prime}=\int_0^{\infty}x^r g(x)dx=\frac{\pi}{2}\vartheta\int_0^{\infty}x^{\gamma+r}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right)dx.$$

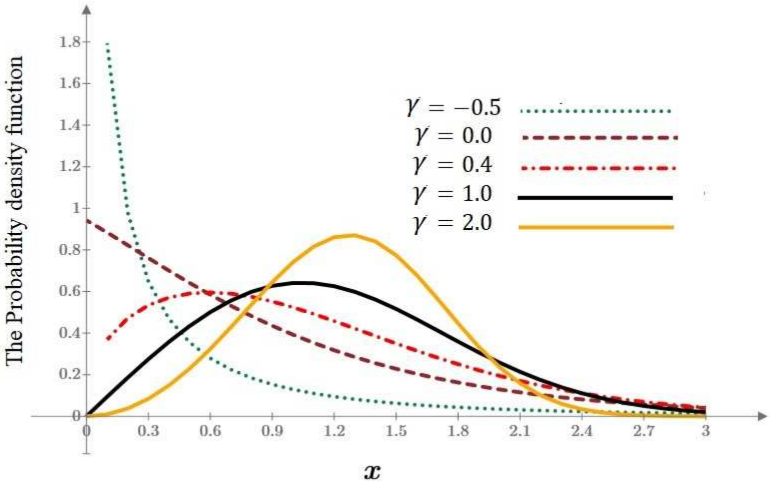

Fig. 1: Visual illustration of $g(x)$ with $\vartheta = 0.6$.

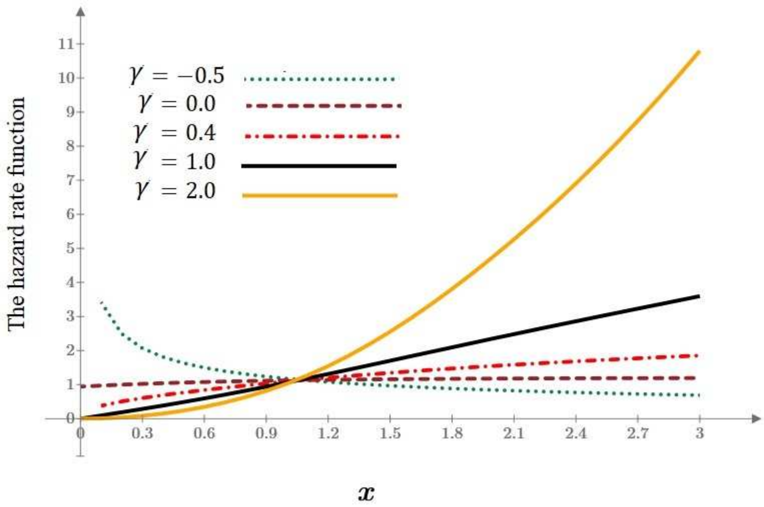

Fig. 2: Visual illustration of $h(x)$ with $\vartheta = 0.6$.

Since

$$\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1},$$

then

$$\mu_r' = \vartheta \sum_{i=0}^{\infty} \frac{(-1)^i (\pi/2)^{2i+2}}{(2i+1)!} \int_0^{\infty} x^{\gamma+r} e^{-\frac{\vartheta(2i+2)}{\gamma+1} x^{\gamma+1}} dx$$

Let $u=\frac{2\vartheta(i+1)}{\gamma+1}x^{\gamma+1}$, then $x=\left(\frac{\gamma+1}{2\vartheta(i+1)}u\right)^{\frac{1}{\gamma+1}}$ and $dx=\frac{1}{\gamma+1}\left(\frac{\gamma+1}{2\vartheta(i+1)}\right)^{\frac{1}{\gamma+1}}u^{\frac{1}{\gamma+1}-1}du$,
therefore,

$$\begin{aligned}\mu_r^{'} &= \frac{\vartheta}{\gamma+1}\sum_{i=0}^{\infty}\frac{(-1)^i(\pi/2)^{2i+2}}{(2i+1)!}\left(\frac{\gamma+1}{2\vartheta(i+1)}\right)^{\frac{r}{\gamma+1}+1}\int_0^{\infty}u^{\frac{r}{\gamma+1}}e^{-u}dx\\ &= \frac{\vartheta}{\gamma+1}\sum_{i=0}^{\infty}\frac{(-1)^i(\pi/2)^{2i+2}}{(2i+1)!}\left(\frac{\gamma+1}{2\vartheta(i+1)}\right)^{\frac{r}{\gamma+1}+1}\Gamma\left(\frac{r}{\gamma+1}+1\right).\end{aligned}$$

