Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170208

Numerical Solution for the Sin-Gordon Equation Using the Finite Difference Method and the Non-Stander Finite Difference Method

K. R. Raslan[1], *A. A. Soliman*[2], *Khalid K. Ali*[1,*], *M. Gaber*[1], *and Soma R. Almhdy*[2]

[1]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt
[2]Department of Mathematics, Faculty of Science, Arish University, Arish,45111, Egypt

Received: 6 Nov. 2022, Revised: 22 Dec. 2022, Accepted: 7 Feb. 2023
Published online: 1 Mar. 2023

Abstract: In this paper, We study the non-linear sin Gordon equation as a numerical solution using two methods.: the finite differences method and the non-standard finite differences method. A comparison is presented between the numerical results of both methods.The stability of two schemes is conditionally stable.Some tables and figures are presented to illustrate our results.

Keywords: Sine-Gorden equation, finite differences methods, non-standard finite differences method, stability

1 Introduction

Differential equations represent the best way to describe most of the most effective engineering, mathematical and scientific problems. This is evident when describing heat transfer processes, inhibitor flow, wave motion in electronic circuits, and the analysis of chemical reactions using mathematics. Applications of the nonlinear sine-Gordon equation include the transmission of fluxions in Josephson junctions [1], differential geometry. fluid motion stability, nonlinear physics, and practical sciences [2]. Numerous problems in numerous branches of applied mathematics, physics, and engineering take the form of non-linear wave problems. One of the most important non-linear wave equations is the one-dimensional sin-Gordon equation there are non-linear problems that do not have known analytic solutions. Ben-Yu et al. [3] proposed two distinct strategies to resolve the finite differences. By Bratsos and Twizell [4] The method of lines is used to transform the initial/boundary-value problem associated with the nonlinear hyperbolic sine-Gordon equation, into a first-order, nonlinear, initial-value problem. In [5], Mohebbi and Dehghan introduced a method based on using the diagonally implicit Runge-Kutta-Nyström (DIRKN) method and the compact finite difference scheme, respectively, for the spatial and temporal components. We apply a limited approximation to the finite difference. And also presented the initial/boundary value problem associated with sin-Gordon equation nonlinear initial value problem by Bratsos and Twizell [6] using the method of lines. Schatzman and Evry [7] Use a chart that is divided into two areas by a nearly vertical line. In A. El-Sayed [8] continuing to solve the initial value problem. In Ketabchi and Moosaei [9] use the optimal correction equation for the absolute value of error. And T.D Taylor[10] the one-dimensional equations of inviscid fluid flow are integrated for shock waves, contact surfaces, and rarefaction waves using the numerical techniques. Both use Shamardan [11] and Christie [12] the finite element technique for nonlinear problems. In A.A. Soliman[13] and [14] use induction using the Van Neumann method that is used. In Khomeriki and Ramaz [15] used the Sine equation Cordon as a non-linear wave equation for the phenomenon of bistable behavior (Bistable Behavior). The explanation for this phenomenon analytically and numerically to study the time system.And also studied Shohet, J. Leon[16] the nonlinear model of the Sine-Gordon equation that describes the magnetic hydrodynamic patterns. In L. Zhang a finite difference technique that is energy-conservative was suggested. It was established that the various solutions were stable and

* Corresponding author e-mail: khalidkaram2012@azhar.edu.eg

converged. In Raslan[18] solving partial differential equations such as the wave equation and other nonlinear equations. Sewell,Granville [19] the 2D sine-Gordon equation has new solutions. In Johnson and Suarez [20] The 2D sine-Gordon equation is domain wall collision has, as far as the authors are aware, not been reported in the literature, is of particular interest. We consider the following one-dimensional sine-Gordon equation

$$v_{tt} - v_{xx} + \sin v = 0. \quad (1)$$

The sine-Gordon equation has been numerically solved using a number of different schemes that have been developed in the literature. to find the numerical solution of the partial differential equation substituted by approximates of the finite difference method or non-stander finite difference method in the partial differential equation. Then we get the difference scheme, which can be solved using the Mathematica program. The structure of this paper is as follows. Section 2 displays the problem of the sin Gordon equation. In section 3, finite difference method. In sections 4 and 5, stability analysis and local truncation error of finite difference for Sin-Gorden equation. In section 6, non-stander finite difference method. In sections 7 and 8, stability analysis and local truncation error of non-stander finite difference for Sin-Gorden equation. where we developed two schemes that are conditionally stable for solving the sin-Gordon equation, this scheme is based on the finite difference method and the nonstandard finite difference method, the stability analysis of the two schemes is discussed, and we show that the two schemes are conditionally stable, the numerical results are obtained and discussed.

