Journal of Statistics Applications & Probability
An International Journal

Hamiltonian Formulation of Generalized Classical Field Systems Using Linear fields' variables (ϕ, A_i, A_j)

http://dx.doi.org/10.18576/jsap/120215

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Received: 24 Nov. 2022, Revised: 2 Dec. 2022, Accepted: 11 Jan. 2023
Published online: 1 May 2023

Abstract: The functional variational principle and differential equations of motion for the Lee–Wick electrodynamics equation are investigated in this study. A functional Hamiltonian principle is built utilizing the concept of a functional derivative. The Hamiltonian formulation of third-order continuous field systems is created using functional derivatives for continuous third-order systems. The formalism is generalized, and this new formulation is used to solve the Lee–Wick electrodynamics equation. We used the Euler-Lagrange equations for these systems to compare the results obtained using Hamilton's equations in terms of functional derivatives. To compare the outcomes of the two methodologies in terms of functional derivatives, one example has been presented. The results of this study show that functional calculus has more flexible models than classical calculus due to the ordering of the functional derivative and the functional operator. This property is critical when developing a novel generalization of the Lee-Wick equation using functional derivatives.

Key Words: Functional derivatives, Hamiltonian Systems, Lee–Wick electrodynamics Equation

1 Introduction

Higher order derivative theories have been discussed in the literature by a large number of authors [1,2,3,4,5,6] mainly due to, the possibility of obtaining finite theories at short distances. An illustrative example of such, a class of theories is the electrodynamics proposed by Lee–Wick. The simplest higher-order derivative gauge theory is the so-called Lee–Wick electrodynamics, which is characterized by the Maxwell Lagrangian and a higher-order derivative kinetic term. Since its proposition, the theory has been standing out by its classical as well as its quantum aspects, as it exposures many interesting peculiarities. For example, the fact that in this electrodynamics the self-energy of a point charge is finite in (3 + 1) dimensions [7,8,9,10,11,12] , it tracks a finite theory Closely related to the Pauli–Villars regularization scheme [13,14,15,16,17,18] where the divergences of the quantum electrodynamics are controllable [19] and it shows classical

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dynamical stability[20]. With the passage of time, systems with higher order Lagrangian have been studied with rising interest because they appear in many pertinent physical problems. More general examples being perhaps higher order regularization of quantum gauge field theories and so-called rigid strings [21, 22], rigid particles [23, 24] , a relativistic particle with curvature and torsion in three dimensional space-time, and the work of Lee–Wick [25] who independently proposed generalization of electrodynamics containing third order derivatives. The effective Lagrangian in gauge theories [26, 27] , dealing with theories of higher order derivatives has first been developed by Ostrogradski [28] ,allowed them to obtain the Hamilton equations of motion and the Euler-Lagrange equations. Chen T.J. et al [29] investigated the linear instability in a non-degenerate higher-derivative theory and found that it can only be removed by the addition of constraints if the original theory's phase space is reduced. The path integral quantization of systems of higher order derivatives were presented in ref [30].

In ref.[31], the action function is examined for both constrained and unconstrained systems in order to discover a solution to the appropriate set of Hamilton-Jacobi partial differential equations. Researchers used the WKB approach to solve the motion equations. Separately, other Scholars [32,33] used Dirac's approach to limited dynamics to examine the Hamiltonian formulation of higher order dynamical systems.

They created the Hamiltonian formulation of conventional higher-order Lagrangians, as well , as Ostrogradski's standard description of such systems. Furthermore, reference [34] investigated a novel development for systems with higher order derivatives. References[35, 36] have reported a new evolution of systems with higher order fractional derivatives in recent decades, and the route integral quantization for both conservative and non-conservative systems has been recovered.

While working on the Hamilton formulation for continuous systems with second order derivatives, researchers defined techniques for system with second order differential equation in reference [37].Muslih and El-Zalan [31,38] recently established unique formulations for dealing with discrete systems with higher order Lagrangians, using path integral quantization to obtain canonical coordinates without the requirement to integrate over higher order derivatives. The authors utilized the formalism to investigate second-order Lagrangian. The research's key contribution is to evaluate Hamilton's equation using its innovative functional derivatives formulation, which is based on previously published data. The characteristics listed below define the new features presented in this article.

- ✓ In this technique, we extend our formulations such that they may be used to continuous systems with third-order derivatives. The purpose of the method is to obtain Lee-Wick generalized electrodynamics.
- ✓ Because The new formulation uses the functional Derivative, they are more challenging to solve in practice. As, a result, we provide a novel and efficient technique.
- ✓ There are three factors that distinguish the functional derivative:
    1. It recently piqued the curiosity of several researchers, and some applications have been updated to reflect this description.
    2. It is a method for simulating system ordinary differential equations such as the Lagrange and Hamiltonian equations.
    3. By combining the Lagrange and Hamiltonian equations into systems-order differential equation models, the functional derivative enables the development of novel comparisons and applications.

For these reasons, we reconstruct the Lee-Wick field using the functional derivative, as used by the authors in equation modeling, and then apply the Generalized functional derivative to produce Hamiltonian equations for this system.

The following is a breakdown of how this work is organized: Section 2 summarizes the Euler-Lagrange equations of motion. Sec. 3 investigates the momentum density form of the Euler-Lagrange equation. Section 4 is devoted to motion equations in terms of Hamiltonian density in functional derivative form. Section 5 explores Hamilton's equation in terms of $(\boldsymbol{\phi}, \boldsymbol{A_i}, \boldsymbol{A_j})$. Section 6 demonstrates how a classical field can be utilized to construct Lee-Wick in functional derivative form.

2 The Euler-Lagrange equations of motion

We start by constructing a classical Lagrangian density $\mathcal{L}(\psi_\mu, \partial_\mu\psi_p, \partial_\mu\partial_\sigma\psi_p, \partial_\mu\partial_\sigma\partial_\varepsilon\psi_p)$ depending on generalized coordinates $\psi_p$and derivatives. Now we can write the L as:

$$L = \int \mathcal{L}(\psi_p, \partial_\lambda\psi_p, \partial_\lambda\partial_\sigma\psi_p, \partial_\lambda\partial_\sigma\partial_\varepsilon\psi_p)d^3x \tag{1}$$

We apply the principle of least action and obtain

$$\delta S = \int \begin{bmatrix} \dfrac{\partial\mathcal{L}}{\partial\psi_\rho}\delta\psi_\rho + \dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\psi_p)}\partial_\lambda(\delta\psi_\rho) + \\ \dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\partial_\sigma\psi_p)}\partial_\lambda\partial_\sigma(\delta\psi_p) + \dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\partial_\sigma\partial_\varepsilon\psi_p)}\partial_\lambda\partial_\sigma\partial_\varepsilon\delta(\psi_p) \end{bmatrix} d^4x = 0 \tag{2}$$

Appendix A illustrates the Euler-Lagrange equations.

$$\frac{\partial\mathcal{L}}{\partial\psi_\rho} - \partial_\lambda\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\psi_p)} + \partial_\lambda\partial_\sigma\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\partial_\sigma\psi_p)} - \partial_\lambda\partial_\sigma\partial_\varepsilon\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\partial_\sigma\partial_\varepsilon\psi_\rho)} = 0 \tag{3}$$

