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Approximating Functions by Fractional Lagrange Polynomials

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Abstract: Approximating functions by matching their values at given points is a key concept in numerical analysis and various fields of mathematics. This technique is often used in interpolation and curve fitting. In this paper, we approximate a function by a new function which can be considered as a generalization of Lagrange and Hermite polynomials. The interpolation error is derived based on the conformable derivative.

Keywords: Conformable derivative, Lagrange and Hermite polynomials.

1 Introduction

The concept of the fractional derivative was inspired by a question posed by L'Hospital in his letter to Leibniz in 1695, [1]. However, intensive studies on fractional calculus were conducted in the last and present centuries in an effort to provide an answer to this question. Several mathematicians, including Liouville, Riemann, Weyl, Fourier, Abel, Leibniz, Grunwald, and Letnikov, made significant contributions to the theory of fractional calculus. For details on the Grunwald-Letnikov, Riemann-Liouville, Caputo, and other definitions, as well as their characteristics, we refer the reader to A. Kilbas, K.S. Miller, and I. Podlubny, [2,3,4]. Calculus provides various tools for studying and understanding the behavior of a significant class of functions, namely continuous and differentiable functions. This creates a challenge in real applications where functions model the relationships between quantities, and the only information available about these functions comes from a set of discrete data points obtained from measurements. The challenge of constructing a continuous function from this data is known as data fitting. The aim of interpolation, regarded as a specific case of data fitting, is to find a linear combination of known functions that fits a set of data exactly rather than approximately. The use of polynomials or other functions for interpolation was first introduced by J. Wallis in 1665, see [5]. Lagrange polynomials are employed for polynomial interpolation based on a given set of distinct points t_i and values y_i. In fact, the Lagrange polynomial is the polynomial of the lowest degree that takes the corresponding value y_i at each point t_i, see [6].

Similar to Lagrange polynomial interpolation, the primary objective of this paper is to utilize the theory of conformable fractional derivatives to interpolate a nonnegative set of data by constructing a function that we refer to as the Fractional Lagrange polynomial.

2 Conformable Fractional Derivative

Most definitions of fractional derivatives rely on integral forms, which introduce non-local behaviors, leading to a variety of interesting applications. However all the existing fractional derivatives have some kind of inconsistency, for example most of them do not adhere to the familiar product, quotient, and chain rules for the derivatives of two functions, and most of them, except for the Caputo derivative, do not satisfy the condition that the derivative of a constant function is zero.

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To avoid some of these difficulties, recently new definitions take the advantages from the limit form as used in the regular derivatives have been developed. In this paper, we use the following definition of fractional derivative of order β:

$$D_\beta(\phi)(t) = \lim_{\varepsilon \to 0} \frac{\phi(t + \varepsilon t^{1-\beta}) - \phi(t)}{\varepsilon}, \quad D_\beta(\phi)(t) = \lim_{t \to 0^+} D_\beta(t). \tag{1}$$

where $t > 0$ and $\beta \in (0,1)$. This definition was first introduced by Khalil and colleagues [7]. For the sake of brevity, we will refer to the derivative mentioned above simply as the conformable derivative. A function ϕ is considered β-differentiable if the conformable fractional derivative of ϕ of order β exists.

Abdeljawad [8] utilized this new definition of the fractional derivative to establish definitions for left and right conformable fractional derivatives, provide Taylor power series representations, develop Laplace transformations of certain functions, formulate fractional integration by parts, and introduce the chain rule and Gronwall inequality.

The authors in [7] demonstrated that the β-fractional derivatives satisfy the product and quotient rules, and they proved several results similar to the Mean Value Theorem and Rolle's Theorem. However, it unfortunately does not adhere to the natural chain rule.

In the sequel, we present some of the results in [7,8,9,10,11,12,13] that we used in our work.

Theorem 1. *Let ϕ be a real valued function with domain $[0,\infty)$ such that ϕ is s β-differentiable at a point $t_0 > 0$, $\beta \in (0,1]$, then ϕ is continuous at t_0.*

Theorem 2. *Let $\beta \in (0,1]$ and ϕ and ψ be $\beta-$differentiable functions at a point $z > 0$. Then,*

(1) $D_\beta(\frac{1}{\beta} z^\beta) = 1$.

(2) $D_\beta(z^n) = n z^{n-\beta}$.