2 The problem

The Sine-Gordon equation formula

$$v_{tt}(x,t) = v_{xx}(x,t) - \sin v, \quad 0 \le x \le l, \quad t > 0,$$

where subscripts x and t stands for difference, subject to the conditions

boundary conditions

$$v(0,t) = v(1,t) = 0, \forall t > 0$$

with initial conditions

$$v(x,0) = f(x),$$

and

$$v_t(x,0) = g(x), 0 \le x \le l$$

the exact solution is:

$$v(x,t) = \frac{1}{2}(\sin \pi(x+t) + \sin \pi(x-t)).$$

The $x-axis$ step size was selected $h = \frac{l}{M}$ where M is a positive integer number, then select a time step size k,$k > 0$.

3 Finite Difference method

solving the Sine-Gordon equation problem using the finite difference approach. First, we present the derivative approximation shown below.

$$(V_i^j)_{tt} = \frac{1}{k^2}[V_i^{j+1} - 2V_i^j + V_i^{j-1}] + O(k)^2 \quad (2)$$

$$(V_i^j)_{xx} = \frac{1}{h^2}[V_{i+1}^j - 2V_i^j + V_{i-1}^j] + O(h^2), \quad (3)$$

where $x_i = ih$, $t_j = jk$, $i = 0,1...$and $j = 0,1...$ where the quantity associated with time level t_j is denoted by the superscript j, and the quantity associated with space mesh point x_i is denoted by the subscript i.

We now suppose v_i^j at the grid point, that is the exact solution. (x_i,t_n) and V_i^j is the approximate numerical values at the same point. When it occurs, the finite difference method for (1) becomes

$$(V_i^j)_{tt} - (V_i^j)_{xx} + \sin(V_i^j) = 0. \quad (4)$$

Using the difference approximations given by relations (2) and (3) into (4), then we obtain:

$$\frac{1}{k^2}[V_i^{j+1} - 2V_i^j + V_i^{j-1}] = \frac{1}{h^2}[V_{i+1}^j - 2V_i^j + V_{i-1}^j] - \sin(V_i^j), \quad (5)$$

which can be written further as:

$$V_i^{j+1} = r^2[V_{i+1}^j - 2V_i^j + V_{i-1}^j] + 2V_i^j - V_i^{j-1} - \sin(V_i^j)$$

The scheme of finite differences is

$$V_i^{j+1} = r^2 V_{i-1}^j + 2(1-r^2)V_i^j + r^2 V_{i+1}^j - V_i^{j-1} - k^2 \sin(V_i^j), \quad r^2 = \frac{k^2}{h^2}. \quad (6)$$

Through this scheme, it is possible to obtain, a system of algebra for the equation that can be solved to option the numerical solution.Now we talk about the stability and local truncation error method.

4 Stability analysis of Finite Difference Scheme

Lemma 1.*The method of finite differences (6) is a conditionally stable.*

*Proof.*The stability of the system will be investigated using the Von-Neumann theory concept of the growth of the Fourier mode using the finite difference method defined as:

$$V_i^j = \xi^j e^{Ikih}, I = \sqrt{-1}, \quad (7)$$

where h is the element size and k is the mode number.

To use the von Neumann method to Eq.(1), we use the following linearization of the nonlinear term $\sin(V_i^j) = V$. The scheme becomes

$$V_i^{j+1} = r^2 V_{i-1}^j + 2(1-r^2)V_i^j + r^2 V_{i+1}^j - V_i^{j-1} - k^2 V_i^j. \quad (8)$$