The mean and the variance of SS-PHD can be derived as follows

$$\mu=\mu_1^{'}=\frac{\vartheta}{\gamma+1}\sum_{i=0}^{\infty}\frac{(-1)^i(\pi/2)^{2i+2}}{(2i+1)!}\left(\frac{\gamma+1}{2\vartheta(i+1)}\right)^{\frac{1}{\gamma+1}+1}\Gamma\left(\frac{1}{\gamma+1}+1\right). \tag{10}$$

and

$$\mu_2^{'}=\frac{\vartheta}{\gamma+1}\sum_{i=0}^{\infty}\frac{(-1)^i(\pi/2)^{2i+2}}{(2i+1)!}\left(\frac{\gamma+1}{2\vartheta(i+1)}\right)^{\frac{2}{\gamma+1}+1}\Gamma\left(\frac{2}{\gamma+1}+1\right),$$

therefore,

$$\begin{aligned}Var(X) &= \mu_2^{'}-\mu_1^{'2}\\ &= \frac{\vartheta}{\gamma+1}\sum_{i=0}^{\infty}\frac{(-1)^i(\pi/2)^{2i+2}}{(2i+1)!}\left(\frac{\gamma+1}{2\vartheta(i+1)}\right)^{\frac{2}{\gamma+1}+1}\Gamma\left(\frac{2}{\gamma+1}+1\right)\\ &\quad -\left[\frac{\vartheta}{\gamma+1}\sum_{i=0}^{\infty}\frac{(-1)^i(\pi/2)^{2i+2}}{(2i+1)!}\left(\frac{\gamma+1}{2\vartheta(i+1)}\right)^{\frac{1}{\gamma+1}+1}\Gamma\left(\frac{1}{\gamma+1}+1\right)\right]^2.\end{aligned} \tag{11}$$

3.2 Skewness and kurtosis

To check the symmetry and flatness of any distribution, skewness and kurtosis are used. The different approaches are available in the literature. One of them is the moments. The formulae for skewness and kurtosis are summarized in (12) and (13) as follows

$$sk=E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]=\frac{\mu_3^{'}-3\mu_2^{'}\mu_1^{'}+2\mu_1^{'3}}{\sigma^{3/2}}, \tag{12}$$

$$ku=E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]=\frac{\mu_4^{'}-4\mu_3^{'}\mu_1^{'}+6\mu_2^{'}\mu_1^{'2}-3\mu_1^{'4}}{\sigma^2}. \tag{13}$$

Where $\mu_r^{'}$ is the rth moments about origin, see (9).

3.3 The quantiles

The q^{th} quantiles of the SS-PHD can be obtained as follows

$$G(x_q)=q,$$

from the (7), we have

$$x_q=\left[-\left(\frac{\gamma+1}{\vartheta}\right)\ln\left(\frac{2}{\pi}\cos^{-1}(q)\right)\right]^{\frac{1}{\gamma+1}}. \tag{14}$$

From (14), we can find the 1^{st} quartile, $q=0.25$, median when $q=0.50$ and 3^{rd} quartile when $q=0.75$. When $q=0.50$, then the median $m=x_{0.5}$ can be obtained as

$$m=\left[-\left(\frac{\gamma+1}{\vartheta}\right)\ln\left(\frac{2}{3}\right)\right]^{\frac{1}{\gamma+1}}. \tag{15}$$

Assume $U \sim \text{Uniform}(0,1)$, then (14) can be used in simulation to generate random number of size n from SS-PH distribution as given below:

$$x_i = \left[-\left(\frac{\gamma+1}{\vartheta}\right) \ln\left(\frac{2}{\pi}\cos^{-1}(u_i)\right)\right]^{\frac{1}{\gamma+1}}, \quad i = 1,2,\cdots,n. \tag{16}$$

3.4 The mode

The mode for the SS-PH distribution can be obtained by differentiating PDF and equating to zero.

$$\frac{d}{dx}g(x) = \frac{\pi}{2}\vartheta x^{\gamma-1}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\left[\gamma\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right) - \vartheta x^{\gamma+1}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right) - \frac{\pi}{2}\vartheta x^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\cos\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right)\right].$$

Put $\frac{d}{dx}g(x) = 0$, we get

$$\frac{\pi}{2}\vartheta x^{\gamma-1}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\left[(\gamma - \vartheta x^{\gamma+1})\tan\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right) - \frac{\pi}{2}\vartheta x^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right] = 0.$$

Therefore, $x = 0$, or

$$(\gamma - \vartheta x^{\gamma+1})\tan\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right) - \frac{\pi}{2}\vartheta x^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}} = 0. \tag{17}$$

This equation has no closed form, so some numerical program technique can be used.

Table 1: Some Statistical measures of SS-PHD for $\vartheta = 0.6$ and $\gamma \in (-0.5, 3.0)$.