(3) $D_\beta[c_1\phi + c_2\psi] = c_1 D_\beta(\phi) + c_2 D_\beta(\psi)$, *for all* $c_1, c_2 \in R$.

(4) $D_\beta(C) = 0$, *for all constant* C.

(5) $D_\beta(\phi\psi) = \psi D_\beta(\phi) + \phi D_\beta(\psi)$.

(6) $D_\beta(\frac{\phi}{\psi}) = \dfrac{\psi D_\beta(\phi) - \phi D_\beta(\psi)}{\psi^2}$.

(7) *If, in addition, ϕ is differentiable, then* $D_\beta(\phi)(z) = z^{1-\beta} \frac{d\phi(z)}{dz}$.

(8) $\lim_{\beta \to 1} D_\beta(\phi)(z) = \phi'(z)$.

Remark. It is evident from Theorem 2.2, part (7), that the conformable derivative for differentiable functions is just a straightforward change of variables. This type of variable change is encountered in many physical phenomena (see, e.g., [14,15,16,17,18]). For more details, refer to [10].

Theorem 3. *Let $\beta \in (0,1]$. If ψ is an $\beta-$differentiable function at a point $t > 0$ and ϕ is a differentiable function at $\psi(t)$ then $\phi \circ \psi$ is $\beta-$differentiable at t and*

$$D_\beta(\phi \circ \psi)(t) = \phi'(\psi(t)) D_\beta(\phi)(t),$$

The following theorem can be considered as a generalization of Rolle's Theorem.

Theorem 4. *Let ϕ be a function that is $m-1$ times continuously $\beta-$differentiable on the interval $[c,d]$ and the mth derivative exists on the interval (c,d), and there are m intervals given by $t_1 < u_1 \le t_2 < u_2 \le \ldots \le t_n < u_n$ in $[a,b]$ such that $\phi(t_k) = \phi(u_k)$ for every k from 1 to m. Then there is a number ς in (c,d) such that the mth $\beta-$derivative of ϕ at ς is zero.*

3 Fractional Lagrange polynomial

For a given set of $n+1$ non-negative distinct numbers, $t_0, t_1, \ldots, t_n$ suppose that values of a function ϕ is known at these numbers, then the Fractional Lagrange polynomial P^β is defined to be a function of the form

$$P_m^\beta(t) = a_m t^{m\beta} + a_{m-1} t^{(m-1)\beta} + \cdots + a_0 \tag{2}$$

such that m is minimum and $P_n^\beta(t_j) = \phi(t_j)$ for all $j = 0, 1, \ldots, n$. The number m will be called the degree of P_n^β. The following theorem describes how we find P_n^β and what m should be.

Theorem 5. *If the set $\{t_0, t_1, ..., t_n\}$ consists of nonnegative constants and suppose $\phi(t_j)$ is known for all $j = 1, 2, \ldots, n$. Then a unique fractional polynomial P_n^β of degree less than $n+1$ exists with $P_n^\beta(t_j) = \phi(t_j)$ for all $j = 0, 1, \ldots, n$. This fractional polynomial can be computed as follows*

$$P_n^\beta(t) = \phi(t_0)L_{n,0}^\beta(t) + \phi(t_1)L_{n,1}^\beta(t) + \cdots + \phi(t_n)L_{n,n}^\beta(t) = \sum_{k=0}^{n} \phi(t_k)L_{n,k}^\beta(t)$$

where

$$L_{n,k}^\beta(t) = \prod_{j=0, j\neq k}^{n} \frac{(t^\beta - t_j^\beta)}{(t_k^\beta - t_j^\beta)}.$$

Moreover, if ϕ is $n+1$ times continuously $\beta-$differentiable on the closed interval $[0,a]$ and $t_0, t_1, \ldots, t_n$ are distinct numbers in $[0,a]$, then to each t in $[0,a]$ there corresponds a point ζ_x in $(0,a)$ such that

$$\phi(t) - P_n^\beta(t) = \frac{D_\beta^{(n+1)}\phi(\zeta_t)}{\beta^{n+1}(n+1)!} \prod_{j=0}^{n} (t^\beta - t_j^\beta). \tag{3}$$

