Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170105

Study of Extended Hermite-Appell Polynomial via Fractional Operators

Anjali Goswami[1], *Mohammad Faisal Khan*[1] *and Mohammad Shadab*[2] *

[1]Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh-11673, Saudi Arabia
[2]Department of Mathematics, School of Basic and Applied Sciences, Lingaya's Vidyapeeth (Deemed to be University), Faridabad-121002, Haryana, India

Received: 2 Oct. 2022, Revised: 23 Nov. 2022, Accepted: 6 Dec. 2022
Published online: 1 Jan. 2023

Abstract: The motive of this research paper is to establish the generalized Hermite and generalized Hermite-Appell polynomials by combining the operational definitions and integral representations. The explicit summation formulae, determinant and recurrence relations for the generalized Hermite-Appell polynomials are derived by applying the integral transforms and appropriate operational rules. For the application purpose, we present the corresponding results for the generalized Hermite-Bernoulli, generalized Hermite-Euler, and Hermite-Genocchi polynomials.

Keywords: Appell polynomials, Hermite-Appell polynomial, Fractional operators, Operational rules, Determinants forms.

1 Introduction and preliminaries

The class of the Appell polynomial sequence is one of the significant classes of polynomial sequences [1,2,3]. The set of Appell polynomial sequence is closed under the operation of umbral composition of polynomial sequences [4,5,6,7,8]. The Appell polynomial sequence can be given by the following generating function

$$A(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}. \qquad (1)$$

The power series $A(t)$ is given by

$$A(t) = A_0 + \frac{t}{1!}A_1 + \frac{t^2}{2!}A_2 + \cdots + \frac{t^n}{n!}A_n, \quad A_0 \neq 0, \qquad (2)$$

where $A_i\{i = 1,2,3,\dots\}$ are real coefficients. It is easy to see that for any $A(t)$, the derivative of $A_n(x)$ satisfies

$$A_n^{'}(x) = nA_{n-1}(x). \qquad (3)$$

The Bernoulli polynomials and numbers [9,10] can be defined by the generating function as follows

$$e^{xt}\left(\frac{t}{e^t - 1}\right) = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \qquad (|t| < 2\pi). \qquad (4)$$

On setting $x = 0$, then the Bernoulli numbers $B_n(0) := B_n$ can be defined by

$$\left(\frac{t}{e^t - 1}\right) = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \qquad (|t| < 2\pi). \qquad (5)$$

The Euler polynomials and numbers [9,10] can be defined by the generating function as follows

$$e^{xt}\left(\frac{2}{e^t + 1}\right) = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \qquad (|t| < \pi). \qquad (6)$$

On setting $x = 0$, then the Euler numbers $E_n(0) := E_n$ can be defined by

$$\left(\frac{2}{e^t + 1}\right) = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}, \qquad (|t| < \pi). \qquad (7)$$

The Genocchi polynomials and numbers can be defined by the generating function as follows

$$e^{xt}\left(\frac{2t}{e^t + 1}\right) = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \qquad (|t| < \pi). \qquad (8)$$

On setting $x = 0$, then the Genocchi numbers $G_n(0) := G_n$ can be defined by

$$\left(\frac{2t}{e^t + 1}\right) = \sum_{n=0}^{\infty} G_n\frac{t^n}{n!}, \qquad (|t| < \pi). \qquad (9)$$

* Corresponding author e-mail: shadabmohd786@gmail.com

Remark. It should be noted that, the Genocchi polynomials $G_n(x)$ do not fulfill all requirements of Appell polynomials, for instance, the degree of $G_n(x)$ is $n-1$, though, the degree of Appell polynomials is n. Therefore, we may put $G_n(x)$ in the class of polynomial sequences which are not considered Appell polynomials in the strong sense [11,12].

In the past decades, mathematics has extensively evaluated generalised and multi-variable special functions. With the use of specific two-variable polynomials, many differential equations in physical problems can be addressed.

These polynomials play a role in a number of core problems, including quantum physics and optics. They are necessary when evaluating the integral involving product of special functions.