The integration over space formula in Eq. (2) can now be converted into summation as follows (**view Appendix A**):

$$\begin{aligned}
&\sum_i \begin{bmatrix} \dfrac{\partial\mathcal{L}}{\partial\psi_0} + -\partial_i\dfrac{\partial\mathcal{L}}{\partial(\partial_i\psi_0)} + \partial_i\partial_0\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\psi_0)} \\ +\partial_0\partial_r\dfrac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\psi_0)} + \partial_i\partial_r\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\psi_0)} \\ -\partial_0\partial_0\partial_f\dfrac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\partial_f\psi_0)} - \partial_0\partial_r\partial_0\dfrac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_0\psi_0)} \\ -\partial_0\partial_r\partial_f\dfrac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_f\psi_0)} - \partial_i\partial_0\partial_0\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_0\psi_0)} \\ -\partial_i\partial_0\partial_f\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_f\psi_0)} - \partial_i\partial_r\partial_0\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\psi_0)} - \partial_i\partial_r\partial_f\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\psi_0)} \end{bmatrix}_i \delta\psi_{p_0}\delta\tau_0 \\
&+ \sum_i \begin{bmatrix} +\dfrac{\partial\mathcal{L}}{\partial\psi_l} - \partial_i\dfrac{\partial\mathcal{L}}{\partial(\partial_i\psi_l)} + \partial_i\partial_0\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\psi_l)} \\ +\partial_0\partial_r\dfrac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\psi_l)} + \partial_i\partial_r\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\psi_l)} - \partial_0\partial_0\partial_f\dfrac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\partial_f\psi_l)} \\ -\partial_0\partial_r\partial_0\dfrac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_0\psi_l)} - \partial_0\partial_r\partial_f\dfrac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_f\psi_l)} - \partial_i\partial_0\partial_0\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_0\psi_l)} \\ -\partial_i\partial_0\partial_f\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_f\psi_l)} - \partial_i\partial_r\partial_0\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\psi_l)} - \partial_i\partial_r\partial_f\dfrac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\psi_l)} \end{bmatrix}_i \delta\psi_{p_i}\delta\tau_i \\
&+ \sum_i \left[\frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\rho)}\right]_i \delta(\dot{\psi_\rho})_i\delta\tau_i + \sum_i \left[\frac{\partial\mathcal{L}}{\partial(\partial_0^2\psi_\rho)}\right]_i \delta(\ddot{\psi_\rho})_i\delta\tau_i + \sum_i \left[\frac{\partial\mathcal{L}}{\partial(\partial_0^3\psi_\rho)}\right]_i \delta(\dddot{\psi_0})_i\delta\tau_i = 0
\end{aligned} \tag{4}$$

By terms of Lagrangian density, we can express Eq. (4) as follows:

$$\sum_i [\boldsymbol{\delta\mathcal{L}}]_i \, \boldsymbol{\delta\tau_i} = \mathbf{0}$$

where the left-hand side in Eqs.(4 and 5) represents the variation of L(i.e. δL) which is now produced by independent variations in $\delta(\psi_\rho)_i, \delta(\partial_0\psi_\rho)_i, \delta(\partial_0^2\psi_\rho)_i, and\ \delta(\partial_0^3\psi_\rho)_i$. Suppose now that all$\delta(\psi_\rho)_i, \delta(\partial_0\psi_\rho)_i, \delta(\partial_0^2\psi_\rho)_i, and\ \delta(\partial_0^3\psi_\rho)_i$are zeros except for a particular $\delta\psi_j$. It is natural to define the functional derivative of the Lagrangian (∂L) with respect to $\delta(\psi_\rho)_i, \delta(\partial_0\psi_\rho)_i, \delta(\partial_0^2\psi_\rho)_i, and\ \delta(\partial_0^3\psi_\rho)_i$for a point in the j-th cell to the ratio of δL to $\delta\psi_j$.

Applying Eq. (4) and noting that now the left side denotes L, we get:

$$\frac{\partial L}{\partial \dot{\psi}} = \lim_{\delta\tau_j \to 0} \frac{\delta\mathcal{L}}{\delta(\dot{\psi}_\rho)_j \delta_{\tau j}} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_p)} \tag{6}$$

$$\frac{\partial L}{\partial \ddot{\psi}} = \lim_{\delta\tau_j \to 0} \frac{\delta\mathcal{L}}{\delta(\ddot{\psi}_\rho)\delta_{\tau j}} = \frac{\partial\mathcal{L}}{\partial(\partial_0^2\psi_p)} \tag{7}$$

$$\frac{\partial L}{\partial \dddot{\psi}} = \lim_{\delta\tau_j \to 0} \frac{\delta\mathcal{L}}{\delta(\dddot{\psi}_\rho)\delta_{\tau j}} = \frac{\partial\mathcal{L}}{\partial(\partial_0^3\psi_p)} \tag{8}$$

and

$$\frac{\partial L}{\partial \psi_\rho} = \left[\begin{array}{c}
\frac{\partial\mathcal{L}}{\partial\psi_0} + \frac{\partial\mathcal{L}}{\partial\psi_l} - \partial_i \frac{\partial\mathcal{L}}{\partial(\partial_i\psi_0)} - \partial_i \frac{\partial\mathcal{L}}{\partial(\partial_i\psi_l)} \\
+\partial_i\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\psi_0)} + \partial_i\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\psi_l)} + \partial_0\partial_r \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\psi_0)} \\
+\partial_0\partial_r \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\psi_l)} + \partial_i\partial_r \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\psi_0)} + \partial_i\partial_r \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\psi_l)} \\
-\partial_0\partial_0\partial_f \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\partial_f\psi_0)} - \partial_0\partial_0\partial_f \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\partial_f\psi_l)} - \partial_0\partial_r\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_0\psi_0)} \\
-\partial_0\partial_r\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_0\psi_l)} - \partial_0\partial_r\partial_f \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_f\psi_0)} - \partial_0\partial_r\partial_f \frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_f\psi_l)} \\
-\partial_i\partial_0\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_0\psi_0)} - \partial_i\partial_0\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_0\psi_l)} - \partial_i\partial_0\partial_f \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_f\psi_0)} \\
-\partial_i\partial_0\partial_f \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_f\psi_l)} - -\partial_i\partial_r\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\psi_0)} - \partial_i\partial_r\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\psi_l)} \\
-\partial_i\partial_r\partial_f \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\psi_0)} - \partial_i\partial_r\partial_f \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\psi_l)}
\end{array}\right] \tag{9}$$

They may now construct Eq. (9), the Euler-Lagrange equation, by using terms of a Lagrangian L and functional derivatives as shown in Eqs. (6),(7), and (8) to obtain:

$$\frac{\partial L}{\partial \psi_\rho} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\psi}_\rho}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial L}{\partial \ddot{\psi}_\rho}\right) - \frac{\partial^3}{\partial t^3}\left(\frac{\partial L}{\partial \dddot{\psi}_\rho}\right) = 0 \tag{10}$$

As a result, Lagrangian variations can be defined in terms of functional derivatives and variants of $\psi, \dot{\psi}, \ddot{\psi}$ and $\dddot{\psi}$ as

$$\delta L = \int \left[\frac{\partial L}{\partial \psi_\rho} \delta\psi_\rho + \frac{\partial L}{\partial \dot{\psi}_\rho} \delta\dot{\psi}_\rho + \frac{\partial L}{\partial \ddot{\psi}_\rho} \delta\ddot{\psi}_\rho + \frac{\partial L}{\partial \dddot{\psi}_\rho} \delta\dddot{\psi}_\rho \right] \mathrm{d}^3\mathrm{r} \quad (11)$$