Next, substitute (7) into (8) yields

$$\xi^{j+1} = g\xi^j, \quad (9)$$

where g is the growth factor and we obtain from (8) and (9

$$g^2 - 2g[1 - \frac{k^2}{2} - 2r^2 \sin^2(\frac{kh}{2})] + 1 = 0. \quad (10)$$

Let the equation's roots be (12) g_1 *and* g_2 so that $g_1 g_2 = 1$, and $g_1 + g_2 = 1 - \frac{k^2}{2} - 2r^2 \sin^2(\frac{kh}{2})$, we can prove that

$$\mid 1 - \frac{k^2}{2} - 2r^2 \sin^2(\frac{kh}{2}) \mid \leq 1 \quad (11)$$

$$-1 \leq 1 - \frac{k^2}{2} - 2r^2 \sin^2(\frac{kh}{2}) \leq 1 \quad (12)$$

That is, $r^2 = \frac{k^2}{h^2}$ cannot be a negative quantity, that is the inequality (11) leads to $r^2 > 0$, and this is true.in order to satisfy the inequality (12) we need

$$-2 \leq -\frac{k^2}{2} - 2r^2 \sin^2(\frac{kh}{2}) \leq 0$$

$$0 \leq \frac{k^2}{2} + 2r^2 \sin^2(\frac{kh}{2}) \leq 2$$

$$0 \leq \frac{k^2}{2} + 2r^2 \sin^2(\frac{kh}{2}) \leq 2$$

$$r^2 = \frac{4-k^2}{4} = 1 - \frac{k^2}{4}, \quad (13)$$

Therefore

$$-\sqrt{1 - \frac{k^2}{4}} \leq r \leq \sqrt{1 - \frac{k^2}{4}}, \quad (14)$$

then the Von-Neumann method can get the quadratic algebraic equation from (10) can be expressed as

$$g^2 - 2g\sin\theta + 1 = 0$$

$$g = \sin\theta \pm \sqrt{\sin\theta^2 - 1} \quad (15)$$

where $\sin\theta = 1 - \frac{k^2}{2} - 2r^2 \sin^2(\frac{kh}{2})$, thus (15) yields $\mid g_1 \mid = \mid g_2 \mid = 1$. The finite difference method is hence conditionally stable.

5 Local Truncation Error of Finite Difference method

Lemma 2. *The finite difference method (6) is truncation error, τ_i^j is of order $(k^2 + h^2)$.*

Proof. To investigate the scheme's accuracy (6) We expand all concepts related to the point (x_i, t_j) using Taylor's series. Now, give the local truncation error

$$\tau_i^j = \frac{1}{k^2}(v_i^{j+1} - v_i^j + v_i^{j-1}) - \frac{1}{(h)^2}[v_{i+1}^j - 2v_i^j + v_{i-1}^j] + \sin v_i^j, \quad (16)$$

using Taylor's series expansion we get

$$\frac{1}{k^2}(v_i^{j+1} - v_i^j + v_i^{j-1}) = (\frac{\partial^2 v}{\partial t^2})_i^j + \frac{\triangle(t)^2}{12!}(\frac{\partial^4 v}{\partial t^4})_i^j + \ldots \quad (17)$$

$$\frac{1}{(h)^2}[v_{i+1}^j - 2v_i^j + v_{i-1}^j] = (\frac{\partial^2 v}{\partial x^2})_i^j + \frac{\triangle(x)^2}{12!}(\frac{\partial^4 v}{\partial x^4})_i^j + \ldots \quad (18)$$

Substitution from (17)and(18) into (16)

$$\tau_i^j = (\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial t^2} + \sin v)_i^j + \frac{\triangle(t)^2}{12!}(\frac{\partial^4 v}{\partial t^4})_i^j - \frac{\triangle(x)^2}{12!}(\frac{\partial^4 v}{\partial x^4})_i^j + \ldots \quad (19)$$

But is the solution of differential (1), so

$$(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial t^2} + \sin v)_i^j = 0.$$

In view of this, the main part of the local truncation error is

$$\frac{\triangle(t)^2}{12!}(\frac{\partial^4 v}{\partial t^4})_i^j - \frac{\triangle(x)^2}{12!}(\frac{\partial^4 v}{\partial x^4})_i^j + \ldots \quad (20)$$

The local truncation error is a result

$$\tau_i^j = O(k^2 + h^2). \quad (21)$$

6 Non-standard Finite Difference method

Solving the Sine-Gordon equation problem using the non-standard finite difference approach. First, we present the derivative approximation shown below.