The generating function of 2-variable Hermite polynomials define by The 2-variable Hermite polynomial can be presented by following formula

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{x^{n-2r}y^r}{r!(n-2r)!}, \tag{10}$$

The differential equation and generating function for the 2-variable Hermite polynomials $H_n(x,y)$ [13, p. 149, (1.10) and (1.14)] can be given as follows

$$\left(2y\frac{\partial^2}{\partial^2 x^2} + x\frac{\partial}{\partial x} - n\right) = 0, \tag{11}$$

and

$$exp(xt+yt^2) = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}, \tag{12}$$

or

$$exp(xt)J_o(2t\sqrt{-y}) = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!},$$

where $C_n(x)$ (or $J_n(x)$) is the n-th order Tricomi (or Bessel) function [14]

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!(n-r)!} = x^{\frac{-n}{2}} J_n(2\sqrt{x}).$$

The operational definition of 2-variable Hermite polynomials [15, (1.13)] is given as

$$H_n(x,y) = exp\left(y\frac{\partial^2}{\partial y^2}\right)\{x\}^n, \tag{13}$$

in the view of the equation $H_n(x,0) = \{x\}^n$.

Now, we recall the 2-variable Hermite-Appell polynomials ${}_HA_n(x,y)$ introduced by Khan et al. [15] in 2009. In this paper, the generating function of 2-variable Hermite based appell polynomials is introduced as

$$A(t)exp(xt)J_0(2t\sqrt{-y}) = \sum_{n=0}^{\infty} {}_HA_n(x,y)\frac{t^n}{n!}, \tag{14}$$

and

$$A(t)exp(xt)exp(D_y^{-1}t^2) = \sum_{n=0}^{\infty} {}_HA_n(x,y)\frac{t^n}{n!}, \tag{15}$$

where D_x^{-1} is the inverse derivative operator.
From equation (15)

$$\frac{\partial^2}{\partial x^2} {}_HA_n(x,y) = n(n-1) {}_HA_{n-2}(x,y)$$

and

$$\frac{\partial}{\partial y} {}_HA_n(x,y) = n(n-1) {}_HA_{n-2}(x,y) \tag{16}$$

which consequently gives

$$\frac{\partial}{\partial y} {}_HA_n(x,y) = \frac{\partial^2}{\partial x^2} {}_HA_n(x,y). \tag{17}$$

From the generating function and $A_n(x,0) = A_n(x)$, the operational rule for the 2-variable Hermite-Appell polynomials can be given as follows

$${}_HA_n(x,y) = exp\left(y\frac{\partial^2}{\partial x^2}\right)\{A_n(x)\}. \tag{18}$$

Dattoli et al. in [16] used the Euler integral to obtain and generalize the new special hybrid polynomials in wide sense. The Euler integral [14,17,18] can be defined as

$$a^{-\nu} = \frac{1}{\Gamma(\nu)} \int_o^{\infty} e^{-at} t^{\nu-1} dt, \quad min\{Re(\nu), Re(a)\} > 0, \tag{19}$$

which consequently yields the following [16]

$$\begin{aligned}\left(\alpha - \frac{\partial}{\partial x}\right)^{-\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_o^{\infty} e^{-at} t^{\nu-1} e^{t\frac{\partial}{\partial x}} f(x)dt \\ &= \int_0^{\infty} \frac{1}{\Gamma(\nu)} \int_o^{\infty} e^{-at} t^{\nu-1} f(x+t)dt.\end{aligned} \tag{20}$$

For the 2^{nd} order derivatives, we have

$$\left(\alpha - \frac{\partial^2}{\partial x^2}\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_o^{\infty} e^{-at} t^{\nu-1} e^{t\frac{\partial^2}{\partial x^2}} f(x)dt. \tag{21}$$

2 Extended Hermite-Appell polynomials

First we derive the operational rule connecting the Appell polynomials and the extended Hermite–Appell polynomials [13,15,19].

