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The s- Generalized Composition Operators from $\mathscr{B}_g^{(m,n)} \to Q_P$ Spaces

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Abstract: In this paper we introduce a new weighted composition operator, called s- generalized composition operator. The boundedness and compactness of the s- generalized composition operator from $\mathscr{B}_g^{(m,n)} \to Q_P$ Spaces are investigated in this paper.

Keywords: The generalized composition operator, $\mathscr{B}_g^{(m,n)}$ Space, compactness, Q_p space.

1 Introduction

In this paper, χ denotes a nonconstant analytic self-map of the unit disk $\mathbb{D}$ in the complex plane C. The purpose of this paper is to study the boundedness and compactness of the s- generalized composition operator from $\mathscr{B}_g^{(m,n)} \to Q_P$ Spaces.

Let $\mathbb{D} = \{\Omega \in \mathbb{C} : |\Omega| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$ and denote the class of all holomorphic functions that belonging to $\mathbb{D}$.

Following [1], for each $a \in \mathbb{D}$, $\Psi_a : \mathbb{D} \to \mathbb{D}$ denotes the Möbius transformations defined by

$$\Psi_a(\Omega) := \frac{a-\Omega}{1-\bar{a}\Omega}, \qquad \text{for every } \Omega \in \mathbb{D}.$$

Green's function of $\mathbb{D}$ with logarithmic singularity at a is defined as follows

$$g(\Omega,a) := \log\left|\frac{1-\bar{a}\Omega}{\Omega-a}\right| = \log\frac{1}{|\Psi_a(\Omega)|}.$$

For each $a \in \mathbb{D}$, the pseudo-hyperbolic disc ,$D(a,r)$, is defined by the following

$$D(a,r) := \{\Omega \in \mathbb{D} : |\Psi_a(\Omega)| < r\},$$

where $0 < r < 1$, (see [2]).

For any analytic self−mapping χ of $\mathbb{D}$. The symbol χ induces a linear composition operator $C_\chi(\rho) := \rho \circ \chi$ from $H(\mathbb{D})$ into itself (see[3]).

The composition operator C_χ is well studied for many years (see[4,5,6,7,8] and others), We refer to (see[9,10, 11]) which are excellent sources for the development of the theory of composition operators in function spaces.

Now, we define a new weighted composition operator called s- generalized composition operators $C_\chi^{h,s}$

Definition 1.*From the recent research on the operator theory of complex-type function spaces, we will introduced the s- generalized composition operators $C_\chi^{h,s}$ which we will used in the current paper as follow :*

$$\left(C_\chi^{h,s}\rho\right)(\Omega) = \int_0^\Omega \rho'(\chi(\xi))h^{(s-1)}(\xi)d\xi, \tag{1}$$

where, $h^{(s-1)}(\Omega) = \frac{d^{s-1}h(\Omega)}{d\Omega^{s-1}}$, with "$s-1$" order derivatives, $s \in \mathbb{N}$.

The papers [12,13,14,15] are defined the analytic Bloch-type space as follow:

$$\mathscr{B} = \{\rho \in H(\mathbb{D}) : \sup_{a\in\mathbb{D}}(1-|\Omega|^2)|\rho'(\Omega)|\}.$$

The analytic little Bloch-type space $\mathscr{B}_0$ is symbolized by $\mathscr{B}_0$, for which

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$$\mathscr{B}_0 = \{\rho \in H(\mathbb{D}) : \lim_{|a|\to 1^-} (1-|\Omega|^2)|\rho'(\Omega)| = 0\}.$$

For numerous global studies on Bloch-type spaces, we refer to [16,17,18,19,20,21,22] and others.
The Q_p spaces has been defined in [23] as follows

Definition 2. *Let $0 < p < \infty$ and ρ is analytic functions on $\mathbb{D}$. Then*

$$Q_p := \{\rho \in H(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\rho'(\Omega)|^2 g^p(\Omega,a)dA(\Omega) < \infty\},$$

which is equivalent to the definition

$$Q_p := \{\rho \in H(\mathbb{D}) :$$

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\rho'(\Omega)|^2 (1-|\chi_a(\Omega)|)^p dA(\Omega) < \infty\},$$

and the little Q_p is given as follows

$$Q_{p,0} := \{\rho \in H(\mathbb{D}) :$$

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |\rho'(\Omega)|^2 g^p(\Omega,\Omega_0)dA(\Omega) = 0\},$$

which is equivalent to the definition

$$Q_{p,0} := \{\rho \in H(\mathbb{D}) :$$

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |\rho'(\Omega)|^2 (1-|\chi_a(\Omega)|)^p dA(\Omega) = 0\}.$$