$$(V_i^j)_{tt} = \frac{1}{\varphi(k^2)}[V_i^{j+1} - 2V_i^j + V_i^{j-1}] + O(k)^2 \quad (22)$$

$$(V_i^j)_{xx} = \frac{1}{\beta(h^2)}[V_{i+1}^j - 2V_i^j + V_{i-1}^j] + O(h^2), \quad (23)$$

where $\varphi(k) = sin(k), \beta(h) = sin(h), x_i = ih, t_j = jk$, $i = 0,1...$and $j = 0,1...$ where the quantity associated with time level t_j is denoted by the superscript j, and the quantity associated with space mesh point x_i is denoted by the subscript i. We now suppose v_i^j at the grid point, that is the exact solution. (x_i,t_n) and V_i^j is the approximate numerical values at the same point. When it occurs, the nonstandard finite difference method for (1) becomes

$$\frac{1}{\varphi(k^2)}[V_i^{j+1} - V_i^j + V_i^{j-1}] = \frac{1}{\beta(h^2)}[V_{i+1}^j - 2V_i^j + V_{i-1}^j] - \sin V_i^j. \quad (24)$$

The scheme of nonstandard finite differences is

$$V_i^{j+1} = r^2 V_{i-1}^j + 2(1-r^2)V_i^j + r^2 V_{i+1}^j - V_i^{j-1} - \varphi(k^2)\sin V_i^j, \ldots r^2 = \frac{\varphi(k^2)}{\beta(h^2)}. \quad (25)$$

Through this scheme, it is possible to obtain, a system of algebra for the equation that can be solved to option the numerical solution. Now we talk about the stability and local truncation error method.

7 Stability analysis of Non-standard Finite Difference Scheme

Lemma 3. *The method of non-standard finite differences (25) is a conditionally stable.*

Proof. The stability of the system will be investigated using the Von-Neumann theory concept of the growth of the Fourier mode using the nonstandard finite difference method as:

$$V_i^j = \xi^j e^{Ikih}, I = \sqrt{-1}, \quad (26)$$

where h is the element size and k is the mode number.

To use the von Neumann method to (1), we use the the following linearization of the nonlinear term $\sin(V_i^j) = V$.

The schem becomes

$$V_i^{j+1} = r^2 V_{i-1}^j + 2(1-r^2)V_i^j + r^2 V_{i+1}^j - V_i^{j-1} - \varphi(k)^2 V_i^j. \quad (27)$$

Next, substitute (26) into (27) yields

$$\xi^{j+1} = g\xi^j, \quad (28)$$

where g is the growth factor and we obtain from (27) and (28)

$$g^2 - 2g[1 - \frac{\varphi(k)^2}{2} - 2r^2\sin^2(\frac{\varphi(k)\beta(h)}{2})] + 1 = 0. \quad (29)$$

Let the equation's roots be (29) g_1 *and* g_2 so that $g_1 g_2 = 1$, and $g_1 + g_2 = 1 - \frac{\varphi(k)^2}{2} - 2r^2\sin^2(\frac{\varphi(k)\beta(h)}{2})$, we can prove that

$$\mid 1 - \frac{\varphi(k)^2}{2} - 2r^2\sin^2(\frac{\varphi(k)\beta(h)}{2}) \mid \leq 1. \quad (30)$$

$$-1 \leq 1 - \frac{\varphi(k)^2}{2} - 2r^2\sin^2(\frac{\varphi(k)\beta(h)}{2}) \leq 1. \quad (31)$$

That is, $r^2 = \frac{k^2}{h^2}$ cannot be a negative quantity, that is the inequality (30) leads to $r^2 > 0$, and this is true.in order to satisfy the inequality (31) we need

$$-2 \leq -\frac{\varphi(k)^2}{2} - 2r^2\sin^2(\frac{\varphi(k)\beta(h)}{2}) \leq 0,$$

$$0 \leq \frac{\varphi(k)^2}{2} + 2r^2\sin^2(\frac{\varphi(k)\beta(h)}{2}) \leq 2,$$

$$r^2 = \frac{4 - \varphi(k)^2}{4} = 1 - \frac{\varphi(k)^2}{4}, \quad (32)$$

$$-\sqrt{1 - \frac{\varphi(k)^2}{4}} \leq r \leq \sqrt{1 - \frac{\varphi(k)^2}{4}}. \quad (33)$$

Then the Von-Neumann method can get the quadratic algebraic equation from (29) can be expressed as

$$g^2 - 2g\sin\theta + 1 = 0,$$

$$g = \sin\theta \pm \sqrt{\sin\theta^2 - 1}, \quad (34)$$

where $\sin\theta = 1 - \frac{\varphi(k)^2}{2} - 2r^2\sin^2(\frac{\varphi(k)\beta(h)}{2})$, and $\varphi(k) = sin(k), \beta(h^2) = sin(h)$. Thus (34) yields $\mid g_1 \mid = \mid g_2 \mid = 1$. The nonstandard finite difference method is hence conditionally stable.