γ	Mean	Median	Variance	C.V.	Sk	Ku
-0.5	0.407	0.114	0.715	2.07758	6.102	75.036
0.0	0.928	0.676	0.767	0.94373	1.846	8.128
0.5	1.135	1.009	0.536	0.64504	0.964	4.099
1.0	1.22	1.163	0.369	0.49791	0.539	3.116
1.5	1.255	1.233	0.263	0.40863	0.273	2.805
2.0	1.268	1.266	0.194	0.34736	0.086	2.726
2.5	1.271	1.279	0.148	0.30268	-0.055	2.743
3.0	1.268	1.282	0.116	0.26860	-0.166	2.804

From Table 1, we can conclude that:

- For $\gamma \geq 0$, the mean, median is increasing, while variance is decreasing.
- For all values of $\gamma \leq 2$, skewness is positive and negative for $\gamma > 2$.
- The coefficient of variance is decreasing when γ is increasing.
- For $\gamma \leq 2$, the kurtosis is decreasing and increasing for $\gamma > 2$.
- For $\gamma > 1.0$, the kurtosis is less than normal distribution.

4 Order Statistics

Suppose $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are ordered statistics of a random sample $X_1, X_2, \cdots, X_n$ drawn from SS-PH distribution, then the PDF of $X_{k:n}$ is

$$f_{k:n}(x) = \frac{1}{B(k,n-k+1)}[G_X(x)]^{k-1}[1-G_X(x)]^{n-k}g_X(x), \quad k = 1,2,\cdots,n. \tag{18}$$

The PDF of k^{th} order statistics for (6) is as follows:

$$f_{k:n}(x) = \frac{\pi\vartheta x^{\gamma}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}}{2B(k,n-k+1)}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i\left[\cos\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right)\right]^{k+i-1}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right), \tag{19}$$

The first order statistic (minimum) is,

$$f_{1:n}(x)=\frac{\pi}{2}n\vartheta x^{\gamma}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{i}\left[\cos\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right)\right]^{i}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right), \tag{20}$$

The largest order statistic (maximum) is

$$f_{n:n}(x)=\frac{\pi}{2}n\vartheta x^{\gamma}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\left[\cos\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right)\right]^{n-1}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right), \tag{21}$$

5 The Inequality Measures

To measure income inequality, Bonferroni and Lorenz curves are applied. These curves have been not applications in economics (to study income and poverty), reliability, finance, and insurance but also in population studies.

Theorem 1. The r.v. $X\sim SS-PHD$, then the Bonferroni curve (BC(q)) and Lorenz curve (LC(q)) are given, respectively, by

$$BC(q)=\frac{\vartheta}{q\mu}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{(2i+1)!k!}\left(\frac{\pi}{2}\right)^{2(i+1)}\left[\frac{2\vartheta(i+1)}{\gamma+1}\right]^{k}\left[\frac{x_q^{(k+1)(\gamma+1)+1}}{(k+1)(\gamma+1)+1}\right], \tag{22}$$

and

$$LC(q)=\frac{\vartheta}{\mu}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{(2i+1)!k!}\left(\frac{\pi}{2}\right)^{2(i+1)}\left[\frac{2\vartheta(i+1)}{\gamma+1}\right]^{k}\left[\frac{x_q^{(k+1)(\gamma+1)+1}}{(k+1)(\gamma+1)+1}\right], \tag{23}$$

where, x_q is the qth quantiles (14).

Proof. From equation (9) and (6), when $r=1$, then the Bonferroni curve is given by

$$\begin{aligned}BC(q)&=\frac{1}{q\mu}\int_0^{x_q}xg(x)dx=\frac{\vartheta\pi}{2p\mu}\int_0^{x_q}x^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x^{\gamma+1}}\right)dx\\&=\frac{\vartheta}{q\mu}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{(2i+1)!}\left(\frac{\pi}{2}\right)^{2i+2}\int_0^{x_q}x^{\gamma+1}e^{-\frac{2\vartheta(i+1)}{\gamma+1}x^{\gamma+1}}dx\\&=\frac{\vartheta}{q\mu}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{(2i+1)!k!}\left(\frac{\pi}{2}\right)^{2(i+1)}\left[\frac{2\vartheta(i+1)}{\gamma+1}\right]^{k}\int_0^{x_q}x^{(k+1)(\gamma+1)}dx\\&=\frac{\vartheta}{q\mu}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{(2i+1)!k!}\left(\frac{\pi}{2}\right)^{2(i+1)}\left[\frac{2\vartheta(i+1)}{\gamma+1}\right]^{k}\left[\frac{x_q^{(k+1)(\gamma+1)+1}}{(k+1)(\gamma+1)+1}\right].\end{aligned}$$

The $LC(q)$ is given by

$$L(q)=\frac{1}{\mu}\int_0^{x_q}xg(x)dx=\frac{\vartheta}{\mu}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{(2i+1)!k!}\left(\frac{\pi}{2}\right)^{2(i+1)}\left[\frac{2\vartheta(i+1)}{\gamma+1}\right]^{k}\left[\frac{x_q^{(k+1)(\gamma+1)+1}}{(k+1)(\gamma+1)+1}\right].$$

6 Estimation of Parameters

This section considers the estimation of SS-PH distribution via maximum likelihood approach.