3 Euler-Lagrange Equation in terms of Momentum Density

Momentum could be represented in the following way:

$$[P_j^a]_1 = \frac{\delta L}{\delta[\dot{\psi}_\rho]_j} \quad (12)$$

$$[P_j^a]_2 = \frac{\delta L}{\delta[\ddot{\psi}_\rho]_j} \quad (13)$$

$$[P_j^a]_3 = \frac{\delta L}{\delta[\dddot{\psi}_\rho]_j} \quad (14)$$

The momentum densities π_1 , π_2 and π_3 can be obtained using Given in eq. (12), (13) and (14). As a result, generalized moments are defined as

$$\pi_1 = \frac{\partial L}{\partial[\dot{\psi}_\rho]_j} = \frac{\partial \mathcal{L}}{\partial\,[\dot{\psi}_\rho]_j} \quad (15)$$

$$\pi_2 = \frac{\partial L}{\partial[\ddot{\psi}_\rho]_j} = \frac{\partial \mathcal{L}}{\partial\,[\ddot{\psi}_\rho]_j} \quad (16)$$

$$\pi_3 = \frac{\partial L}{\partial[\dddot{\psi}_\rho]_j} = \frac{\partial \mathcal{L}}{\partial\,[\dddot{\psi}_\rho]_j} \quad (17)$$

From Eq. (9)

$$\frac{\partial L}{\partial \psi_\rho} = \dot{\pi}_1 - \ddot{\pi}_2 + \dddot{\pi}_3 \quad (18)$$

The above equation represents the form of Euler- Lagrange terms of momentum density and the functional derivative of the Lagrangian.

4 Equations of Motion in terms of Hamiltonian Density in Functional Form

The following are the definitions of Hamiltonian formulations:

$$\mathcal{H} = \pi_1\dot{\psi}_\rho + \pi_2\ddot{\psi}_\rho + \pi_3\dddot{\psi}_\rho - \mathcal{L} \quad (19)$$

The Hamiltonian $\mathcal{H}$ can alternatively be expressed in terms of the Hamiltonian density H, as shown below:

$$H = \sum_i \mathcal{H}_i\,\delta\tau_i \quad (20)$$

Substituting Eqs. (19) in to Eq. (20), one gets:

$$H = \sum_i \left[(\pi_1)_i(\dot{\psi}_\rho)_i + (\pi_2)(\ddot{\psi}_\rho)_i + (\pi_3)(\dddot{\psi}_\rho)_i\right]\delta\tau_i - \sum_{\mathrm{i}} \mathcal{L}\,\delta\tau_i$$

The following equation is written in continuous form as follows:

$$H = \int [(\pi_1)_i(\dot{\psi}_\rho) + (\pi_2)(\ddot{\psi}_\rho) + (\pi_3)(\dddot{\psi}_\rho)]d^3r - \int \mathcal{L}\,d^3r \quad (21)$$

Using Eqs. 11 and 18, we can determine the following from variation of H:

$$\delta\mathcal{H} = \int \delta\left[\pi_1\dot{\psi}_\rho + \pi_2\ddot{\psi}_\rho + \pi_3\dddot{\psi}_\rho\right] \mathrm{d}^3\mathrm{r} - \delta L \quad (22)$$

Using Eqs. (15) , (16), (17) and (18), we rewrite the variation of the Lagrangian(given by Eq. (11)) as:

$$\delta L = \int\left[(\dot{\pi}_1 - \ddot{\pi}_2 + \dddot{\pi}_3)\delta\psi_\rho + \pi_1\delta\dot{\psi}_\rho + \pi_2\delta\ddot{\psi}_\rho + \pi_3\delta\dddot{\psi}_\rho\right] \mathrm{d}^3\mathrm{r} \quad (23)$$

When Eq. (23) is substituted for Eq. (22), the following result is obtained:

$$\delta\mathcal{H} = \int\left[(-\dot{\pi}_1 + \ddot{\pi}_2 - \dddot{\pi}_3)\delta\psi_\rho + \dot{\psi}_\rho\delta\pi_1 + \ddot{\psi}_\rho\delta\pi_2 + \dddot{\psi}_\rho\delta\pi_3\right] \mathrm{d}^3\mathrm{r} \quad (24)$$

Because of the similarity to the variation in L (i.e. Eq. (11)), we may write the variation of Hamiltonian produced by variations in independent variables in terms of functional derivative, as illustrated in cases 1 and 2.

Case 1:All variables are independent(ψ_ρ , π_1, π_2, π_3)

$$\delta H = \int\left[\frac{\partial H}{\partial \psi_\rho}\delta\psi_\rho + \frac{\partial H}{\partial \pi_1}\delta\pi_1 + \frac{\partial H}{\partial \pi_2}\delta\pi_2 + \frac{\partial H}{\partial \pi_3}\delta\pi_3\right] \mathrm{d}^3\mathrm{r} \quad (25)$$

We obtain the separate equations of motion in terms of the Hamiltonian by comparing Eq. (25) with Eq. (24). (see Appendix B).

Case 2: π_1 depend on (ψ_ρ),π_2 depend ($\partial_0\psi_\rho$) and π_3 depend ($\partial_0\partial_0\psi_\rho$) ; so thatwe take the variation just only for independent variables ψ_ρ , $\partial_0\psi_\rho$and$\partial_0\partial_0\psi_\rho$we can

$$\delta H = \int\left[\frac{\partial H}{\partial \psi_\rho}\delta\psi_\rho + \frac{\partial H}{\partial (\partial_0\psi_\rho)}\delta(\partial_0\psi_\rho) + \frac{\partial H}{\partial (\partial_0^2\psi_\rho)}\delta(\partial_0^2\psi_\rho)\right] d^3r \quad (26)$$

Appendix C includes detailed equations of motion derived from Eq (26)

5- The Hamilton's equation in terms$(\boldsymbol{\phi}, \boldsymbol{A_i}, \boldsymbol{A_j})$

We could obtain the following two cases by defining the fields' variables ψ_ρ $(\phi\,, A_i \text{ and } A_j)$ and then writing the Hamilton's equation in terms of functional derivatives as follows (cases 1 and 2):

Case 1: All variables are independent

$$\begin{bmatrix} \frac{\partial h}{\partial\phi} - \partial_i\frac{\partial h}{\partial(\partial_i\phi)} + \partial_0\partial_r\frac{\partial h}{\partial(\partial_0\partial_r\phi)} + \partial_i\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\phi)} + \partial_i\partial_r\frac{\partial h}{\partial(\partial_i\partial_r\phi)} \\ -\partial_0\partial_0\partial_f\frac{\partial h}{\partial(\partial_0\partial_0\partial_f\phi)} - \partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0\partial_r\partial_0\phi)} - \partial_0\partial_r\partial_f\frac{\partial h}{\partial(\partial_0\partial_r\partial_f\phi)} \\ -\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\partial_0\phi)} - \partial_i\partial_0\partial_f\frac{\partial h}{\partial(\partial_i\partial_0\partial_f\phi)} - \partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0\phi)} \\ -\partial_i\partial_r\partial_f\frac{\partial h}{\partial(\partial_i\partial_r\partial_f\phi)} \end{bmatrix} = -\partial_0\pi_1^1 + \partial_0^2\pi_2^1 - \partial_0^3\pi_3^1 \quad (27)$$