Theorem 1. *For the generalized Hermite polynomials ${}_\nu H_n(x,y;\alpha)$, following operational rule holds true*

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-\nu} \{x^n\} = {}_\nu H_n(x,y;\alpha). \tag{22}$$

*Proof.*By taking $a = \left(\alpha - (y\frac{\partial^2}{\partial x^2})\right)$ in integral and operating the resultant equation on x^n, we get

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-v}\{x^n\} = \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}exp\left(ty\frac{\partial^2}{\partial x^2}\right)\{x^n\}dt, \quad (23)$$

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-v}\{x^n\} = \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}H_n(x,yt)dt. \quad (24)$$

The right hand side of equation (24) represents a new class of polynomials. This class of special polynomials is known as extended Hermite polynomials, and denoted by ${}_vH_n(x,y;\alpha)$ then we have

$${}_vH_n(x,y;\alpha) = \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}H_n(x,yt)dt. \quad (25)$$

On using the equations (24) and (25), we get (22).

Theorem 2.*The generalized Hermite polynomials ${}_vH_n(x,y;\alpha)$ satisfy following generating function*

$$\frac{exp(xu)}{(\alpha - (D_y^{-1})u^2)^v} = \sum_{n=0}^\infty {}_vH_n(x,y;\alpha)\frac{u^n}{n!}. \quad (26)$$

*Proof.*By multiplying $\frac{u^n}{n!}$ both sides of (25) and summing over n, we get

$${}_vH_n(x,y;\alpha)\frac{u^n}{n!} = \sum_{n=0}^\infty \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}H_n(x,yt)\frac{u^n}{n!}dt. (27)$$

Using the equation (12) in above equation (27), we get

$${}_vH_n(x,y;\alpha)\frac{u^n}{n!} = \frac{exp(xu)}{\Gamma(v)}\int_0^\infty e^{-(\alpha - D_y^{-1}u^2)}t^{v-1}dt. \quad (28)$$

By using equation (19) in right hand side of the above equation, we get (26).

Theorem 3.*For the extended Hermite based Appell polynomials ${}_HA_n(x,y;\alpha)$, following operational rule holds true*

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-v}\{A_n(x)\} = {}_HA_n(x,y,\alpha). \quad (29)$$

*Proof.*By taking $a = \left(\alpha - (y\frac{\partial^2}{\partial x^2})\right)$ in integral and operating the resultant equation on $A_n(x)$, we get

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-v}\{A_n(x)\} = \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}exp\left(ty\frac{\partial^2}{\partial x^2}\right)\{A_n(x)\}dt (30)$$

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-v}\{A_n(x)\} = \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}{}_HA_n(x,yt)dt. \quad (31)$$

The right hand side of equation (31) represents a new class of special polynomials. This class of special polynomials denoted by ${}_HA_n(x,y;\alpha)$ and it is known as extended Hermite polynomials, we have

$${}_HA_n(x,y;\alpha) = \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}{}_HA_n(x,yt)dt. \quad (32)$$

On using the equations (31) and (32), we get (29).

Theorem 4.*The generalized Hermite based Appell polynomials ${}_HA_n(x,y;\alpha)$ satisfy following generating function*

$$A(u)\frac{exp(xu)}{(\alpha - (D_y^{-1})u^2)^v} = \sum_{n=0}^\infty {}_HA_n(x,y;\alpha)\frac{u^n}{n!}. \quad (33)$$

*Proof.*By multiplying $\frac{u^n}{n!}$ both sides of (32) and summating over n, we get

$${}_HA_n(x,y;\alpha)\frac{u^n}{n!} = \sum_{n=0}^\infty \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}{}_HA_n(x,yt)\frac{u^n}{n!}dt (34)$$

Using equation (15) in above equation, we have

$${}_HA_n(x,y;\alpha)\frac{u^n}{n!} = \frac{exp(xu)}{\Gamma(v)}\int_0^\infty e^{-(\alpha - D_y^{-1}u^2)}t^{v-1}dt. \quad (35)$$

By using equation (19) in right hand side of above equation, we get (33).

Here we notice that for $\alpha = v = 1$ and $y - D_y^{-1}$, the generalized Hermite and generalized Hermite-Appell polynomial reduce to Hermite and Hermite-Appell polynomials respectively.

Next, we derive an explicit summation equation for the extended Hermite-Appell polynomials.