8 Local Truncation Error of Non-standard Finite Difference method

Lemma 4. *The nonstandard finite difference method (25) is truncation error, τ_i^j is of order $(k^2 + h^2)$. nonstandard*

Proof. To investigate the scheme's accuracy (25) We expand all concepts related to the point (x_i,t_j) using Taylor's series. Now, give the local truncation error

$$\tau_i^j = \frac{1}{k^2}(v_i^{j+1} - v_i^j + v_i^{j-1}) - \frac{1}{(h)^2}[v_{i+1}^j - 2v_i^j + v_{i-1}^j] + \sin v_i^j. \quad (35)$$

Using Taylor's series expansion we get

$$\frac{1}{k^2}(v_i^{j+1}-v_i^j+v_i^{j-1})=(\frac{\partial^2 v}{\partial t^2})_i^j+\frac{\triangle(t)^2}{12!}(\frac{\partial^4 v}{\partial t^4})_i^j+\dots \quad (36)$$

$$\frac{1}{(h)^2}[v_{i+1}^j-2v_i^j+v_{i-1}^j]=(\frac{\partial^2 v}{\partial x^2})_i^j+\frac{\triangle(x)^2}{12!}(\frac{\partial^4 v}{\partial x^4})_i^j+\dots \quad (37)$$

Substitution from (36) and (37) into (35)

$$\tau_i^j=(\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial t^2}+\sin v)_i^j+\frac{\triangle(t)^2}{12!}(\frac{\partial^4 v}{\partial t^4})_i^j-\frac{\triangle(x)^2}{12!}(\frac{\partial^4 v}{\partial x^4})_i^j+\dots \quad (38)$$

But is the solution of differential (1), so

$$(\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial t^2}+\sin v)_i^j=0.$$

Then,

$$\frac{\triangle(t)^2}{12!}(\frac{\partial^4 v}{\partial t^4})_i^j-\frac{\triangle(x)^2}{12!}(\frac{\partial^4 v}{\partial x^4})_i^j+\dots \quad (39)$$

The local truncation error is a result

$$\tau_i^j=O(k^2+h^2). \quad (40)$$

9 Numerical solution and results

In this section, the Sine-Gordon equation's numerical solutions for a test problem are obtaines and used our scheme of the nonstandard finite difference method and the finite difference method solve problems of partial differential equation.to illustrate The precision and effectiveness of the scheme. We use the absolute value of the error [16] to means the difference between the numerical and exact solution, the results obtained are tabulated inTables 1-4, all computations for these examples are performed by using Mathematica 11.

examples: we considerThe Sine-Gordon equation formula

$$v_{tt}(x,t)=v_{xx}(x,t)-\sin v, \quad 0\le x\le 1, \forall t>0$$

bondary conditions

$$v(0,t)=v(1,t)=0, \forall t>0$$

with initial conditions

$$v(x,0)=f(x),$$

and

$$v_t(x,0)=g(x), 0\le x\le 1.$$

The exact solution is:

$$v(x,t)=\frac{1}{2}(\sin\pi(x+t)+\sin\pi(x-t)).$$

Table 1: Exact solution and numerical results using FDM (case 1)

x_i	Exact solution	FDM	Absolute error
0	0	0	0
0.1	0.305212	0.304948	0.000263988
0.2	0.580549	0.580075	0.000473499
0.3	0.799057	0.798452	0.000604315
0.4	0.939347	0.938681	0.000666564
0.5	0.987688	0.987005	0.000683555
0.6	0.939347	0.938681	0.000666564
0.7	0.799057	0.798452	0.000604315
0.8	0.580549	0.580075	0.000473499
0.9	0.305212	0.304948	0.000263988
1	0	0	0

Table 2: Exact solution and numerical results using NSFDM (case) 1

x_i	Exact solution	Ns-FDM	Absolute error
0	0	0	0
0.1	0.305212	0.304936	0.000276306
0.2	0.580549	0.580052	0.000496933
0.3	0.799057	0.79842	0.000636573
0.4	0.939347	0.938643	0.000704488
0.5	0.987688	0.986965	0.000723432
0.6	0.939347	0.938643	0.000704488
0.7	0.799057	0.79842	0.000636573
0.8	0.580549	0.580052	0.000496933
0.9	0.305212	0.304936	0.000276306
1	0	0	0

– Case 1: use sizes of steps $h=0.1$ and $k=0.001, M=50, N=10$.