For intensive research on analytic $Q_p$spaces, we may refer to [24,25,26,27] and others. Also, there are certain specific generalizations of these weighted classes of analytic functions in C^n [28,29]. On the other hand, there are some interesting extensions using quaternion-valued functions setting Hereafter, we set

$$\Psi_{\Omega_0}(\Omega) := \frac{\Omega_0 - \Omega}{1-\bar{\Omega}_0\Omega}, \qquad \text{for every } \Omega \neq \Omega_0.$$

and set

$$\Psi_{\Omega_0}(\Omega) = C < 1, \text{ when } \Omega = \Omega_0$$

The modified Green's function is introduced by

$$g(\Omega,\Omega_0) := \ln\left|\frac{1-\bar{\Omega}_0\Omega}{\Omega_0 - \Omega}\right| = \ln\frac{1}{|\Psi_{\Omega_0}(\Omega)|}.$$

Motivated by the modified Green's function, the following definitions can be presented.

Definition 3. *Let $0 < m < \infty$ and $0 < n < \infty$. For the function $\rho \in H(D)$, we define the analytic g-Bloch space $\mathscr{B}_g^{(m,n)}$ as follows:*

$$\mathscr{B}_g^{(m,n)} = \{\rho : \sup_{\Omega,\Omega_0\in\mathbb{D}} \frac{(1-|\Omega|^2)^n}{g^m(\Omega,\Omega_0)}|\rho'| < \infty\}.$$

Furthermore, assume that

$$\mathscr{B}_g^{(m,n)}(\rho) = \sup_{\Omega,\Omega_0\in\mathbb{D}} \frac{(1-|\Omega|^2)^n}{g^m(\Omega,\Omega_0)}|\rho'|.$$

The paper is organized as follows, the second section, in which we present some of the requirements required in the paper. In the third section, we show boundedness. In the fourth section, we present compactness. In the fifth section, we display the part of the application. In the end, we present the conclusion of the work and our findings.

2 Auxiliary results

In this section, we state several results, which are used in the main result proofs. Now, we will introduce the definition of boundedness and compactness of the operator $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$.

Definition 4. *The operator $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$ is said to be bounded, if there is a positive constant C such that $||C_\chi^{h,s}\rho||_{Q_p} \le C||\mathscr{B}_g^{(m,n)}||$ for all $\rho \in \mathscr{B}_g^{(m,n)}$.*

Definition 5. *The operator $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$ is said to be compact, if it maps any unit disc in $\mathscr{B}_g^{(m,n)}$ onto a pre-compact set in Q_p.*

we list up following lemmas which are needed to prove our main results.

Lemma 1. *Assume that $\rho_1, \rho_2 \in \mathscr{B}_g^{(m,n)}$ and $n,m \in \mathbb{N}$, such that*

$$\Big[\,|\,\rho_1'(\Omega)\,| + |\,\rho_2'(\Omega)\,|\,\Big] \leqslant \frac{Cg^m(\Omega,\Omega_0)}{(1-|\,\Omega\,|^2)^n}.$$

The proof is similar to that of the Lemma 1 in ([30]) , so we omit it here.

Lemma 2. *Assume that $\rho \in \mathscr{B}$. Then for each $n,m \in \mathbb{N}$.*

$$\|\,\rho\,\|_{\mathscr{B}_g^{(m,n)}} = \rho(0) + \sup_{\Omega\in\mathbb{D}} \frac{\rho'(\chi(\Omega))(1-|\,\Omega\,|^2)^n}{g^m(\Omega,\Omega_0)}.$$

The proof is similar to that of the Lemma 1 in [6], so we omit it here.

3 The boundedness of the operator $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$

In this section, we characterize the operators $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$. Moreover, we give the conditions which prove the boundedness of the operators C_χ. Now we will introduce the main results of boundedness .