Proof. From the definition of $L_{n,k}^\beta(t)$ we have $L_{n,k}^\beta(t_j) = \delta_{kj}$, where δ_{kj} is the Kronecker delta function defined by $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ if $k \neq j$. Thus $P_n^\beta(t_j) = \phi(t_j)$. If $x = x_j$, for some $j \in \{0, 1, \ldots, n\}$, then equation (3) is true since both sides of equation (3) reduce to 0. So, let t be any point other than a node. Define

$$w(t) = \prod_{j=0}^{n} (t^\beta - t_j^\beta)$$

and

$$\varphi(\tau) = \phi(\tau) - P_n^\beta(\tau) - \frac{\phi(t) - P_n^\beta(t)}{w(t)} w(\tau).$$

Since φ is $n+1$ times continuously $\beta-$differentiable on the interval $[0,a]$ and vanishes at the $n+2$ points $t, t_0, t_1, \ldots, t_n$, then by Theorem 4, $D_\beta^{(n+1)}\varphi$ has at least one zero, say ζ_t, in $(0,a)$. Now

$$D_\beta^{(n+1)}\varphi(\tau) = D_\beta^{(n+1)}\phi(\tau) - D_\beta^{(n+1)}P_n^\beta(\tau) - \frac{f(t) - P_n^\beta(t)}{w(t)} D_\beta^{(n+1)}w(\tau).$$

But $D_\beta^{(n+1)}P_n^\beta(\tau) = 0$ and $D_\beta^{(n+1)}w(t)(\tau) = \beta^{n+1}(n+1)!$, so

$$D_\beta^{(n+1)}\varphi(\zeta_t) = D_\beta^{(n+1)}\phi(\zeta_t) - \frac{\phi(t) - P_n^\beta(\tau)}{w(t)} \beta^{n+1}(n+1)! = 0.$$

Solve for $\phi(t)$ to obtain

$$\phi(t) - P_n^\beta(t) = \frac{D_\beta^{(n+1)}\phi(\zeta_t)}{\beta^{n+1}(n+1)!} \prod_{j=0}^{n} (t^\beta - t_j^\beta).$$

4 Divided Difference

There is another way of computing the Lagrange interpolating polynomial. Here we assume

$$P_n^\beta(t) = a_0 + \sum_{k=1}^{n} a_k (t^\beta - t_0^\beta) \ldots (t^\beta - t_{k-1}^\beta) \tag{4}$$

Then, one can notice that $a_0 = \phi(t_0), a_1 = \frac{\phi(t_1) - \phi(t_0)}{t_1^\beta - t_0^\beta}$. Evidently, a_k is a linear combination of $\phi(t_0), \phi(t_1), \ldots, \phi(t_k)$ with coefficients that depend on $t_0^\beta, t_1^\beta, \ldots, t_k^\beta$. Therefore, we introduce the notation

$$a_k = \phi^\beta[t_0, t_1, \ldots, t_k], \qquad k = 0, 1, 2, \ldots,$$

for this linear combination, and call the right-hand side the nth fractional divided difference of f relative to the nodes $t_0, t_1, \dots, t_n$. To compute $\phi^\beta[t_0, t_1, \dots, t_k]$, we have the following theorem.

Theorem 6. *If $t_0, t_1, ..., t_n$ are $n+1$ nonnegative distinct numbers and ϕ is a function whose values are given at these numbers, then*

$$\phi^\beta[t_0, t_1, \dots, t_n] = \frac{\phi^\beta[t_1, t_2, \dots, t_n] - \phi^\beta[t_0, t_1, \dots, t_{n-1}]}{t_n^\beta - t_0^\beta}. \tag{5}$$

and if ϕ is $n+1$ times continuously $\beta-$ differentiable on the closed interval $[0,a]$ and $t_0, t_1, \dots, t_n$ are distinct numbers in $[0,a]$, then a number ζ exists in $(0,\infty)$ with $\phi^\beta[t_0, t_1, \dots, t_n] = \frac{D_\beta^{(n)}\phi(\zeta_t)}{\beta^{n+1} n!}$.