– Case 2: when using sizes of steps $h=0.1$,and $k=0.00001, M=100, N=10$.

Now, we use the finite difference method and the non-standard finite difference method to find the numerical solution for this example. Table $[1-4]$ and Figure $[1-4]$ below give the approximate solution for this example and its comparison to the exact solution. and also find the absolute error.

Table 3: Exact solution and numerical results using FDM (case 2)

x_i	Exact solution	FDM	Absolute error
0	0	0	0
0.1	0.309015	0.309015	1.22915×10^{-7}
0.2	0.587782	0.587782	2.2194×10^{-7}
0.3	0.809013	0.809013	2.85862×10^{-7}
0.4	0.951052	0.951052	3.17923×10^{-7}
0.5	0.999995	0.999995	3.27131×10^{-7}
0.6	0.951052	0.951052	3.17923×10^{-7}
0.7	0.809013	0.809013	2.85862×10^{-7}
0.8	0.587782	0.587782	2.2194×10^{-7}
0.9	0.309015	0.309015	1.22915×10^{-7}
1	0	0	0

Table 4: Exact solution and numerical results using NSFDM (case 2)

x_i	Exact solution	Ns-FDM	Absolute error
0	0	0	0
0.1	0.309015	0.309015	1.27916×10^{-7}
0.2	0.587782	0.587782	2.31453×10^{-7}
0.3	0.809013	0.809013	2.98955×10^{-7}
0.4	0.951052	0.951052	3.33314×10^{-7}
0.5	0.999995	0.999995	3.43314×10^{-7}
0.6	0.951052	0.951052	3.33314×10^{-7}
0.7	0.809013	0.809013	2.98955×10^{-7}
0.8	0.587782	0.587782	2.31453×10^{-7}
0.9	0.309015	0.309015	1.27916×10^{-7}
1	0	0	0

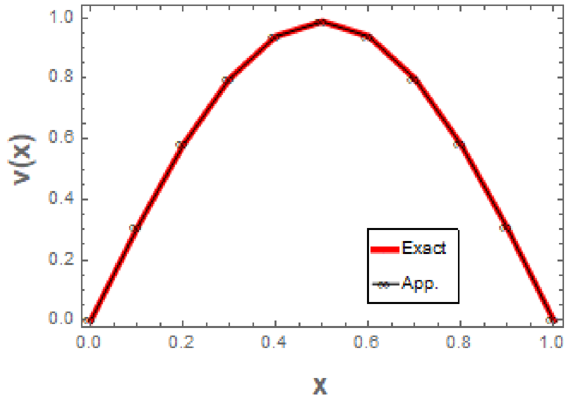


Fig. 1: Approximate and exact solution Using FDM at $k = 0.001, h = 0.1$.

10 Conclusions

In this paper, we studied the solution nonlinear sin Gordon equation using the finite difference method with

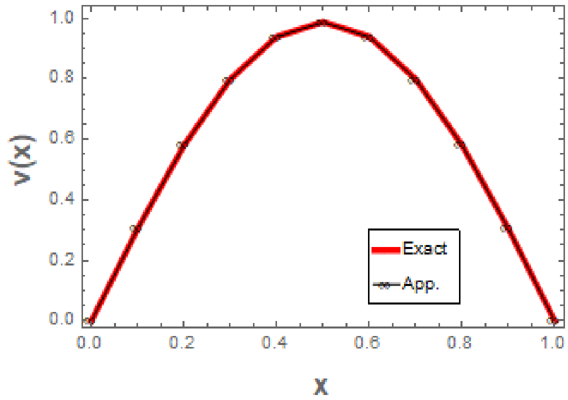


Fig. 2: Approximate and exact solution using NSFDM at $k = 0.001, h = 0.1$.