Theorem 1.*Suppose $g \in H(\mathbb{D})$ and χ denotes an analytic self-map of $\mathbb{D}$ then the $C_\chi^{h,s}\mathscr{B}_g^{(m,n)} \to Q_p$ is said to be bounded if and only if*

$$\sup_{\Omega\in\mathbb{D}} \frac{|h^{(s-1)}(\Omega)|^2 g^{(2m+p)}(\Omega,\Omega_0)}{(1-|\Omega|^2)^{2n}} dA(\Omega) < \infty. \quad (2)$$

*Proof.*For $\rho \in \mathscr{B}_g^{(m,n)}$ with $\|\rho\|_{\mathscr{B}_g^{(m,n)}} \leq 1$ then in view of Theorem 4.8, we obtain

$$\begin{aligned}
&\| C_\chi^{h,s}\rho(\Omega)\|_{Q_p} \\
&= \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} |\rho'(\chi(\Omega))h^{(s-1)}(\Omega)|^2 g^p(\Omega,\Omega_0)dA(\Omega) \\
&\leq \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} |\rho'(\chi(\Omega))|^2 |h^{(s-1)}(\Omega)|^2 g^p(\Omega,\Omega_0)dA(\Omega) \\
&\leq \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} \Bigg((|\rho'(\chi(\Omega))|^2 |h^{(s-1)}(\Omega)|^2) \\
&\quad \frac{(1-|\Omega|^2)^{2n} g^{p+2m}(\Omega,\Omega_0)}{(1-|\Omega|^2)^{2n} g^{2m}(\Omega,\Omega_0)} \Bigg) dA(\Omega) \\
&\leq \|\rho(\chi(\Omega))\|^2_{\mathscr{B}_g^{(m,n)}} \\
&\quad \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} \frac{|h^{(s-1)}(\Omega)|^2 g^{p+2m}(\Omega,\Omega_0)}{(1-|\Omega|^2)^{2n}} dA(\Omega) \\
&< \infty. \qquad (3)
\end{aligned}$$

For the other direction we use the fact that for each function $\rho \in \mathscr{B}_g^{(m,n)}$, the analytic function $C_\chi^{h,s}\rho(\Omega) \in Q_p$. Then using the functions of Lemma 1 we get the following:

$$\begin{aligned}
&4\ \{\|C_\chi^{h,s}\rho_1(\Omega)\|_{Q_p} + \|C_\chi^{h,s}\rho_2(\Omega)\|_{Q_p}\} \\
&= 4 \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} \Bigg[|\rho_1'(\chi(\Omega))h^{(s-1)}(\Omega)|^2 \\
&\quad + |\rho_2'(\chi(\Omega))h^{(s-1)}(\Omega)|^2 \Bigg] g^p(\Omega,\Omega_0)dA(\Omega) \\
&= 4 \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} \Bigg[|(\rho_1'(\chi(\Omega))h^{(s-1)}(\Omega))|^2 \\
&\quad + |(\rho_2'(\chi(\Omega))h^{(s-1)}(\Omega))|^2 \Bigg] g^p(\Omega,\Omega_0)dA(\Omega) \\
&= \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} \Bigg[|\rho_1'(\chi(\Omega))|^2 + |\rho_2'(\chi(\Omega))|^2 \Bigg] \\
&\quad |h^{(s-1)}(\Omega)|^2 g^p(\Omega,\Omega_0)dA(\Omega) \\
&\geqslant 4 \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} \Bigg[|\rho_1'(\chi(\Omega))| + |\rho_2'(\chi(\Omega))| \Bigg]^2 \\
&\quad \frac{|h^{(s-1)}(\Omega)|^2 (1-|\Omega|^2)^{2n} g^{p+2m}(\Omega,\Omega_0)}{(1-|\Omega|^2)^{2n} g^{2m}(\Omega,\Omega_0)} dA(\Omega) \\
&\geqslant C \sup_{\Omega\in\mathbb{D}} \int_{\mathbb{D}} \frac{|h^{(s-1)}(\Omega)|^2 g^{p+2m}(\Omega,\Omega_0)}{(1-|\Omega|^2)^{2n}} dA(\Omega). \qquad (4)
\end{aligned}$$

Hence $C_\chi^{h,s}$ is bounded, then (2) holds. The proof is completed.

The composition operator $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$ is compact if and only if for every sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset Q_p$ bounded in Q_p norm and $\rho_n \to 0, n \to \infty$, uniformly on compact subset of the unit disk (where $\mathbb{N}$ be the set of all natural numbers), hence

$$\|C_\chi^{h,s}(\rho_n)\|_{Q_p} \to 0, n \to \infty.$$

Now, we describe compactness in the following result.