Proof. It is clear that $\phi^\beta[t_0, t_1, \dots, t_k]$ is the leading coefficient of $P_n^\beta(t)$ that is given by equation (4). Let q and r denote the functions of form (5) that interpolates ϕ at $t_1, t_2, \dots, t_n$ and $t_0, t_2, \dots, t_{n-1}$, respectively. Note that both q and r are of degree at most $n-1$. Define

$$Q^\beta(t) = q(t) + \frac{t^\beta - t_n^\beta}{t_n^\beta - t_0^\beta}\big(q(t) - r(t)\big).$$

Then the degree of $Q^\beta(t)$ is at most n and it interpolates ϕ at $t_0, t_1, \dots, t_n$. But by Theorem 5, there is a unique function with such properties, so $Q^\beta(t)$ has to be of form (4). Hence, the leading coefficient of the left side is $\phi^\beta[t_0, t_1, \dots, t_n]$ and it is not difficult to show that the leading coefficient of the right side is $\frac{\phi^\beta[t_1, t_2, \dots, t_n] - \phi^\beta[t_0, t_1, \dots, t_{n-1}]}{t_n^\beta - t_0^\beta}$ and so we arrive at equation (5). Now consider the fractional Lagrange polynomial $P_{n-1}^\beta(t)$ at $t_0, t_1, \dots, x_{n-1}$, then from (3)

$$\phi(t_n) = P_{n-1}^\beta(t_n) + \frac{D_\beta^n \phi(\zeta_t)}{\beta^n n!} \prod_{j=0}^{n-1} (t_n^\beta - t_j^\beta).$$

However,

$$P_n^\beta(t_n) = \phi^\beta[t_0] + \sum_{k=0}^{n-1} \phi^\beta[t_0, t_1, \dots, t_k](t_n^\beta - t_0^\beta)\dots(t_n^\beta - t_{k-1}^\beta)$$

and

$$P_n^\beta(t_n) = P_{n-1}^\beta(t_n) + \phi^\beta[t_0, t_1, \dots, t_n] \prod_{j=0}^{n-1} (x_n^\beta - t_j^\beta) = \phi(t_n)$$

Thus

$$\phi^\beta[t_0, t_1, \dots, t_n] = \frac{D_\beta^n \phi(\zeta_t)}{\beta^n n!}. \tag{6}$$

Formula (5) can be used to create the fractional divided differences table in the manner shown below.

Table 4.1: fractional divided differences table

x	$\phi^\beta[t_i]$	$\phi^\beta[t_i, t_{i+1}]$	$\phi^\beta[t_i, t_{i+1}, t_{i+2}]$	$\phi^\beta[t_i, t_{i+1}, t_{i+2}, t_{i+3}]$
t_0	$\phi^\beta[t_0]$			
t_1	$\phi^\beta[t_1]$	$\phi^\beta[t_0, t_1]$		
t_2	$\phi^\beta[t_2]$	$\phi^\beta[t_1, t_2]$	$\phi^\beta[t_0, t_1, t_2]$	
t_3	$\phi^\beta[t_3]$	$\phi^\beta[t_2, t_3]$	$\phi^\beta[t_1, t_2, t_3]$	$\phi^\beta[t_0, t_1, t_2, t_3]$
...	...	...	...	...

The fractional divided differences that appears in equation (5) are exactly the first $n+1$ diagonal entries in the table.

5 Hermite interpolation

Given $t_0, t_1, \ldots, t_n$ are $n+1$ distinct nonnegative integers and corresponding integers $\mu_r \geq 1$ and suppose a function ϕ is $N-1$ times continuously $\beta-$ differentiable where $N = \max_r \mu_r$. We generalize the Hermite as follows: We look for a fractional polynomial H_β of minimum degree such that, for $k = 0, 1, ..., n$,

$$H_\beta^{(i)}(t_k) = D_\beta^i \phi(t_k), \qquad i = 0, 1, ..., \mu_r - 1.$$

To get $H_\beta^{(i)}(t_k)$, we can either use an approach similar to Theorem 5, or we can view this problem as a limiting case of fractional Lagrange interpolation through the convergence of μ_r different points into a single point t_r. For more clarification, we construct the table of divided differences by entering each point t_r exactly μ_r times in the first column of the table. With the aid of equation (6), we can initialize the fractional divided differences for these points. These points can be utilized to set up the fractional divided differences. For example, if $\mu_r = 4$, then we have the table.

Table 5.1: fractional divided differences table with $\mu_k = 4$

$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
t_k	$\phi^\beta[t_k]$				$\cdots$
t_k	$\phi^\beta[t_k]$	$\frac{1}{\beta} D_\beta \phi t_k$			$\cdots$
t_k	$\phi^\beta[t_k]$	$\frac{1}{\beta} D_\beta \phi t_k$	$\frac{1}{2!\beta^2} D_\beta^2 \phi t_k$		$\cdots$
t_k	$\phi^\beta[t_k]$	$\frac{1}{\beta} D_\beta \phi t_k$	$\frac{1}{2!\beta^2} D_\beta^2 \phi t_k$	$\frac{1}{3!\beta^3} D_\beta^3 \phi t_k$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

There is another method to derive the fractional polynomial H_β when $m_r = 1$ for each $r = 0, 1, \ldots, n$. In this scenario, we refer to the polynomial H_β as the fractional Hermite polynomials. The following theorem precisely describes the form of fractional Hermite polynomials. The proof closely resembles that of Theorem 5.