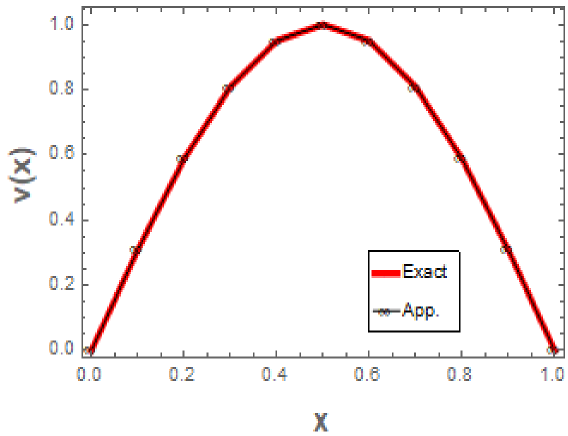


Fig. 3: Approximate and exact solution using FDM at $k = 0.00001, h = 0.1$.

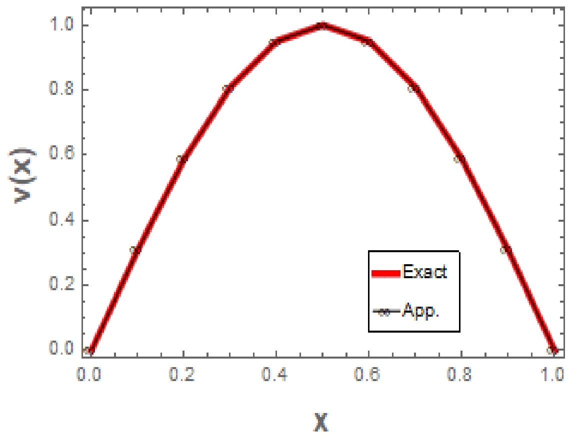


Fig. 4: Approximate and exact solution using NSFDM at $k = 0.00001, h = 0.1$.

the non-nonstandard finite difference method. Two Cases are presentedin the example and the exact solution and approximate solution to the problem are shown as shown in Case 1 and Case 2 was obtained by our method. and we made a comparison of their result with the exact solution to demonstrate the efficiency of the proposed method and show that the approximate solution converges to the exact solution. The absolute error obtained indicated that our method can give a good approximation of the finite difference method of the Sine-Gorden equation. The stability analysis of both methods used to solve the Sein-Gordon equation using Von-Neumann theory is found to be conditionally stable.

Acknowledgement

We would like to thank the reviewers for helpful and constructive comments, which have made great contributions to the improvement of the paper.

References

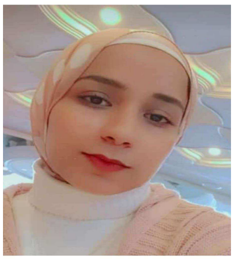
[1] J. K. Perring, T. H. R. Skyrme, A model unified field equation, Nuclear Physics, **31**, 550–555, (1962).
[2] A. Barone, F. Esposito, C. J. Magee, A. C. Scott, Theory and applications of the sine-Gordon equation, La Rivista del Nuovo (1971-1977), **1**(2), 227-267, (1971).
[3] G. Ben-Yu, P. J. Pascual, M. J. Rodriguez, and L. Vazquez, Numerical solution of the sine-Gordon equation, Applied Mathematics and Computation, **18**(1), 1-14, (1986).
[4] A. G. Bratsos, E. H. Twizell, The solution of the sineGordon equation using the method of lines, International Journal of Computer Mathematics, **61**(3-4), 271-292, (1996).
[5] A. Mohebbi, M. Dehghan, High-order solution of one-dimensional sine-Gordon equation using compact finite difference and DIRKN methods, Mathematical and Computer Modelling, **51**(5-6), 537-549 (2010).
[6] A. G. Bratsos, E. H. Twizell, A family of parametric finite-difference methods for the solution of the sine-Gordon equation, Applied Mathematics and Computation, **93**(2-3), 117-137, (1998).
[7] Sc. Evry, A theory of the role of magnetic activity during star formation. Annales d'Astrophysique, **25**, (1962).
[8] A. M. A. El-Sayed, On the fractional differential equations, Appl. Math. Comp, **49**(2-3), (1992).
[9] Ketabchi S., Moosaei H., Fallahi S., Optimal error correction of the absolute value equation using a genetic algorithm, Mathematical and Computer Modelling, **57**(9-10), 2339-2342, (2013).
[10] T. D. Taylor, E. Ndefo, B. S. Masson, A study of numerical methods for solving viscous and inviscid flow problems, Journal of Computational Physics, **9**(1), 99-119, (1972).
[11] A. B. Shamardan, Y. M. Essa, Multi-level adaptive solutions to initial-value problems. Korean Journal of Computational and Applied Mathematics, **7**(1), 215-222, (2000)
[12] Ch. Ian, Product approximation for non-linear problems in the finite element method. IMA Journal of Numerical Analysis, **1**(3), 253-266, (1981).
[13] A. A. Soliman, Exact travelling wave solution of nonlinear variants of the RLW and the PHI-four equations, Physics Letters A, **368**(5), 383-390, (2007).
[14] A. A. Soliman, Exact solutions of KdV-Burgers' equation by Exp-function method, Chaos, Solitons & Fractals, **41** (2), 1034-1039, (2009).
[15] K. Ramaz, J. Leon, Bistability in the sine-Gordon equation: The ideal switch. Physical Review E, **71**(5), (2005).
[16] S. J. Leon, The sine-Gordon equation in reversed-field pinch experiments. Physics of Plasmas, **11**(8), 3877-3887, (2004).
[17] L. Zhang, A finite difference scheme for generalized long wave equation, Appl. Math. Comput, **168**(2), 962-972, (2005).
[18] K. R. Raslan, Numerical methods for partial differential equations. PhD thesis, AL-Azhar University, Cairo, (1999).
[19] S. Granville, The numerical solution of ordinary and partial differential equations, John Wiley & Sons, **75**, (2005).
[20] Johnson S., P. Suarez, and A. Biswas. New exact solutions for the sine-Gordon equation in 2+ 1 dimensions. Computational Mathematics and Mathematical Physics, **52**(1), 98-104, (2012).