6.1 Maximum Likelihood Estimation

Let $x_1,x_2,\cdots,x_n$ denote a random sample of complete data from the SS-PH distribution. The likelihood function (LF) is given as

$$L(\vartheta,\gamma)=\prod_{i=1}^{n}g(x_i,\vartheta,\gamma)=\prod_{i=1}^{n}\left[\frac{\pi}{2}\vartheta x_i^{\gamma}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)\right]. \tag{24}$$

The log-LF is

$$logL(\vartheta,\gamma)=n\ln\left(\frac{\pi}{2}\right)+n\ln(\vartheta)+\gamma\sum_{i=1}^{n}\ln(x_i)-\frac{\vartheta}{\gamma+1}\sum_{i=1}^{n}x_i^{\gamma+1}+\sum_{i=1}^{n}\ln\left[\sin\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)\right]. \tag{25}$$

The partial derivatives of (25) are as follows

$$\begin{aligned}\frac{\partial}{\partial\vartheta}logL(\vartheta,\gamma) &= \frac{n}{\vartheta}-\frac{1}{\gamma+1}\sum_{i=1}^{n}x_i^{\gamma+1}-\frac{\pi}{2(\gamma+1)}\sum_{i=1}^{n}\left[x_i^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\cot\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)\right],\\ \frac{\partial}{\partial\gamma}logL(\vartheta,\gamma) &= \sum_{i=1}^{n}\ln(x_i)+\frac{\vartheta}{\gamma+1)^2}\sum_{i=1}^{n}x_i^{\gamma+1}-\frac{\vartheta}{\gamma+1}\sum_{i=1}^{n}x_i^{\gamma+1}\ln(x_i)+\frac{\pi\vartheta}{2(\gamma+1)^2}\times\\ &\quad\sum_{i=1}^{n}[1-(\gamma+1)\ln(x_i)]\,x_i^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\cot\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right).\end{aligned}$$

The MLEs of ϑ and γ can be obtained as follows

$$\frac{n}{\vartheta}-\frac{1}{\gamma+1}\sum_{i=1}^{n}x_i^{\gamma+1}-\frac{\pi}{2(\gamma+1)}\sum_{i=1}^{n}\left[x_i^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\cot\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)\right]=0, \tag{26}$$

$$\sum_{i=1}^{n}\ln(x_i)+\frac{\vartheta}{\gamma+1)^2}\sum_{i=1}^{n}x_i^{\gamma+1}-\frac{\vartheta}{\gamma+1}\sum_{i=1}^{n}x_i^{\gamma+1}\ln(x_i)+\frac{\pi\vartheta}{2(\gamma+1)^2}\sum_{i=1}^{n}[1-(\gamma+1)\ln(x_i)]\,x_i^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\cot\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)=0. \tag{27}$$

This system of non-linear equations has no closed form solution in ϑ and γ, so we shall use a numerical program system to find its solution with respect to ϑ and γ.

6.2 Asymptotic confidence bounds

Since the MLEs of unknown parameters are not in closed form, so, we derive asymptotic confidence intervals (C.I.) of these parameters by using variance covariance (Var-Cov) matrix $\boldsymbol{V}$, see [23].
The ML estimators are asymptotically normally distributed with multivariate normal distribution given by [23].

$$(\hat{\vartheta},\hat{\gamma})\sim N_2(\Theta,\boldsymbol{V}),$$

where $\Theta=(\vartheta,\gamma)$ and $\boldsymbol{V}$ is given as follows

$$\boldsymbol{V}=\begin{pmatrix}-\frac{\partial^2 LogL}{\partial\vartheta^2} & -\frac{\partial^2 LogL}{\partial\vartheta\partial\gamma}\\ -\frac{\partial^2 LogL}{\partial\vartheta\partial\gamma} & -\frac{\partial^2 LogL}{\partial\gamma^2}\end{pmatrix}^{-1}_{\Theta\to\hat{\Theta}},$$