$$\begin{bmatrix} \frac{\partial h}{\partial A_i} - \partial_i \frac{\partial h}{\partial(\partial_i A_i)} + \partial_0\partial_r \frac{\partial h}{\partial(\partial_0\partial_r A_i)} + \partial_i\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0 A_i)} + \\ +\partial_i\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0 A_i)} + \partial_i\partial_r \frac{\partial h}{\partial(\partial_i\partial_r A_i)} - \partial_0\partial_0\partial_f \frac{\partial h}{\partial\left(\partial_0\partial_0\partial_f A_i\right)} \\ -\partial_0\partial_r\partial_0 \frac{\partial h}{\partial(\partial_0\partial_r\partial_0 A_i)} - \partial_0\partial_r\partial_f \frac{\partial h}{\partial\left(\partial_0\partial_r\partial_f A_i\right)} - \partial_i\partial_0\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0\partial_0 A_i)} \\ -\partial_i\partial_0\partial_f \frac{\partial h}{\partial\left(\partial_i\partial_0\partial_f A_i\right)} - \partial_i\partial_r\partial_0 \frac{\partial h}{\partial(\partial_i\partial_r\partial_0 A_i)} - \partial_i\partial_r\partial_f \frac{\partial h}{\partial\left(\partial_i\partial_r\partial_f A_i\right)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial h}{\partial A_i} - \partial_i \frac{\partial h}{\partial(\partial_i A_i)} + \partial_0\partial_r \frac{\partial h}{\partial(\partial_0\partial_r A_i)} + \partial_i\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0 A_i)} + \\ +\partial_i\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0 A_i)} + \partial_i\partial_r \frac{\partial h}{\partial(\partial_i\partial_r A_i)} - \partial_0\partial_0\partial_f \frac{\partial h}{\partial\left(\partial_0\partial_0\partial_f A_i\right)} \\ -\partial_0\partial_r\partial_0 \frac{\partial h}{\partial(\partial_0\partial_r\partial_0 A_i)} - \partial_0\partial_r\partial_f \frac{\partial h}{\partial\left(\partial_0\partial_r\partial_f A_i\right)} - \partial_i\partial_0\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0\partial_0 A_i)} \\ -\partial_i\partial_0\partial_f \frac{\partial h}{\partial\left(\partial_i\partial_0\partial_f A_i\right)} - \partial_i\partial_r\partial_0 \frac{\partial h}{\partial(\partial_i\partial_r\partial_0 A_i)} - \partial_i\partial_r\partial_f \frac{\partial h}{\partial\left(\partial_i\partial_r\partial_f A_i\right)} \end{bmatrix} = -\partial_0\pi_1^2 + \partial_0^2\pi_2^2 - \partial_0^3\pi_3^2 \tag{28}$$

$$\begin{bmatrix} \frac{\partial h}{\partial A_j} - \partial_i \frac{\partial h}{\partial\left(\partial_i A_j\right)} + \partial_0\partial_r \frac{\partial h}{\partial\left(\partial_0\partial_r A_j\right)} + \partial_i\partial_0 \frac{\partial h}{\partial\left(\partial_i\partial_0 A_j\right)} + \\ +\partial_i\partial_0 \frac{\partial h}{\partial\left(\partial_i\partial_0 A_j\right)} + \partial_i\partial_r \frac{\partial h}{\partial\left(\partial_i\partial_r A_j\right)} - \partial_0\partial_0\partial_f \frac{\partial h}{\partial\left(\partial_0\partial_0\partial_f A_j\right)} \\ -\partial_0\partial_r\partial_0 \frac{\partial h}{\partial\left(\partial_0\partial_r\partial_0 A_j\right)} - \partial_0\partial_r\partial_f \frac{\partial h}{\partial\left(\partial_0\partial_r\partial_f A_j\right)} - \partial_i\partial_0\partial_0 \frac{\partial h}{\partial\left(\partial_i\partial_0\partial_0 A_j\right)} \\ -\partial_i\partial_0\partial_f \frac{\partial h}{\partial\left(\partial_i\partial_0\partial_f A_j\right)} - \partial_i\partial_r\partial_0 \frac{\partial h}{\partial\left(\partial_i\partial_r\partial_0 A_j\right)} - \partial_i\partial_r\partial_f \frac{\partial h}{\partial\left(\partial_i\partial_r\partial_f A_j\right)} \end{bmatrix} = -\partial_0\pi_1^3 + \partial_0^2\pi_2^3 - \partial_0^3\pi_3^3 \tag{29}$$

$$\begin{bmatrix} \frac{\partial h}{\partial \phi} - \partial_i \frac{\partial h}{\partial(\partial_i \phi)} + \partial_0\partial_r \frac{\partial h}{\partial(\partial_0\partial_r \phi)} + \partial_i\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0 \phi)} + \partial_i\partial_r \frac{\partial h}{\partial(\partial_i\partial_r \phi)} \\ -\partial_0\partial_0\partial_f \frac{\partial h}{\partial\left(\partial_0\partial_0\partial_f \phi\right)} - \partial_0\partial_r\partial_0 \frac{\partial h}{\partial(\partial_0\partial_r\partial_0 \phi)} - \partial_0\partial_r\partial_f \frac{\partial h}{\partial\left(\partial_0\partial_r\partial_f \phi\right)} \\ -\partial_i\partial_0\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0\partial_0 \phi)} - \partial_i\partial_0\partial_f \frac{\partial h}{\partial\left(\partial_i\partial_0\partial_f \phi\right)} - \partial_i\partial_r\partial_0 \frac{\partial h}{\partial(\partial_i\partial_r\partial_0 \phi)} \\ -\partial_i\partial_r\partial_f \frac{\partial h}{\partial\left(\partial_i\partial_r\partial_f \phi\right)} \end{bmatrix}$$

$$= -\partial_0\pi_1^1 + \partial_0^2\pi_2^1 - \partial_0^3\pi_3^1 + \partial_0\phi\left(\frac{\partial g_1^1}{\partial \phi}\right) \tag{30}$$

$$\begin{bmatrix} \frac{\partial h}{\partial A_i} - \partial_i \frac{\partial h}{\partial(\partial_i A_i)} + \partial_0\partial_r \frac{\partial h}{\partial(\partial_0\partial_r A_i)} + \\ \partial_i\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0 A_i)} + \partial_i\partial_r \frac{\partial h}{\partial(\partial_i\partial_r A_i)} - \partial_0\partial_0\partial_f \frac{\partial h}{\partial(\partial_0\partial_0\partial_f A_i)} \\ -\partial_0\partial_r\partial_0 \frac{\partial h}{\partial(\partial_0\partial_r\partial_0 A_i)} - \partial_0\partial_r\partial_f \frac{\partial h}{\partial(\partial_0\partial_r\partial_f A_i)} - \partial_i\partial_0\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0\partial_0 A_i)} \\ -\partial_i\partial_0\partial_f \frac{\partial h}{\partial(\partial_i\partial_0\partial_f A_i)} - \partial_i\partial_r\partial_0 \frac{\partial h}{\partial(\partial_i\partial_r\partial_0 A_i)} - \partial_i\partial_r\partial_f \frac{\partial h}{\partial(\partial_i\partial_r\partial_0 A_i)} \end{bmatrix}$$

$$= \partial_0\pi_1^2 + \partial_0^2\pi_2^2 - \partial_0^3\pi_3^2 + \partial_0 A_i\left(\frac{\partial g_1^2}{\partial A_i}\right) \quad (31)$$

$$\begin{bmatrix} \frac{\partial h}{\partial A_j} - \partial_i \frac{\partial h}{\partial(\partial_i A_j)} + \partial_0\partial_r \frac{\partial h}{\partial(\partial_0\partial_r A_j)} + \\ \partial_i\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0 A_j)} + \partial_i\partial_r \frac{\partial h}{\partial(\partial_i\partial_r A_j)} - \partial_0\partial_0\partial_f \frac{\partial h}{\partial(\partial_0\partial_0\partial_f A_j)} \\ -\partial_0\partial_r\partial_0 \frac{\partial h}{\partial(\partial_0\partial_r\partial_0 A_j)} - \partial_0\partial_r\partial_f \frac{\partial h}{\partial(\partial_0\partial_r\partial_f A_j)} - \partial_i\partial_0\partial_0 \frac{\partial h}{\partial(\partial_i\partial_0\partial_0 A_j)} \\ -\partial_i\partial_0\partial_f \frac{\partial h}{\partial(\partial_i\partial_0\partial_f A_j)} - \partial_i\partial_r\partial_0 \frac{\partial h}{\partial(\partial_i\partial_r\partial_0 A_j)} - \partial_i\partial_r\partial_f \frac{\partial h}{\partial(\partial_i\partial_r\partial_0 A_j)} \end{bmatrix}.$$

$$= \partial_0\pi_1^3 + \partial_0^2\pi_2^3 - \partial_0^3\pi_3^3 + \partial_0 A_j\left(\frac{\partial g_1^2}{\partial A_j}\right) \quad (32)$$