Theorem 5.*Following summation formula for the generalized Hermite polynomials $H_n(x,y;\alpha)$ and Appell polynomials $A_n(y)$ satisfy the generalized Hermite based Appell polynomials*

$${}_{vH}A_n(x,y;\alpha) = \sum_{k=0}^n\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} (-1)^k q^k A_r(q){}_vH_{n-k-r}(x,y;\alpha). \quad (36)$$

*Proof.*By the product of generating function (1) and (26) in the subsequent form

$$A(t)e^{(qt)}(\alpha - (D_y^{-1})t^2))^{-v}exp(xt) = \sum_{n=0}^\infty\sum_{r=0}^\infty A_r(q){}_vH_{n-k-r}(x,y;\alpha)\frac{t^{n+r}}{n!r!}. \quad (37)$$

Now, we can take $n = n - r$ in r.h.s of the above equation and moving the first exponential to the r.h.s, it gives that

$$A(t)\left(\alpha - (D_y^{-1}t^2)\right)^{-v} exp(xt) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(-1)^k q^k A_r(q)_v H_{n-r}(x,y;\alpha)\frac{t^n}{n!},$$

by replacing n by $n-k$

$$A(t)\left(\alpha - (D_y^{-1}t^2)\right)^{-v} exp(xt) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k q^k A_r(q)_v H_{n-k-r}(x,y;\alpha)\frac{t^n}{n!}.$$

In the end, by the use of the generating function (33) in the right hand side of equation (38) and after that equating the coefficients of same powers of t in the resultant equation, we get following the equation.

$${}_{vH}A_n(x,y;\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k q^k A_r(q)_v H_{n-k-r}(x,y;\alpha).$$

*Remark.*By making $A(u) = \left(\frac{u}{e^u-1}\right)$ and $A_n(u) = B_n(u)$ in equations (29)(33), and in explicit summation of Hermite based Appell polynomials, we find that for the generalized Hermite-Appell polynomials ${}_{vH}B_n(x,y,\alpha)$

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-v}\{B_n(x)\} = {}_{vH}B_n(x,y,\alpha), \tag{38}$$

$$\left(\frac{u}{e^u-1}\right)\frac{exp(xu)}{(\alpha - yu^2)^v} = \sum_{n=0}^{\infty} {}_{vH}B_n(x,y,\alpha)\frac{u^n}{n!}, \tag{39}$$

$${}_{vH}B_n(x,y,\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} \times(-1)^k q^k B_r(q)_v H_{n-k-r}(x,y;\alpha). \tag{40}$$

*Remark.*By making $A(u) = \left(\frac{2}{e^u+1}\right)$ and $A_n(u) = E_n(u)$ in equations (29)(33), and in explicit summation of Hermite based Appell polynomials, we find that for the generalized Hermite-Appell polynomials ${}_{vH}E_n(x,y,\alpha)$

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-v}\{E_n(x)\} = {}_{vH}E_n(x,y,\alpha), \tag{41}$$

$$\left(\frac{2}{e^u+1}\right)\frac{exp(xu)}{(\alpha - yu^2)^v} = \sum_{n=0}^{\infty} {}_{vH}E_n(x,y,\alpha)\frac{u^n}{n!}, \tag{42}$$

$${}_{vH}E_n(x,y,\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} \times(-1)^k q^k E_r(q)_v H_{n-k-r}(x,y;\alpha). \tag{43}$$

*Remark.*By making $A(u) = \left(\frac{2u}{e^u+1}\right)$ and $A_n(u) = G_n(u)$ in equations (29)(33),and in explicit summation of Hermite based Appell polynomials we find that for the generalized Hermite-Appell polynomials ${}_{vH}G_n(x,y,\alpha)$.

$$\left(\alpha - y\left(\frac{\partial^2}{\partial x^2}\right)\right)^{-v}\{G_n(x)\} = {}_{vH}G_n(x,y,\alpha), \tag{44}$$

$$\left(\frac{2u}{e^u+1}\right)\frac{e^{(xu)}}{(\alpha - yu^2)^v} = \sum_{n=0}^{\infty} {}_{vH}G_n(x,y,\alpha)\frac{u^n}{n!}, \tag{45}$$

$${}_{vH}G_n(x,y,\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} \times(-1)^k q^k G_r(q)_v H_{n-k-r}(x,y;\alpha). \tag{46}$$