where,

$$\begin{aligned}\frac{\partial^2}{\partial\vartheta^2}LogL(\vartheta,\gamma) &= -\frac{n}{\vartheta^2}+\frac{\pi}{2(\gamma+1)^2}\sum_{i=1}^{n}\left[x_i^{2(\gamma+1)}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\cot\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)\right]\\ &\quad-\frac{\pi^2}{4(\gamma+1)^2}\sum_{i=1}^{n}\left[x_i^{2(\gamma+1)}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\csc^2\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)\right],\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial^2}{\partial\vartheta\partial\gamma}LogL(\vartheta,\gamma) &= \frac{1}{(\gamma+1)^2}\sum_{i=1}^{n}[1-(\gamma+1)\ln(x_i)]\,x_i^{\gamma+1}+\\ &\quad\frac{\pi}{2(\gamma+1)^3}\sum_{i=1}^{n}[1-(\gamma+1)\ln(x_i)]\left[(\gamma+1)-\vartheta x_i^{\gamma+1}\right]x_i^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\cot\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)\\ &\quad+\frac{\vartheta\pi^2}{4(\gamma+1)^3}\sum_{i=1}^{n}[1-(\gamma+1)\ln(x_i)]\,x_i^{2(\gamma+1)}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\csc^2\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right),\end{aligned} \tag{29}$$

$$\frac{\partial^2}{\partial\gamma^2}LogL(\vartheta,\gamma) = -\frac{\vartheta}{(\gamma+1)^3}\sum_{i=1}^{n}\left[2-2(\gamma+1)\ln(x_i)+(\gamma+1)^2\ln(x_i)^2\right]x_i^{\gamma+1} - \frac{\vartheta\pi}{2(\gamma+1)^3}\times$$
$$\sum_{i=1}^{n}\left[2-2(\gamma+1)\ln(x_i)+(\gamma+1)^2\ln(x_i)^2\right]x_i^{\gamma+1}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\cot\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)$$
$$+\frac{\vartheta^2\pi}{2(\gamma+1)^4}\sum_{i=1}^{n}\left[1-(\gamma+1)\ln(x_i)\right]^2 x_i^{2(\gamma+1)}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\cot\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right)$$
$$-\frac{\vartheta^2\pi^2}{4(\gamma+1)^4}\sum_{i=1}^{n}\left[1-(\gamma+1)\ln(x_i)\right]^2 x_i^{2(\gamma+1)}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\csc^2\left(\frac{\pi}{2}e^{-\frac{\vartheta}{\gamma+1}x_i^{\gamma+1}}\right). \tag{30}$$

A $100(1-\alpha)\%$ confidence interval for $\Theta = (\vartheta,\gamma)$, can be approximated by

$$\hat{\vartheta} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\vartheta}}, \quad \text{and} \quad \hat{\gamma} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\gamma}},$$

where $z_{\frac{\alpha}{2}}$ is the upper $100\frac{\alpha}{2}$-th percentile of $N(0,1)$, and $var(\hat{\Theta}_i)$ is the diagonal i-th element in $\boldsymbol{V}$.

7 Applications

We will analysis some real data using our proposed model and compare it with some other models. For comparison some criteria such as, Kolmogorov Smirnov (K-S) statistic, Akaike information criterion (AIC), [24], Akaike information criterion with correction (AICC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) can be used.

$$K-S = \sup_x |F_m(x)-\hat{F}(x)|, \qquad AIC = 2k-2\ell,$$

$$AAIC = AIC + \frac{2k(k+1)}{m-k+1}, \qquad BIC = k\ln(m)-2\ell,$$

where k and m are the number of parameters and observed data, $\ell = LogL(\hat{\vartheta},\hat{\gamma})$, $\hat{F}(x)$ is estimated CDF and $F_m(x)$ is the empirical CDF,

$$\bar{F}(x) = \frac{1}{m}\sum_{i=1}^{m}\hat{F}(x_i), \qquad F_m(x) = \frac{1}{m}\sum_{i=1}^{m}I\left(x_{(i)} \le x\right),$$

and

$$I\left(x_{(i)} \le x\right) = \begin{cases} 1, \text{ if } x_{(i)} \le x \\ 0, \text{ otherwise} \end{cases}.$$

If the data have a smaller value of AIC, AAIC, BIC and K-S, it indicates that proposed model can be taken as a best fit.

Example 7.1: Bladder Cancer Patients Data.
The data set listed below provided by [25], it consists of remission times (in months) of 128 patients.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98
6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28
9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
1.46	18.10	11.79	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25
8.37	12.02	2.02	13.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	12.07
6.76	21.73	2.07	3.36	6.93	8.65	12.63	22.69				

Table 2: MLEs of the parameters and K-S.