5 Example

Take the following Lagrangian[26] as an example of a continuous system:

$$\mathcal{L}_{LW} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4m^2}F_{\mu\nu}\partial_\alpha\partial^\alpha F^{\mu\nu} - \frac{\partial_\mu A^\mu}{2\xi} - J_\mu A^\mu \quad (33)$$

where m is a parameter that has mass dimension, J_μ is an external source and ξ is a gauge fixing parameter. $F^{\mu\nu}$ is a four dimension antisymmetric second rank tensor and A^μ is a the four – vector potential .

Using,

$$\left.\begin{matrix} F_{\mu\nu} = \partial_\mu A_\upsilon - \partial_\upsilon A_\mu \\ F^{\mu\nu} = \partial^\mu A^\upsilon - \partial^\upsilon A^\mu \end{matrix}\right] \quad (34)$$

μ, ν, α can be expanded into $(0, i)$, $(0, j)$ *and* $(0, k)$, correspondingly; Lagrange (33) can be written as

$$\mathcal{L}_{LW} = \begin{bmatrix} -\frac{1}{4m^2}\begin{pmatrix} -2(F_{0i})\partial_0^2(F_{0i}) + 2(F_{0i})\partial_k^2(F_{0i}) \\ -(F_{ij})\partial_0^2(F_{ij}) + (F_{ij})\partial_k^2(F_{ij}) \end{pmatrix} \\ \frac{1}{4}\left(2(F_{0i})^2 + (F_{ij})^2\right) + \frac{\partial_0 A_0}{2\xi} + \frac{\partial_i A_i}{2\xi} + J_0 A_0 - J_i A_i \end{bmatrix} \quad (35)$$

The following are the outcomes of Equations (A1, A2, and A3): (see Appendix A).

$$\begin{bmatrix} J_0 + \partial_i(F_{0i}) - \frac{2}{4m^2}\partial_i\left(\partial_0^2(F_{0i}) - \partial_k^2(F_{0i})\right) - \frac{2}{4m^2}\partial_f\left(\partial_0^2(F_{0f}) + \partial_k^2(F_{0f})\right) + \\ J_i - \partial_0(F_{0i}) - \frac{2}{4m^2}\partial_0\left(\partial_0^2(F_{0i}) - \partial_k^2(F_{0i})\right) - \frac{1}{4m^2}\partial_0^2\left(2\partial_0(F_{0i}) - \partial_f(F_{if})\right) + \\ \frac{2}{4}\partial_i(F_{ij}) - \frac{1}{4m^2}\partial_k^2\left(2\partial_0(F_{0i}) + \partial_f(F_{if})\right) - \frac{1}{4m^2}\partial_i\left(\partial_0^2(F_{ij}) + \partial_k^2(F_{ij})\right) \\ -\frac{1}{4m^2}\partial_f\left(\partial_0^2 F_{if} + \partial_k^2 F_{fj}\right) = 0 \end{bmatrix} \quad (36)$$

The momenta π_1^1, π_1^2 are given as

$$\pi_1^1 = \frac{\partial L}{\partial(\partial_0 A_0)} = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_0)} = -\frac{1}{2\xi} \tag{37}$$

$$\pi_1^2 = \frac{\partial L}{\partial(\partial_0 A_i)} = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} = (F_{0i}) + \frac{2}{4m^2}\left[\partial_0^2(F_{0i}) - \partial_k^2(F_{0i})\right] \tag{38}$$

$\pi_1^3 = \pi_2^1 = \pi_2^2 = \pi_2^3 = \pi_3^1 = \pi_3^2$ and $=\pi_3^3 = 0$ are commonly set to zero in the literature by a wide range of authors. Because of the Lagrangian equation, $\frac{\partial \mathcal{L}}{\partial (\partial_0 A_j)} = \frac{\partial \mathcal{L}}{\partial (\partial_0^2 A_0)} = \frac{\partial \mathcal{L}}{\partial \left(\partial_0^2 A_i\right)} = \frac{\partial \mathcal{L}}{\partial (\partial_0^2 A_j)} = \frac{\partial \mathcal{L}}{\partial (\partial_0^3 A_0)} = \frac{\partial \mathcal{L}}{\partial \left(\partial_0^3 A_i\right)}$ and $\frac{\partial \mathcal{L}}{\partial (\partial_0^3 A_j)} =0$

As a result, $\pi_1^3 = \pi_2^1 = \pi_2^2 = \pi_2^3 = \pi_3^1 = \pi_3^2$ and $=\pi_3^3 = 0$

The Hamiltonian density can therefore be expressed as:

$$h = \pi_1^1\partial_0 A_0 + \pi_1^2\partial_0 A_i + \pi_1^3\partial_0 A_j + \pi_2^1\partial_0^2 A_0 + \pi_2^2\partial_0^2 A_i + \pi_2^3\partial_0^2 A_j + \pi_3^1\partial_0^3 A_0 + \pi_3^2\partial_0^3 A_i + \pi_3^3\partial_0^3 A_i - \mathcal{L} \tag{39}$$

Substituting the Lagrangian, we get

$$h = \begin{bmatrix} \frac{1}{2\xi}(\partial_0 A_0) + \left(\frac{1}{4}(4F_{0i}) - \frac{2}{4m^2}\left(-\partial_0^2(F_{0i}) + \partial_k^2(F_{0i})\right)\right)(\partial_0 A_i) \\ -\frac{1}{4}\left(2(F_{0i})^2 + \left(F_{ij}\right)^2\right) + \frac{1}{4m^2}\begin{pmatrix} -2(F_{0i})\partial_0^2(F_{0i}) + 2(F_{0i})\partial_k^2(F_{0i}) \\ -\left(F_{ij}\right)\partial_0^2\left(F_{ij}\right) + \left(F_{ij}\right)\partial_k^2\left(F_{ij}\right) \end{pmatrix} \\ +\frac{\partial_0 A_0}{2\xi} - \frac{\partial_i A_i}{2\xi} - J_0 A_0 + J_i A_i \end{bmatrix} \tag{40}$$

Using Eqs. (27, 28 and 29) in case (1) respectively gives:

$$\begin{bmatrix} J_0 + \partial_i(F_{0i}) - \frac{2}{4m^2}\partial_i\left(\partial_0^2(F_{0i}) - \partial_k^2(F_{0i})\right) - \frac{2}{4m^2}\partial_f\left(\partial_0^2\left(F_{0f}\right) + \partial_k^2\left(F_{0f}\right)\right) + \\ J_i - \partial_0(F_{0i}) - \frac{2}{4m^2}\partial_0\left(\partial_0^2(F_{0i}) - \partial_k^2(F_{0i})\right) - \frac{1}{4m^2}\partial_0^2\left(2\partial_0(F_{0i}) - \partial_f\left(F_{if}\right)\right) + \\ \frac{2}{4}\partial_i\left(F_{ij}\right) - \frac{1}{4m^2}\partial_k^2\left(2\partial_0(F_{0i}) + \partial_f\left(F_{if}\right)\right) - \frac{1}{4m^2}\partial_i\left(\partial_0^2\left(F_{ij}\right) + \partial_k^2\left(F_{ij}\right)\right) \\ -\frac{1}{4m^2}\partial_f\left(\partial_0^2 F_{if} + \partial_k^2 F_{fj}\right) = 0 \end{bmatrix} \tag{41}$$

This is similar to the result obtained using Euler-Lagrange, see Eq (36).