3 Determinant form and recurrence relations

Theorem 6.*Following determinant form for generalized Hermite-Appell polynomials holds true*

$${}_{vH}A_0(x,y;\alpha) = \frac{1}{\beta_0}, \tag{47}$$

$${}_{vH}A_n(x,y;\alpha) = \frac{(-1)^n}{(\beta_0)^{n+1}} \times \begin{vmatrix} 1 & {}_vH_1(x,y;\alpha) & {}_vH_2(x,y;\alpha) & \dots & {}_vH_{n-1}(x,y;\alpha) & {}_vH_n(x,y;\alpha) \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \dots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ 0 & 0 & 0 & \dots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix},$$

where $n = 1,2,3,.....;\beta_0,\beta_1,....,\beta_n$ *and*

$$\beta_n = -\frac{1}{A_0}\left(\sum_{k=0}^{n}\binom{n}{k}A_k\beta_{n-k}\right), n = 1,2,... \tag{48}$$

*Proof.*We recall the following determinant definition for the Appell polynomials [21]

$$A_0(x) = \frac{1}{\beta_0}, \tag{49}$$

$$A_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & y & y^2 & \cdots\cdots & y^{n-1} & y^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots\cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots\cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots\cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 . \end{vmatrix} \tag{50}$$

Taking $n = 0$ in explicit summation formula of generalized Hermite-Appell polynomials, and then using equations (49) and (50), we get assertion (47).

After that expanding determinant (50) with respect to the first row and then operating $\left(\alpha - \left(y\frac{\partial^2}{\partial x^2}\right)\right)$ on each aspects of the resultant equation and using equation (22) and (29), we get

$$_{vH}A_n(x,y;\alpha) = \frac{(-1)^n {}_vH_0(x,y;\alpha)}{(\beta_0)^{n+1}}$$

$$\times \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & . & \\ . & . & . & . & \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 , \end{vmatrix}$$

$$- \frac{(-1)^n {}_vH_1(x,y;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & . & \\ . & . & . & . & \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 , \end{vmatrix}$$

$$+ \frac{(-1)^n {}_vH_2(x,y;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & . & \\ . & . & . & . & \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 , \end{vmatrix} + \ldots$$

$$+ \frac{(-1)^n {}_vH_{n-1}(x,y;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n}{2}\beta_{n-2} \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\beta_1 , \end{vmatrix}$$

$$+ \frac{(-1)^n {}_vH_n(x,y;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\beta_1 . \end{vmatrix}$$

Combining the terms of r.h.s of the above equation, we get the assertion (48).

*Remark.*For the determinant form of generalized Hermite-Bernoulli polynomials we take $\beta_0 = 1$, $\beta_i = \frac{1}{i+1}(i = 1,2,\ldots,n)$ (the determinant form of Appell polynomials reduce to Bernoulli polynomials $B_n(y)$ [20, 21]) in equations (47) and (48).

*Remark.*For the determinant form of generalized Hermite-Euler polynomials we take $\beta_0 = 1$ and $\beta_i = \frac{1}{2}(i = 1,2,\ldots,n)$ (the determinant form of the Appell polynomials $A_n(y)$ reduce to the Euler polynomials $E_n(y)$ [20,21,22]) in equations (47) and (48).

*Remark.*For the determinant form of generalized Hermite-Euler polynomials we take $\beta_0 = 1$ and $\beta_i = \frac{1}{2(i+1)}(i = 1,2,...,n)$ (the determinant form of the Appell polynomials $A_n(y)$ reduce to the Bernoulli polynomials $G_n(y)$ [20,21,22]) in equations (47) and (48).

Then, we determine the recurrence relations for the extended Hermite-Appell polynomials $_HA_{n,v}(x,y;\alpha)$ by their generating function.