Models	$\hat{\vartheta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\theta}$	K-S
SS-PHD	0.06	-0.003	–	–	0.068559
SS-Exp(θ)	–	–	–	0.059	0.068684
TIWD	0.833	–	-0.855	1.720	0.119000
TIRD	–	–	0.954	0.748	0.676000
TIED	–	–	0.859	1.688	0.155000
IWD	0.75	–	–	3.288	0.131000

Table 3: The LogL , AIC, AICC, BIC and HQIC

Models	LogL	AIC	AICC	BIC	HQIC
SS-PHD	-415.313	834.625	834.721	840.329	836.943
SS-Exp	-415.314	832.628	832.659	835.48	833.786
TIWD	-438.481	882.963	883.157	891.519	886.439
TIRD	-714.278	1433.00	1433.00	1438.00	1435.00
TIED	-444.835	893.67	893.766	899.374	895.987
IWD	-445.794	895.589	895.685	901.293	897.906

We have extracted the values of MLEs of parameters, K-S test values, log likelihood, AIC, AICC, BIC, and HQIC for SS-PHD, TIWD, TIRD, TIED and IWD for the above considered data and present their values in the Tables 2-3. It has been noticed that SS-PH distribution provides a better fit than the other lifetime distributions for the above data.
where, TIWD: Transmuted inverse Weibull distribution, TIRD: Transmuted inverse Rayleigh distribution, TIED: Transmuted inverted exponential distribution, IWD: Inverse Weibull distribution.
The Var-Cov matrix is given as

$$\boldsymbol{V} = \begin{pmatrix} 6.935\times 10^{-5} & -4.377\times 10^{-4} \\ -4.337\times 10^{-4} & 0.004 \end{pmatrix}.$$

Then the 95% C.I. for ϑ and γ for SS-PH distribution are (0.043, 0.076) and (-0.132, 0.125), respectively. It is shown that the LF has a unique solution by Figure 3.

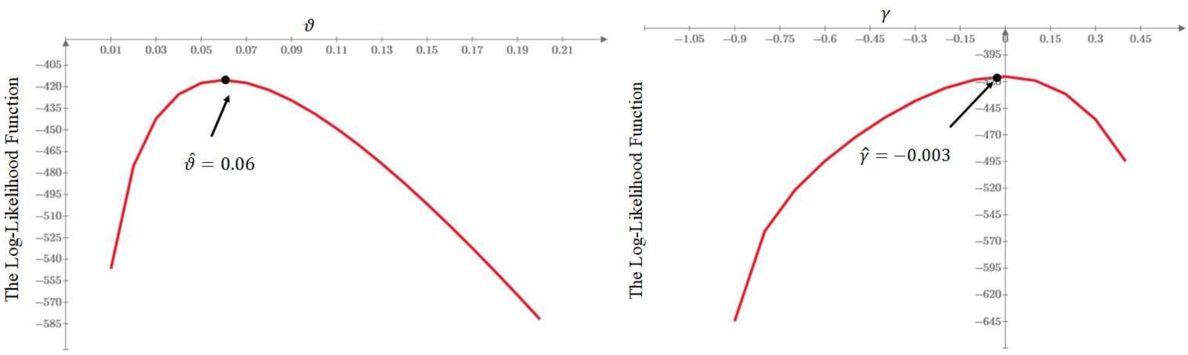

Fig. 3: The profile of the log-LF of ϑ and γ.

Some measures of the SS-PHD (For $\hat{\vartheta} = 0.06$ and $\hat{\gamma} = -0.003$) are displayed in Table 4.

Table 4: Some statistical measures for SS-PHD.

Mean	Median	Variance	C.V.	Sk	Ku
9.325	6.776	77.944	0.946	1.854	8.181

SS-PH model is right skewed heavy-tailed distribution.

A comparison of the SS-PH model and some of its sub-models is shown in Tables 5-6.

Table 5: MLEs of the parameters, K-S, and p-value

Models	$\hat{\vartheta}$	$\hat{\gamma}$	$\hat{\nu}$	K-S	p-value
SS-PHD	0.06	-0.003	–	0.068559	0.586012
SS-EX(ϑ)	0.059	–	–	0.068684	0.583388
SS-WD (ϑ,1)	0.23	–	–	0.787518	0.00
SS-RD (ν)	–	–	13.607	0.374310	0.00

Table 6: The LogL , AIC, AICC, BIC, RMSE and R^2.

Models	LogL	AIC	AICC	BIC	HQIC	RMSE	R^2
SS-PHD	-415.313	834.625	834.721	840.329	836.943	0.032621922	0.98545
SS-EX(ϑ)	-415.314	832.628	832.659	835.48	833.786	0.034535464	0.983713
SS-WD (ϑ,1)	-644.852	1292	1292	1295	1293	0.491078676	0.012209
SS-RD (ν)	-502.815	1008	1008	101	1009	0.253353552	0.601041

On comparing both Tables 5 and 6, we come across SS-PHD gives better fit to the data over its sub model.