Theorem 7. *If ϕ is $\beta-$ differentiable on the interval $[0,a]$ and $t_0, t_1, \ldots, t_n$ are distinct numbers in $[0,a]$, the unique polynomial of minimum degree agreeing with ϕ and $D_\beta \phi$ at $t_0, ..., t_n$ is the fractional Hermite polynomial of degree less than $2n+2$ is given by*

$$H_\beta(t) = \sum_{j=0}^{n} \phi(t_j) H_{\beta,j}(t) + \sum_{j=0}^{n} D_\beta \phi(t_j) H_{\beta,j}(t),$$

where

$$H_{\beta,j}(t) = [1 - \frac{2}{\beta}(t^\beta - t_j^\beta) D_\beta L_{n,k}^\beta(t_j)](L_{n,j}^\beta(t))^2.$$

and

$$\hat{H}_{\beta,j}(t) = \frac{1}{\beta}(t^\beta - t_j^\beta)(L_{n,j}^\beta(t))^2$$

and $L_{n,j}^\beta(t)$ denotes the jth Lagrange coefficient polynomial of degree n.

Moreover, If ϕ is $2n+1$ times continuously $\beta-$ differentiable on the closed interval $[0,a]$ and $t_0, t_1, \ldots, t_n$ are distinct numbers in $[0,a]$, then

$$\phi(t) - H_\beta(t) = \frac{D_\beta^{(2n+2)} \phi(\xi_t)}{\beta^{2n+2}(2n+2)!} \prod_{j=0}^{n} (t^{2\beta} - t_j^{2\beta}). \tag{7}$$

6 Example

Example 1. In the following example, we approximate the function $f(t) = (t \ln t)^2, t \in [1,2]$ by $P_n^{0.25}(t)$ of degree n. We take $t_i = 1 + \frac{i}{n+1}$, $i = 0, 1, \ldots, n$. See Figures 5.1 and 5.2.

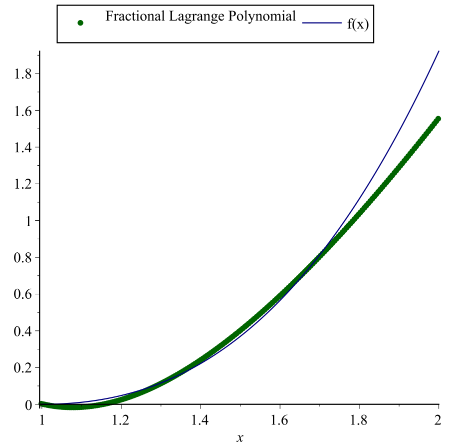

Figure 5.1: Comparison of the function $f(t)$ and $P_2^{0.25}(t)$ of degree 2.

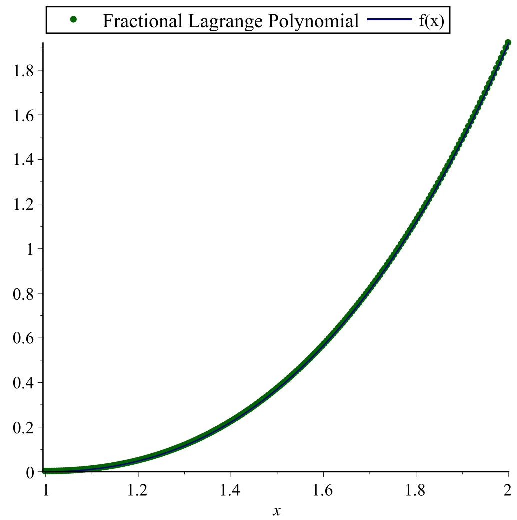

Figure 5.2: Comparison of the function $f(t)$ and $P_5^{0.25}(t)$ of degree 5.

7 Conclusion

In the present paper, the fractional Lagrange polynomial is introduced to approximate functions. We study the interpolation error based on the conformable fractional derivative. In the future, we are interesting to use this polynomial to derive new numerical quadrature formulas for integration.

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