Differentiating the generating function, w.r.t x, y, and α we get the following differential recurrence relations of the extended Hermite–Appell polynomials.

$$\frac{\partial}{\partial x} {}_{vH}A_n(x,y;\alpha) = n_{vH}A_n(x,y;\alpha), \tag{51}$$

$$\frac{\partial}{\partial y} {}_{vH}A_n(x,y;\alpha) = vn(n-1)_{v+1H}A_n(x,y;\alpha), \tag{52}$$

$$\frac{\partial}{\partial \alpha} {}_{vH}A_n(x,y;\alpha) = -v_{v+1H}A_n(x,y;\alpha), \tag{53}$$

consequently,

$$\frac{\partial}{\partial y}\left({}_{vH}A_n(x,y;\alpha)\right) = \frac{\partial^3}{\partial x^2 \partial \alpha} {}_{vH}A_n(x,y;\alpha). \tag{54}$$

The consolidated utilization of integral transforms and special polynomials gives a amazing apparatus to manage fractional operators [16]. The generating function, summation formula, and recurrence relations for the extended Hermite–Appell polynomials are determine here. These results may be useful in the investigation of other useful properties of these polynomials and may have applications in different engineering sciences.

In the following segment, we consider the extended types of the Hermite–Bernoulli, Hermite–Euler polynomials as members of the extended Hermite–Appell family.

(o) operating $(\alpha - y\frac{\delta^2}{\delta x^2})^{-v}$ on each aspects of a given relation.
First, we consider the following results for the Appell polynomials $A_n(x)$ [21]:

$$A_n(x) = \frac{1}{\beta_0}\left(x^n - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}A_k(x)\right), \tag{55}$$

$$x^n = \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}A_k(x), \tag{56}$$

$n = 1,2,\ldots$ Performing operation (O) on each sides of the above equations after that by the use of operational definitions (22) and (29), we obtained

$$_{vH}A_n(x,y;\alpha) = \frac{1}{\beta_0}$$
$$\left({}_vH_n(x,y;\alpha) - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k\,vH}A_k(x,y;\alpha)\right). \tag{57}$$

Next, we define the functional equations for Bernoulli polynomials $B_n(x)$ [9,14,18]

$$B_n(x+1)-B_n(x)=nx^{n-1}, \tag{58}$$

$$\sum_{m=0}^{n-1}\binom{n}{m}B_m(x)=nx^{n-1}, \tag{59}$$

$$B_m(mx)=m^{n-1}\sum_{k=0}^{m-1}B_n\left(x+\frac{k}{m}\right), \tag{60}$$

$n=0,1.2,...,m=1,2,3,...$

Again, performing operation (O) on each sides of the above equations and then using operational definitions (22) and (38), the following identities related to generalized Hermite-Bernoulli polynomials are obtained.

$${}_{\nu H}B_n(x+1,y;\alpha)-{}_{\nu H}B_n(x,y;\alpha)=n_\nu H_{n-1}(x,y;\alpha), \tag{61}$$

$$\sum_{m=0}^{n-1}\binom{n}{m}{}_{\nu H}B_m(x,y;\alpha)=n_\nu H_{n-1}(x,y;\alpha), \tag{62}$$

$${}_{\nu H}B_m(mx,my;\alpha)=m^{n-1}\sum_{k=0}^{m-1}{}_{\nu H}B_n\left(x,y+\frac{k}{m};\alpha\right), \tag{63}$$

$n=0,1.2,...,m=1,2,...$

Further, performing operation (O) with use of operational rules (22), (38)and (41) on the following functional equations involving Euler polynomials $E_n(y)$ [9,14,18] and Genocchi polynomials $G_n(y)$ [14,18,23]

$$E_n(x+1)+E_n(x)=2x^n, \tag{64}$$

$$E_n(mx)=m^n\sum_{k=0}^{m-1}(-1)^kE_n\left(x+\frac{k}{m}\right) \tag{65}$$

$$G_{n+1}(x)+G_n(x)=2nx^{n-1}, \tag{66}$$

$n=0,1,2,...;m$ being odd yields the following identities related to the generalized Hermite-Euler polynomials and generalized Hermite-Genocchi polynomials

$${}_{\nu H}E_n(x+1,y;\alpha)+{}_{\nu H}E_n(x,y;\alpha)=2_\nu H_n(x,y;\alpha), \tag{67}$$

$${}_{\nu H}E_n(mx,my;\alpha)=m^n\sum_{k=0}^{m-1}(-1)^k{}_{\nu H}E_n\left(x,y+\frac{k}{m};\alpha\right) \tag{68}$$

$${}_{\nu H}G_{n+1}(x,y;\alpha)+{}_{\nu H}G_n(x)=2_\nu H_{n-1}(x,y;\alpha). \tag{69}$$

$n=0,1,2,..;m$ being odd.