Example 7.2: The following data

4.571	7.201	3.606	8.479	11.410	8.961	10.919	10.908	6.503	18.474	11.010
16.561	13.226	15.137	8.697	15.787	13.333	11.822	14.242	11.273	14.330	16.046
10.282	11.775	10.138	9.037	12.396	10.644	8.646	8.905	8.906	7.407	7.445
6.194	4.640	5.452	5.073	4.416	4.859	4.408	4.639	3.148	4.040	4.253
3.564	3.827	3.134	2.780	2.881	3.341	2.686	2.814	2.508	2.450	1.518
17.337	11.950	7.214	4.011							

belongs to Italy COVID-19 mortality rates for 59 days. It is noted from (Feb. 27 to Apr. 27, 2020), see [26]. Tables 7-8 reports the comparison criteria,

Table 7: MLEs of the parameters and the K-S, and p-value.

Models	$\hat{\vartheta}$	$\hat{\gamma}$	$\hat{\nu}$	K-S	p-value
SS-PHD	0.018	0.826	–	0.1230440	0.3156159
SS-EX(ϑ)	0.07	–	–	0.2275707	0.0036147
SS-WD (ϑ,1)	0.232	–	–	0.8796020	0.000
SS-RD (ν)	–	–	8.869	0.1563720	0.1013753

Table 8: The LogL, AIC, AICC, BIC, RMSE and R^2.

Models	LogL	AIC	AICC	BIC	HQIC	RMSE	R^2
SS-PHD	-167.933	339.865	340.079	344.02	341.487	0.049283319	0.975105
SS-EX(ϑ)	-180.612	363.224	363.294	365.301	364.035	0.106196991	0.799626
SS-WD (ϑ,1)	-307.616	617.233	617.303	619.31	618.044	0.508417667	0.002451
SS-RD (ν)	-168.317	338.634	338.704	340.712	339.445	0.142698112	0.834532

On comparing both Tables 7 and 8, we come across SS-PHD gives better fit to the data over its sub models. The Var-Cov matrix is

$$\boldsymbol{V} = \begin{pmatrix} 4.794\times 10^{-5} & -0.001 \\ -0.001 & 0.037 \end{pmatrix}.$$

Then the 95% C.I. for ϑ and γ for SS-PHD are (0.004, 0.031) and (0.448, 1.204), respectively. The LF has a unique solution, see Figure 4.

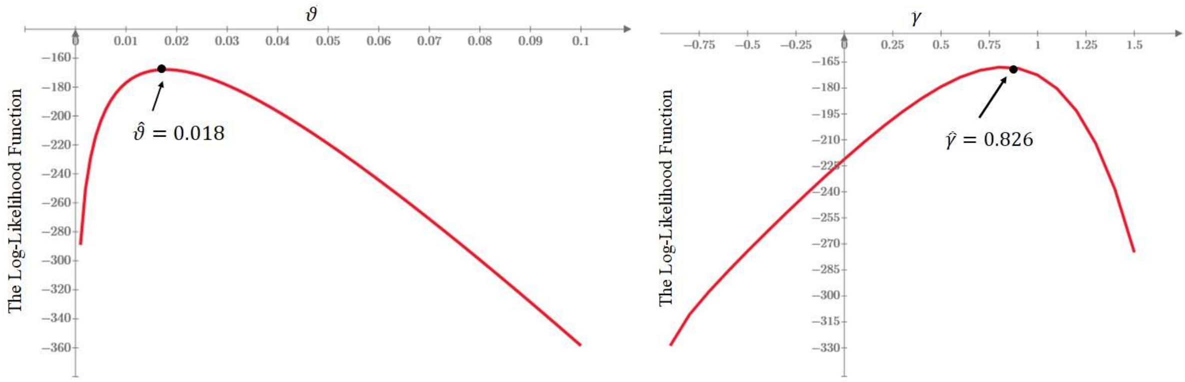

Fig. 4: The profile of the log-LF of ϑ and γ.

Table 9 reports some statistical measures of SS-PHD ($\hat{\vartheta} = 0.018$ and $\hat{\gamma} = 0.826$).

Table 9: Some statistical measures for SS-PHD.

Mean	Median	Variance	C.V.	Sk	Ku
8.174	7.656	19.487	0.540	0.661	3.338

We notice that SS-PHD is right skewed heavy-tailed distribution.

Example 7.3: The following data is the COVID-19 mortality rate for Mexico during 108 days, (from 4 March to 20 July 2020), see [26].