Applying Hamilton's Eqs. (30,31, and 32), we could obtain:

This is similar to the result obtained using Euler-Lagrange, see Eq (36).

Applying Hamilton's Eqs. (30,31, and 32), we could obtain:

$$\begin{bmatrix} J_0 + \partial_i(F_{0i}) - \frac{2}{4m^2}\partial_i\left(\partial_0^2(F_{0i}) - \partial_k^2(F_{0i})\right) - \frac{2}{4m^2}\partial_f\left(\partial_0^2\left(F_{0f}\right) + \partial_k^2\left(F_{0f}\right)\right) + \\ J_i - \partial_0(F_{0i}) - \frac{2}{4m^2}\partial_0\left(\partial_0^2(F_{0i}) - \partial_k^2(F_{0i})\right) - \frac{1}{4m^2}\partial_0^2\left(2\partial_0(F_{0i}) - \partial_f\left(F_{if}\right)\right) + \\ \frac{2}{4}\partial_i\left(F_{ij}\right) - \frac{1}{4m^2}\partial_k^2\left(2\partial_0(F_{0i}) + \partial_f\left(F_{if}\right)\right) - \frac{1}{4m^2}\partial_i\left(\partial_0^2\left(F_{ij}\right) + \partial_k^2\left(F_{ij}\right)\right) \\ -\frac{1}{4m^2}\partial_f\left(\partial_0^2 F_{if} + \partial_k^2 F_{fj}\right) = 0 \end{bmatrix} \tag{42}$$

For n=1, the result in (41 and 42) is identical to the result in [37].

6 Conclusion

The Hamiltonian formulation for continuous systems with third order derivatives was investigated in this paper. Two cases are considered: (a) independent conjugate momenta, and (b) dependent conjugate momenta. For these systems, the Euler-

Lagrange equation is derived. They also, calculated the Hamiltonian for these systems and used functional derivatives to obtain the Hamiltonian equation of motion for these systems. Our findings are, as expected, the same as those obtained using the Euler-Lagrange method. Considering **case n ⟶ 1**, our results would be similar with those obtained in [37, 38]. This study concludes with some examples of applications and their practical implications. First, we look at how the functional -order derivative affects the shape and structure of the first-order relativistic wave equation with interacting fields, as well as states with definite energy-momentum and spin projections from the functional order Lee–Wick equation. Second, in higher-order electromagnetic theory, we use the functional derivative approach. To be compatible with special relativity, functional derivatives are formed by generalizing electrostatic laws, which are rules derived from the generalized Coulomb's law and the superposition principle. This accomplishment will almost certainly pave the way for further advances in functional relativistic quantum mechanics

Using this method, the energy of interaction between a stationary point charge and a conducting plate can be computed. This technique could be used to estimate overall interaction for two charged conducting parallel plates.

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Appendix A

1-Euler-Lagrange Equations of Lee- Wick Lagrangian Density

Let us start with the definition of Lee -Wick Lagrangian density and use the generalization formula of Euler – Lagrange equation (5) to obtain the equations of motion from Lee- Wick Lagrangian density.

Take the first field variable ϕ , then

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\phi}+-\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}-\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}+\partial_i\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\phi)} \\ +\partial_0\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\phi)}+\partial_0\partial_r\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\phi)}+\partial_i\partial_r\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\phi)} \\ -\partial_0\partial_0\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\partial_0\phi)}-\partial_0\partial_0\partial_f\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\partial_f\phi)}-\partial_0\partial_r\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_0\phi)} \\ -\partial_0\partial_r\partial_f\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_f\phi)}-\partial_i\partial_0\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_0\phi)}-\partial_i\partial_0\partial_f\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_f\phi)} \\ -\partial_i\partial_r\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0\phi)}-\partial_i\partial_r\partial_f\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_f\phi)} \end{bmatrix}=0 \qquad (A1)$$

and use the general formula (5) to obtain other equations of motion from the other fields' variables A^i, A^j

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial A^i}-\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 A^i)}-\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i A^i)}+\partial_i\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0 A^i)} \\ +\partial_0\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0 A^i)}+\partial_0\partial_r\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r A^i)}+\partial_i\partial_r\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r A^i)} \\ -\partial_0\partial_0\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\partial_0 A^i)}-\partial_0\partial_0\partial_f\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0\partial_f A^i)}-\partial_0\partial_r\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_0 A^i)} \\ -\partial_0\partial_r\partial_f\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_r\partial_f A^i)}-\partial_i\partial_0\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_0 A^i)}-\partial_i\partial_0\partial_f\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_0\partial_f A^i)} \\ -\partial_i\partial_r\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_0 A^i)}-\partial_i\partial_r\partial_f\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_r\partial_f A^i)} \end{bmatrix}=0 \qquad (A2)$$

And

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial A^j} - \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 A^j)} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A^j)} + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 A^j)} \\ + \partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 A^j)} + \partial_0 \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r A^j)} + \partial_i \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r A^j)} \\ - \partial_0 \partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 \partial_0 A^j)} - \partial_0 \partial_0 \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 \partial_f A^j)} - \partial_0 \partial_r \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_0 A^j)} \\ - \partial_0 \partial_r \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_f A^j)} - \partial_i \partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_0 A^j)} - \partial_i \partial_0 \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_f A^j)} \\ - \partial_i \partial_r \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 A^j)} - \partial_i \partial_r \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_f A^j)} \end{bmatrix} = 0 \qquad (A3)$$

1-Euler-Lagrangian Equation in terms of Functional derivative

Expanding equation (5), where $(\rho = 0, l)$, $(\mu = 0, f)$, $(\sigma = 0, r)\,, (\varepsilon = 0, i)\ and$ $\rho, \sigma, \varepsilon, \mu$ in terms of $(0, l), (0, r)\,, (0, i)\ and (0, f)$ respectively ,we get:

$$\iint \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_0} \delta\psi_0 + \frac{\partial \mathcal{L}}{\partial \psi_l} \delta\psi_l - \frac{\partial \mathcal{L}}{\partial(\partial_i \psi_0)} \partial_i \delta\psi_0 - \frac{\partial \mathcal{L}}{\partial(\partial_i \psi_l)} \partial_i \delta\psi_l + \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_p)} \partial_0 \delta\psi_p + \\ \frac{\partial \mathcal{L}}{\partial(\partial_0^2 \psi_p)} \partial_0^2 \delta\psi_p + \frac{\partial \mathcal{L}}{\partial(\partial_0^3 \psi_p)} \partial_0^3 \delta\psi_p + \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \psi_0)} \partial_i \partial_0 \delta\psi_0 \\ + \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \psi_l)} \partial_i \partial_0 \delta\psi_l + \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \psi_0)} \partial_0 \partial_r \delta\psi_0 + \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \psi_l)} \partial_0 \partial_r \delta\psi_l + \\ \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \psi_0)} \partial_i \partial_r \delta\psi_0 + \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \psi_l)} \partial_i \partial_r \delta\psi_l - \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 \partial_f \psi_0)} \partial_0 \partial_0 \partial_f \delta\psi_0 \\ - \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 \partial_f \psi_l)} \partial_0 \partial_0 \partial_f \delta\psi_l - \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_0 \psi_0)} \partial_0 \partial_r \partial_0 \delta\psi_0 - \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_f \psi_l)} \partial_0 \partial_r \partial_f \delta\psi_l \\ - \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_0 \psi_l)} \partial_0 \partial_r \partial_0 \delta\psi_l - \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_f \psi_0)} \partial_0 \partial_r \partial_f \delta\psi_0 - \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_0 \psi_0)} \partial_i \partial_0 \partial_0 \delta\psi_0 \\ - \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_0 \psi_l)} \partial_i \partial_0 \partial_0 \delta\psi_l - \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_f \psi_0)} \partial_i \partial_0 \partial_f \delta\psi_0 - \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_f \psi_l)} \partial_i \partial_0 \partial_f \delta\psi_l \\ - \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_0)} \partial_i \partial_r \partial_0 \delta\psi_0 - \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_l)} \partial_i \partial_r \partial_0 \delta\psi_l - \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_0)} \partial_i \partial_r \partial_0 \delta\psi_0 \\ - \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_l)} \partial_i \partial_r \partial_0 \delta\psi_l \end{bmatrix} d^3r\, dt \qquad (A4)$$