Finally, considering the following connection formulae involving the Bernoulli and Euler polynomials [9,24]:

$$B_n(x)=2^{-n}\sum_{m=0}^{n}\binom{n}{m}B_{n-m}E_m(2x), \tag{70}$$

$$E_n(x)=\frac{2^{n+1}}{n+1}[B_{n+1}\frac{x+1}{2}-B_{n+1}\frac{x}{2}], \tag{71}$$

$$E_n(mx)=-\frac{2^m}{n+1}\sum_{k=0}^{m-1}(-1)^kB_{n+1}(\frac{x+k}{m}), \tag{72}$$

$n=0,1,2,...;m$ being even. which on performing operation (O) after that by the use of operational definitions yields the following connection formulae related to the generalized Hermite-Bernoulli and Hermite-Euler polynomials

$${}_{\nu H}B_n(x,y;\alpha)=2^{-n}\sum_{m=0}^{n}\binom{n}{m}B_{n-m\nu H}E_m(2x,2y;\alpha), \tag{73}$$

$${}_{\nu H}E_n(x,y;\alpha)=\frac{2^{n+1}}{n+1}[{}_{\nu H}E_{n+1}(\frac{x+1}{2},\frac{y}{2};\alpha)-{}_{\nu H}E_{n+1}(\frac{x}{2},\frac{y}{2};\alpha)], \tag{74}$$

$${}_{\nu H}E_n(mx,my;\alpha)=-\frac{2m^n}{n+1}\sum_{k=0}^{m-1}(-1)^k{}_{\nu H}B_{n+1}(\frac{x+k}{m},\frac{y}{m};\alpha), \tag{75}$$

$n=0,1,2,....m$ being even.

4 Conclusion and observation:

We derived generalized Hermite and generalized Hermite-Appell polynomials by combining the operational definitions and integral representations. We also discusses their explicit summation formulae, determinant and recurrence relations for the generalized Hermite-Appell polynomials. For the application purpose, we presented the corresponding results for the generalized Hermite-Bernoulli, generalized Hermite-Euler, and Hermite-Genocchi polynomials.
Recently, some new families of 3-variables related to Hermite polynomials are introduced. This approach can be extended to derive many new vital identities for 3-variables families involving hybrid polynomials.

Acknowledgement

The authors extend their appreciation to the Deanship of Scientific Research at Saudi Electronic University for funding this research work through the project number(8137).

References

[1] P. Appell, Sur une classe de polynômes, *Ann. Sci. Éc. Norm. Supér Appl.*, **(2)9**, 119-144 (1880).
[2] M.X. He and P.E. Ricci, Differential equation of the Appell polynomials via factorization method, *J. Comput. Appl. Math.*, **139(2)**, 231-237 (2002).
[3] A. Pinter and H.M. Srivastava, Addition theorem for the Appell polynomials and the assosiated classes of polynomials expansions, *Aequat. Math.*, **85**, 483-495 (2013).