Kamal R. Raslan received the M.Sc. and Ph.D. degrees from the Faculty of Science, Menoufia University and Al-Azhar University, Egypt, in 1996 and 1999, respectively. He is currently a full Professor of Mathematics with the Faculty of Science, Al-Azhar University, Egypt. He has authored/coauthored over 114 scientific papers in top-ranked International Journals and Conference Proceedings. His research interests include Numerical Analysis, Finite Difference Methods, Finite Element Methods, Approximation Theory, and Computational Mathematics.

A. A. Soliman Abdel-Maksoud A. Soliman. Professor of Numerical Analysis, Department of Mathematics, Faculty of Science, Arish University. I got my B.Sc. in May 1988 and M.Sc. in 1994, both from Menoufia University. My Ph.D. degree in Numerical Analysis was received in Dec. 1998 from Menoufia University in co-operation with the University of Cincinnati, USA (according to Channel system). I was visiting scholar at the University of Cincinnati from Oct. 1996 to Sep. 1998. I have published several papers in international journals and participated in many research projects funded by King Khalid University and the University of Bisha either as a principal researcher or a co-researcher. I'm also a member of some journals' editorial boards: the editor-in-chief of the "Numerical and Computational Methods in Sciences and Engineering" (by Natural Science Publishing) and an editor in the "Journal of Applied Mathematics", the "Abstract and Applied Analysis" (both by Hindawi Publishing Cooperation), and the "International Journal of Applied Mathematical Research".

Khalid K. Ali MSc in pure Mathematics, Mathematics Department 2015 from Faculty of Science, Al-Azhar University, Cairo, Egypt. Ph.D. in pure Mathematics 2018 from Faculty of Science, Al-Azhar University, Cairo, Egypt. He has authored/co-authored over 110 scientific papers in top-ranked International Journals and Conference Proceedings. His research interests include Numerical Analysis, Finite Difference Methods, Finite Element Methods, and Computational Mathematics.

M. Gaber He holds a Bachelor of Science from the Faculty of Science - Alexandria University - May 1993 with a grade of very good with honors and a master's degree in pure mathematics - Faculty of Science - Alexandria University - 2000 and a Ph. 2007. Currently a teacher of pure mathematics at the Faculty of Science, Arish University. Coordinator of the Financial and Administrative Support Committee of the Quality and Qualification Unit for accreditation



Soma. R. Almhdy degree in pure mathematics 2018, from mathematics department, faculty of science – Arish University - North Sinai - Egypt. Registered Master. At the University of Arish - North Sinai - Egypt in 2020 and still until now.