Integrating (A4) by parts with respect to space, we get:

$$\int dt \int \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_0} + -\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \psi_0)} + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \psi_0)} + \partial_0 \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \psi_0)} \\ + \partial_i \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \psi_0)} - \partial_0 \partial_0 \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 \partial_f \psi_0)} - \partial_0 \partial_r \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_0 \psi_0)} \\ - \partial_0 \partial_r \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_f \psi_0)} - \partial_i \partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_0 \psi_0)} - \partial_i \partial_0 \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_f \psi_0)} \\ - \partial_i \partial_r \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_0)} - \partial_i \partial_r \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_0)} \end{bmatrix} d^3r\, \delta\psi_0 +$$

$$\int dt \int \begin{bmatrix} +\frac{\partial \mathcal{L}}{\partial \psi_l} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \psi_l)} + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \psi_l)} + \partial_0 \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \psi_l)} + \partial_i \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \psi_l)} \\ -\partial_0 \partial_0 \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 \partial_f \psi_l)} - \partial_0 \partial_r \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_0 \psi_l)} - \partial_0 \partial_r \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_r \partial_f \psi_l)} \\ -\partial_i \partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_0 \psi_l)} - \partial_i \partial_0 \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_0 \partial_f \psi_l)} - \partial_i \partial_r \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_l)} \\ -\partial_i \partial_r \partial_f \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_l)} \end{bmatrix} d^3 r \delta \psi_l +$$

$$\int dt \int \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\rho)} \delta(\partial_0 \psi_\rho) + \int dt \int \frac{\partial \mathcal{L}}{\partial(\partial_0^2 \psi_\rho)} \delta(\partial_0^2 \psi_{0\rho}) + \int dt \int \frac{\partial \mathcal{L}}{\partial(\partial_0^3 \psi_\rho)} \delta(\partial_0^3 \psi_\rho) \quad (A5)$$

Appendix B

The Hamilton's equations of motion (all variables are independent ($\psi_\rho, \pi_1, \pi_2, \pi_3$)).

In this Appendix we obtain expressions B1, B2, B3 and B4 in terms of functional derivatives. First, comparing (25) and (26), we get the equations of motion in terms of Hamiltonian as:

$$\frac{\partial H}{\partial \psi_\rho} = -\dot{\pi}_1 + \ddot{\pi}_2 - \dddot{\pi}_3 \quad (B1)$$

By analogy with Eq. (9) for functional derivative of Lagrangian in terms of derivative of Lagrangian density, we can simply define the functional derivative of H in terms of derivative of Hamiltonian density with respect to the general variable field φ as

$$\frac{\partial H}{\partial \varphi_\rho} = \begin{bmatrix} \frac{\partial h}{\partial \varphi_0} + \frac{\partial h}{\partial \varphi_l} - \partial_i \frac{\partial h}{\partial(\partial_i \varphi_0)} - \partial_i \frac{\partial h}{\partial(\partial_i \varphi_l)} + \partial_0 \partial_r \frac{\partial h}{\partial(\partial_0 \partial_r \varphi_0)} \\ +\partial_0 \partial_r \frac{\partial h}{\partial(\partial_0 \partial_r \varphi_l)} + \partial_i \partial_0 \frac{\partial h}{\partial(\partial_i \partial_0 \varphi_0)} + \partial_i \partial_0 \frac{\partial h}{\partial(\partial_i \partial_0 \varphi_l)} \\ +\partial_i \partial_r \frac{\partial h}{\partial(\partial_i \partial_r \varphi_0)} + \partial_i \partial_r \frac{\partial h}{\partial(\partial_i \partial_r \varphi_l)} - \partial_0 \partial_0 \partial_f \frac{\partial h}{\partial(\partial_0 \partial_0 \partial_f \varphi_0)} \\ -\partial_0 \partial_0 \partial_f \frac{\partial h}{\partial(\partial_0 \partial_0 \partial_f \varphi_l)} - \partial_0 \partial_r \partial_0 \frac{\partial h}{\partial(\partial_0 \partial_r \partial_0 \varphi_0)} - \partial_0 \partial_r \partial_0 \frac{\partial h}{\partial(\partial_0 \partial_r \partial_0 \varphi_l)} \\ -\partial_0 \partial_r \partial_f \frac{\partial h}{\partial(\partial_0 \partial_r \partial_f \varphi_0)} - \partial_0 \partial_r \partial_f \frac{\partial \mathcal{H}}{\partial(\partial_0 \partial_r \partial_f \varphi_l)} - \partial_i \partial_0 \partial_0 \frac{\partial \mathcal{H}}{\partial(\partial_i \partial_0 \partial_0 \varphi_0)} \\ -\partial_i \partial_0 \partial_0 \frac{\partial h}{\partial(\partial_i \partial_0 \partial_0 \varphi_l)} - \partial_i \partial_0 \partial_f \frac{\partial h}{\partial(\partial_i \partial_0 \partial_f \varphi_0)} - \partial_i \partial_0 \partial_f \frac{\partial h}{\partial(\partial_i \partial_0 \partial_f \varphi_l)} \\ -\partial_i \partial_r \partial_0 \frac{\partial h}{\partial(\partial_i \partial_r \partial_0 \varphi_0)} - \partial_i \partial_r \partial_0 \frac{\partial h}{\partial(\partial_i \partial_r \partial_0 \varphi_l)} - \partial_i \partial_r \partial_f \frac{\partial h}{\partial(\partial_i \partial_r \partial_f)} \\ -\partial_i \partial_r \partial_f \frac{\partial h}{\partial(\partial_i \partial_r \partial_0 \varphi_l)} \end{bmatrix} \quad (B2)$$