4 The compactness of the operator $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$

Lemma 3.*Assume that χ is a analytic self-map of $\mathbb{D}$ and $h \in H(\mathbb{D})$. Then $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$ is compact if and only if $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$ is bounded and for any bounded sequence $\{\rho_i\}_{i\in N} \in \mathscr{B}_g^{(m,n)}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ we have $\lim_{i\to\infty} ||C_\chi^{h,s}\rho_i||_{Q_p} = 0$.*

Theorem 2.*If χ is an analytic self-map of the unit disk, then the induced composition operator $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$ is compact if and only if $\chi \in Q_p$ and*

$$\lim_{r\to 1} \sup_{\Omega\in\mathbb{D}} \frac{|h^{(s-1)}(\Omega)|^2 g^{(2m+p)}(\Omega,\Omega_0)}{(1-|\Omega|^2)^{2n}} dA(\Omega) = 0. \quad (5)$$

*Proof.*Let $C_\chi^{h,s} : \mathscr{B}_g^{(m,n)} \to Q_p$ be compact. This means that $\chi \in Q_p$.
Let $U_r^1 = \{\Omega : |\chi(\Omega)| > r,\ \ r \in (0,1)\}$,
and $U_r^2 = \{\Omega : |\chi(\Omega)| \leqslant r,\ \ r \in (0,1)\}$.
Let $\rho_n(\Omega) = \frac{\Omega^n}{n}$, since $\|\rho\|_{\mathscr{B}_g^{(m,n)}} \leqslant M$ and $\rho_n(\Omega) \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then

$$\|C_\chi^{h,s}(\rho_n)\|_{Q_p} \to 0, n \to \infty.$$

This means that for each $r \in (0,1)$ and for all $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that if $n \geqslant N$, then

$$\frac{N^p}{r^{p(1-N)}} \sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |h^{(s-1)}(\Omega)|^2 (1-|\Omega|^2)^{2n} g^p(\Omega,\Omega_0)dA(\Omega) < \varepsilon,$$

if we choose r so that $\frac{N^p}{r^{p(1-N)}} = 1$ then

$$\sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |h^{(s-1)}(\Omega)|^2 (1-|\Omega|^2)^{2n} g^p(\Omega,\Omega_0)dA(\Omega) < \varepsilon. \qquad (6)$$

Let now ρ with $\|\rho\|_{\mathscr{B}_g^{(m,n)}} \leqslant 1$. We consider the functions $\rho_t(\Omega) = \rho(t\Omega),\ \ t \in (0,1)$. Then $\rho_t \to \rho$ uniformly on compact subset of the unit disk as $t \to 1$ and the family (ρ_t) is bounded on $B_g^{(m,n)}$, thus

$$\| ((\rho_t'(\chi(\Omega))h^{(s-1)}(\Omega))^2 - (\rho'(\chi(\Omega))h^{(s-1)}(\Omega))^2 \| \to 0.$$

Due to compactness of C_χ we get that, for $\varepsilon > 0$ there is a $t \in (0,1)$ such that

$$\sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho_t'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega) < \varepsilon,$$

where

$$(\rho_t'(\chi(\Omega))h^{(s-1)}(\Omega))^2 = (\rho'(\chi(\Omega))h^{(s-1)}(\Omega))^2 - (\rho_t'(\chi(\Omega))h^{(s-1)}(\Omega))^2.$$

Thus, if we fix t, then

$$\begin{aligned}
&\sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega)\\
&= \sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho'(\chi(\Omega))\ |^2 |\ h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega)\\
&\le 4 \sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho_t'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega)\\
&+ 4 \sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho_t'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega)\\
&= 4\varepsilon + 4 \sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho_t'(\chi(\Omega))\ |^2 |\ h^{(s-1)}(\Omega)\ |^2\\
&\quad g^p(\Omega,\Omega_0)dA(\Omega)\\
&\le 4\varepsilon + 4 \|\ \rho_t'(\chi(\Omega))\ \|_\infty^2 \sup_{\Omega\in\mathbb{D}}\\
&\quad \int_{U_r^1} |\ h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega)\\
&\le 4\varepsilon + 4 \|\ \rho_t'(\chi(\Omega))\ \|_\infty^2, \qquad (7)
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega)\\
&\le \varepsilon + 2 \|\ \rho_t'(\chi(\Omega))\ \|_\infty^2\ . \qquad (8)
\end{aligned}$$

where we have used (6). On the other hand, for each $\|\ \rho\ \|_{B_g^{(m,n)}}$ and $\varepsilon > 0$, there exists a δ depending on ρ, ε, such that for $r \in [\delta, 1)$,

$$\sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega) < \varepsilon. \qquad (9)$$