[4] N. Ahmad, R. Sabri, M.F. Khan, M. Shadab and A. Gupta, Relevance of factorization method to differential and integral equations associated with hybrid class of polynomials, *Fractal and Fract.*, 6(1) (2022). (https://doi.org/10.3390/fractalfract6010005).
[5] G. Dattoli, C. Cesarano and D. Sacchetti, A note on truncated polynomials, *Comput. Appl. Math.*, 134(2-3), 595-605 (2003).
[6] G. Dattoli, M Migliorati, H.M. Srivastava, A class of Bessel summation formulas and associated operational method, *Frac. Calcu .Appl .Anal.*, **7(2)**, 169-176 (2004).
[7] F. Marcellan, S. Jabee and M. Sahdab, Analytic properties of Touchard based hybrid polynomials via operational techniques, *Bull. Malaysian Math. Sci. Soc.*, **44**, 223-242 (2021).
[8] S. Roman, *The umbral calculus*, Springer New York, 2005.
[9] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966.
[10] H.M. Srivastava and C. Vignat, Probabilistic proofs of some relationships between the Bernoulli and Euler polynomials, *European J. Pure Appl. Math.*, (5), 97-107 (2012).
[11] L. Aceto, H.R Maionek and G. Tomaz, A unified matrix approach to the representation of Appell polynomials, *Integral Transf. Spec. Funct.*, **26(6)**, 426-441 (2015).
[12] J.F. Steffensen, The poweroid, an extension of the mathematical notion of power, *Acta Math.*, **73**, 333-366 (1941).
[13] G. Dattoli, Hermite–Bessel and Laguerre–Bessel functions: A by-product of the monomiality principle, in: *Advanced Special Functions and Applications.* Melfi(1999), Proc. Melfi Sch. Adv. Top. Math. Phys. 1, Aracne, Rome , 147-164 (2000).
[14] L.C. Andrews, *Special functions for engineers and Applied mathematicians.*, Macmillan, New York,(1985).
[15] S. Khan, G. Yasmin, R. khan and N.A.M. Hasan, Hermite-based Appell polynomials: properties and applications, *J. Math. Anal. Appl.*, **351**, 756-764 (2009).
[16] G. Dattoli, P.E. Cesarano and C.Vazquez, Special polynomials and fractional calculas, *Math. comput. Modelling* **37**, 729-733, (2003).
[17] S. Jabee, M. Shadab and R.B. Paris, Certain results on Euler-type integrals and their applications, *The Ramanujan J.*, **54**, 245-260 (2021).
[18] H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press–Ellis Horwood Limited–John Wiley and Sons, New York–Chichester–Brisbane–Toronto, 1984.
[19] H.M. Srivastava, M.A. Ozarslan and B. Yilmaz, Some families of differential equation associated with the Hermite based Appell polynomials and other classes of the Hermite-based polynomials, *Filomat.*, **28(4)**, 695–708 (2014).
[20] F.A. Costabile, F. Dell'Accio and M.I. Gualtier, A new approach to Bernoulli polynomials, *Rend. Mat. Appl.*, **26(1)**, 1-12 (2006).
[21] F.A. Costabile, and E. Longo, A determinant approach to Appell polynomials, *J. Comput. Appl. Math.*, **235(5)**, 1528-1542 (2010).
[22] S. Khan and M. Riyasat, Differential and integral equation for 2-iterated Appell polynomials, *J. Comput. Appl. Math. Anal. Appl.* **387**, 116-132 (2012).
[23] G. Dattoli, M. Migliorati and H.M. Srivastva, Sheffer polynomials, monomiality principle, algebric method and theory of classical polynomials, *Math. Comput. Modelling*, **45**, 1033-1040 (2007).
[24] B. Yılmaz and M.A. Özarslan, Differential equations for the extended 2D Bernoulli and Euler polynomials, *Adv. Difference Equ.*, **107**, p. 1-16 (2013).

Anjali Goswami is presently working as an Associate Professor in department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh, Saudi Arabia. She received her Ph.D. in year 2007 from India. Previously she worked as an Associate Professor, HOD at the Department of Mathematics, Jagannath Institute of Engineering and Technology, Jaipur, Rajasthan, India. She has more than 20-years of teaching and Research experience. She has published two books and more than 30 research papers in various referred reputed journals.

Mohammad Faisal Khan is presently working as an Associate Professor in Departments of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh, Saudi Arabia. He received his Ph.D. in 2012 from India. Previously he worked as an Assistant Professor at the Department of Mathematics, Aligarh Muslim University, Aligarh, India. He has more than 10-year experience in teaching and research. He has published two books and more than 50 research papers in various refereed reputed journals.

Mohammad Shadab is presently working as an Assistant Professor in Departments of Mathematics, School of Basic and Applied Sciences, Lingaya's Vidyapeeth (Deemed University), Haryana, India. He received his Ph.D. in 2018 from Jamia Millia Islamia (A Central University), New Delhi, India. He has published more than 30 research papers in the journals of international repute. He has guided 2 Ph.D., 3 M.Phil., many graduate and undergraduate students. He has more than four years experience of research and teaching.