8.826	6.105	10.383	7.267	13.220	6.015	10.855	6.122	10.685	10.035	5.242	7.630
14.604	7.903	6.327	9.391	14.962	4.730	3.215	16.498	11.665	9.284	12.878	6.656
3.440	5.854	8.813	10.043	7.260	5.985	4.424	4.344	5.143	9.935	7.840	9.550
6.968	6.370	3.537	3.286	10.158	8.108	6.697	7.151	6.560	2.988	3.336	6.814
8.325	7.854	8.551	3.228	3.499	3.751	7.486	6.625	6.140	4.909	4.661	1.867
2.838	5.392	12.042	8.696	6.412	3.395	1.815	3.327	5.406	6.182	4.949	4.089
3.359	2.070	3.298	5.317	5.442	4.557	4.292	2.500	6.535	4.648	4.697	5.459
4.120	3.922	3.219	1.402	2.438	3.257	3.632	3.233	3.027	2.352	1.205	2.077
3.778	3.218	2.926	2.601	2.065	1.041	1.800	3.029	2.058	2.326	2.506	1.923

The comparison criterion is given in Tables 10-11.

On comparing both Tables 10 and 11, we come across SS-PHD gives better fit to the data over PHD. The Var-Cov matrix is given as

$$\boldsymbol{V} = \begin{pmatrix} 6.489\times 10^{-5} & -9.799\times 10^{-4} \\ -9.799\times 10^{-4} & 0.018 \end{pmatrix}.$$

The 95% C.I. for ϑ and γ for SS-PH distribution are (0.02, 0.051) and (0.532, 1.053), respectively. The LF has a unique solution can be shown by Figure 5.

For $\hat{\vartheta} = 0.035$ and $\hat{\gamma} = 0.792$, Some measures of the SS-PHD are displayed in Table 12.
We notice that SS-PHD is right skewed heavy-tailed distribution

Table 10: MLEs of the parameters and the K-S, and p-value.

Models	$\hat{\vartheta}$	$\hat{\gamma}$	$\hat{\nu}$	K-S	p-value
SS-PHD	0.035	0.792	–	0.077275	0.53489546
SS-EX(ϑ)	0.099	–	–	0.2253037	0.00002705
SS-WD (ϑ,1)	0.276	–	–	0.857184	0.00
SS-RD (ν)	–	–	6.279	0.114144	0.112552065

Table 11: The LogL , AIC, AICC, BIC, RMSE and R^2.

Models	LogL	AIC	AICC	BIC	HQIC	RMSE	R^2
SS-PHD	-269.465	542.931	543.045	548.295	545.106	0.035547008	0.984713
SS-EX(ϑ)	-292.866	587.731	587.769	590.414	588.819	0.117324507	0.741391
SS-WD (ϑ,1)	-507.622	1017	1017	1019	1018	0.508414595	0.003227
SS-RD (ν)	-270.622	543.245	543.282	545.927	544.332	0.107930853	0.885457

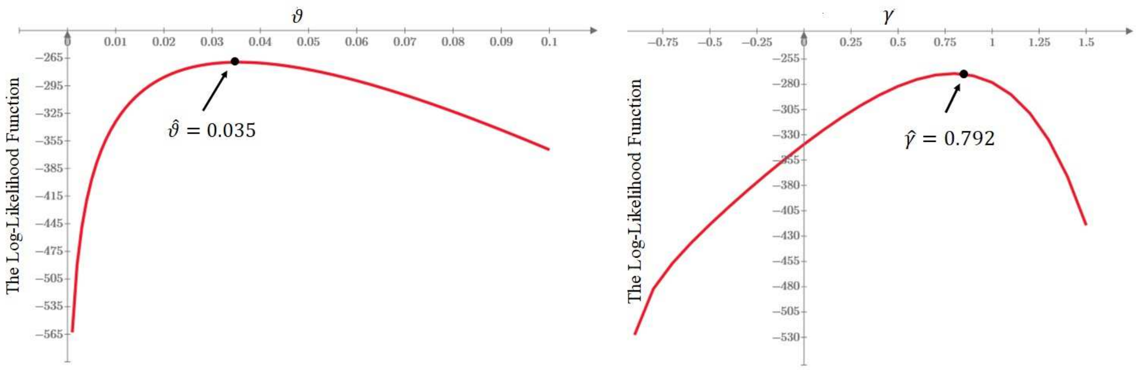

Fig. 5: The profile of the log-LF of ϑ and γ.

Table 12: Some statistical measures for SS-PHD.

Mean	Median	Variance	C.V.	Sk	Ku
5.825	7.656	10.234	0.5491	0.688	3.392

8 Concluding remarks

In this paper, we presented a new generalization of power hazard distribution using the sine function. The proposed distribution is named SS-PH distribution. Some statistical properties of SS-PHD have been studied. The estimators of SS-PHD are also attained. A comparison was conducted with three real data sets of COVID-19 mortality rates. These data showed that SS-PHD provides the most suitable model with compared to the other competing models.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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