Since $C_\chi^{h,s}$ is compact, then it maps the unit ball of $B_g^{(m,n)}$ to a relatively compact subset of Q_p. Thus for each $\varepsilon > 0$ there exists a finite collection of functions $\rho_1, \rho_2, ..., \rho_n$ in the unit ball of $B_g^{(m,n)}$ such that for each $\|\ \rho\ \|_{B_g^{(m,n)}} \le 1$, there is $k \in 1,2,3,...,n$ such that

$$\sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho_k'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega) < \varepsilon,$$

where

$$(\rho_k'(\chi(\Omega))h^{(s-1)}(\Omega))^2 = (\rho'(\chi(\Omega))h^{(s-1)}(\Omega))^2 - (\rho_k'(\chi(\Omega))h^{(s-1)}(\Omega))^2.$$

Using also (9), we get for $\delta = max_{1\le k\le n}\delta(\rho_k, \varepsilon)$ and $r \in [\delta, 1)$, that

$$\sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho_k'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega) < \varepsilon.$$

Hence for any $\rho, \|\ \rho\ \|_{B_g^{(m,n)}} \le 1$, combining the two relations as above we get that

$$\sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega) < 2\varepsilon.$$

Therefore, we get that (5) holds. For the sufficiency we use that $\chi \in Q_p$ and (5) holds. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of functions in the unit ball of $B_g^{(m,n)}$, such that $\rho_n \to 0$ as $n \to \infty$, uniformly on the compact subsets of the unit disk. Let also $r \in (0,1)$. Then

$$\begin{aligned}
\|\ C_\chi^{h,s}\rho_n(\Omega)\ \|_{Q_p} &\le 2\ |\ \rho_n(\chi(0))\ |\\
&+ 2 \sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho_n'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\\
&\quad g^p(\Omega,\Omega_0)dA(\Omega)\\
&+ 2 \sup_{\Omega\in\mathbb{D}} \int_{U_r^2} |\ \rho_n'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\\
&\quad g^p(\Omega,\Omega_0)dA(\Omega)\\
&= 2(I_1 + I_2 + I_3). \qquad (10)
\end{aligned}$$

Since $\rho_n \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then $I_1 = |\rho_n(\chi(0))|$ goes to zero as $n \to \infty$ and for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for each $n > N$,

$$\begin{aligned}
I_2 &= \sup_{\Omega\in\mathbb{D}} \int_{U_r^2} |\ \rho_n'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega)\\
&\le \varepsilon \|\ \chi\ \|_{Q_p}\ . \qquad (11)
\end{aligned}$$

We also observe that

$$\begin{aligned}
I_3 &= \sup_{\Omega\in\mathbb{D}} \int_{U_r^1} |\ \rho_n'(\chi(\Omega))h^{(s-1)}(\Omega)\ |^2\ g^p(\Omega,\Omega_0)dA(\Omega)\\
&\le \|\ \rho_n(\chi(\Omega))\ \|_{B_g^{(m,n)}}\\
&\quad \sup_{\Omega\in\mathbb{D}} \int_{U_r^1} \frac{|\ h^{(s-1)}(\Omega)\ |^2\ g^{(2m+p)}(\Omega,\Omega_0)}{(1-|\ \Omega\ |^2)^{2n}} dA(\Omega). \qquad (12)
\end{aligned}$$

Under the assumption that (5) holds, then for every $n > N$ and for every $\varepsilon > 0$ there exists r_1 such that for every $r > r_1$, $I_3 < \varepsilon$. Thus if $\chi \in Q_p$ we ge

$$\begin{aligned}
\|\ C_\chi^{h,s}\rho_n(\Omega)\ \|_{Q_p} &\le 2(0 + \varepsilon \|\ \chi\ \|_{Q_p} + \varepsilon)\\
&\le C\varepsilon. \qquad (13)
\end{aligned}$$

Combining the above, we get that

$$\|\ C_\chi^{h,s}(\rho_n)\ \|_{Q_p} \to 0, n \to \infty,$$

which proves compactness. The proof of our theorem is therefore established.

5 Applications

Different branches of mathematics, including dynamical systems, semigroup theory, isometries, and quantum physics, have been actively embracing operator theory on various spaces of analytic functions. To achieve comprehensive and original characterizations of many classes of functions, our findings in this study can be generalized and applied to some analytic and hyperbolic classes.

6 Conclusion

In this paper, we investigate the boundedness and compactness of s- generalized composition operator from $\mathscr{B}_g^{(m,n)} \to Q_P$ Spaces on unit disk. Also we obtained necessary conditions and sufficient conditions for from $\mathscr{B}_g^{(m,n)} \to Q_P$ Spaces.

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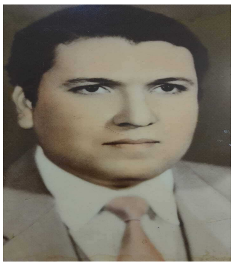

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