Using the definition given in Eq. (B2) above, we can rewrite the equations of motion ($B1, B2, B3, B4$) in terms of Hamiltonian density such that

$$\begin{bmatrix} \frac{\partial h}{\partial \psi_0}+\frac{\partial h}{\partial \psi_l}-\partial_i\frac{\partial h}{\partial(\partial_i\psi_0)}-\partial_i\frac{\partial h}{\partial(\partial_i\psi_l)}+\partial_i\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\psi_0)}+\partial_i\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\psi_l)} \\ +\partial_0\partial_r\frac{\partial h}{\partial(\partial_0\partial_r\psi_0)}+\partial_0\partial_r\frac{\partial h}{\partial(\partial_0\partial_r\psi_l)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_i\partial_r\psi_0)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_i\partial_r\psi_l)} \\ -\partial_0\partial_0\partial_f\frac{\partial h}{\partial(\partial_0\partial_0\partial_f\psi_0)}-\partial_0\partial_0\partial_f\frac{\partial h}{\partial(\partial_0\partial_0\partial_f\psi_l)}-\partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0\partial_r\partial_0\psi_0)} \\ -\partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0\partial_r\partial_0\psi_l)}-\partial_0\partial_r\partial_f\frac{\partial h}{\partial(\partial_0\partial_r\partial_f\psi_0)}-\partial_0\partial_r\partial_f\frac{\partial h}{\partial(\partial_0\partial_r\partial_f\psi_l)} \\ -\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\partial_0\psi_0)}-\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\partial_0\psi_l)}-\partial_i\partial_0\partial_f\frac{\partial h}{\partial(\partial_i\partial_0\partial_f\psi_0)}- \\ \partial_i\partial_0\partial_f\frac{\partial h}{\partial(\partial_i\partial_0\partial_f\psi_l)}-\partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0\psi_0)}-\partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0\psi_l)} \\ -\partial_i\partial_r\partial_f\frac{\partial h}{\partial(\partial_i\partial_r\partial_f\psi_0)}-\partial_i\partial_r\partial_f\frac{\partial h}{\partial(\partial_i\partial_r\partial_f\psi_l)} \end{bmatrix} = -\dot{\pi}_1+\ddot{\pi}_2-\dddot{\pi}_3 \qquad (B6)$$

Appendix C

The Hamilton's equations of motion (all variables are dependent ($\psi_\rho, \pi_1, \pi_2, \pi_3$)).

In this Appendix, the conjugate momenta are field dependent**,** where π_1 depends on ψ_ρ, π_2 depends on $\dot{\psi}_\rho$ *and* π_3 depends on $\ddot{\psi}_\rho$.let us define $\pi_1 = g(\psi_\rho)$,$\pi_2 = f(\partial_0\psi_\rho)$ and $\pi_3 = k(\partial_0\partial_0\psi_\rho)$. So that, we can write their variations as:

$$\delta\pi_1 = \frac{\partial g}{\partial \psi_\rho}\delta\psi_\rho \qquad (C1)$$

$$\delta\pi_2 = \frac{\partial f}{\partial(\partial_0\psi_\rho)}\delta(\partial_0\psi_\rho) \qquad (C2)$$

$$\delta\pi_3 = \frac{\partial k}{\partial(\partial_0^2\psi_\rho)}\delta(\partial_0^2\psi_\rho) \qquad (C3)$$

Now, substituting Eqs.$(C1)$, $(C2)$ and $(C3)$ into Eq.(26) , and comparing with Eq.(11), we get the general equations of the Hamiltonian density for this case:

$$\begin{bmatrix} \frac{\partial h}{\partial \psi_0}+\frac{\partial h}{\partial \psi_l}-\partial_i\frac{\partial h}{\partial(\partial_i\psi_0)}-\partial_i\frac{\partial h}{\partial(\partial_i\psi_l)}+\partial_0\partial_r\frac{\partial h}{\partial(\partial_0\partial_r\psi_l)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_i\partial_r\psi_0)} \\ +\partial_i\partial_r\frac{\partial h}{\partial(\partial_i\partial_r\psi_l)}+\partial_0\partial_r\frac{\partial h}{\partial(\partial_0\partial_r\psi_l)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_i\partial_r\psi_0)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_i\partial_r\psi_l)} \\ -\partial_0\partial_0\partial_f\frac{\partial h}{\partial(\partial_0\partial_0\partial_f\psi_0)}-\partial_0\partial_0\partial_f\frac{\partial h}{\partial(\partial_0\partial_0\partial_f\psi_l)}-\partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0\partial_r\partial_0\psi_0)} \\ -\partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0\partial_r\partial_0\psi_l)}-\partial_0\partial_r\partial_f\frac{\partial h}{\partial(\partial_0\partial_r\partial_f\psi_0)}-\partial_0\partial_r\partial_f\frac{\partial h}{\partial(\partial_0\partial_r\partial_f\psi_l)} \\ -\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\partial_0\psi_0)}-\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\partial_0\psi_l)}-\partial_i\partial_0\partial_f\frac{\partial h}{\partial(\partial_i\partial_0\partial_f\psi_0)}- \\ \partial_i\partial_0\partial_f\frac{\partial h}{\partial(\partial_i\partial_0\partial_f\psi_l)}-\partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0\psi_0)}-\partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0\psi_l)} \\ -\partial_i\partial_r\partial_f\frac{\partial h}{\partial(\partial_i\partial_r\partial_f\psi_0)}-\partial_i\partial_r\partial_f\frac{\partial h}{\partial(\partial_i\partial_r\partial_f\psi_l)} \end{bmatrix} = -\dot{\pi}_1+\ddot{\pi}_2-\dddot{\pi}_3+\dot{\psi}_\rho\frac{\partial g}{\partial \psi_\rho} \qquad (C4)$$

$$\begin{bmatrix} \frac{\partial h}{\partial\partial_0\psi_0}+\frac{\partial h}{\partial\partial_0\psi_l}-\partial_i\frac{\partial h}{\partial(\partial_i\partial_0\psi_0)}-\partial_i\frac{\partial h}{\partial(\partial_i\partial_0\psi_l)}+\partial_i\partial_0\frac{\partial h}{\partial(\partial_i\partial_0^2\psi_0)}+\partial_i\partial_0\frac{\partial h}{\partial(\partial_i\partial_0^2\psi_l)} \\ +\partial_0\partial_r\frac{\partial h}{\partial(\partial_0^2\partial_r\psi_0)}+\partial_0\partial_r\frac{\partial h}{\partial(\partial_0^2\partial_r\psi_l)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_0\partial_i\partial_r\psi_0)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_0\partial_i\partial_r\psi_l)} \\ -\partial_0\partial_0\partial_f\frac{\partial h}{\partial\left(\partial_0^3\partial_f\psi_0\right)}-\partial_0\partial_0\partial_f\frac{\partial h}{\partial\left(\partial_0^3\partial_f\psi_l\right)}-\partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0^3\partial_r\psi_0)} \\ -\partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0^3\partial_r\psi_l)}-\partial_0\partial_r\partial_f\frac{\partial h}{\partial\left(\partial_0^2\partial_r\partial_f\psi_0\right)}-\partial_0\partial_r\partial_f\frac{\partial h}{\partial\left(\partial_0^2\partial_r\partial_f\psi_l\right)} \\ -\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0^3\psi_0)}-\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0^3\psi_l)}-\partial_i\partial_0\partial_f\frac{\partial h}{\partial\left(\partial_i\partial_0^3\partial_f\psi_0\right)}- \\ \partial_i\partial_0\partial_f\frac{\partial h}{\partial\left(\partial_i\partial_0^2\partial_f\psi_l\right)}-\partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0^2\psi_0)}-\partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0^2\psi_l)} \\ -\partial_i\partial_r\partial_f\frac{\partial h}{\partial\left(\partial_0\partial_i\partial_r\partial_f\psi_0\right)}-\partial_i\partial_r\partial_f\frac{\partial h}{\partial\left(\partial_0\partial_i\partial_r\partial_f\psi_l\right)} \end{bmatrix}=\partial_0^2\psi_\rho\frac{\partial f}{\partial\left(\partial_0\psi_\rho\right)} \qquad (C5)$$

$$\begin{bmatrix} \frac{\partial h}{\partial\partial_0^2\psi_0}+\frac{\partial h}{\partial\psi_l}-\partial_i\frac{\partial h}{\partial(\partial_0^2\partial_i\psi_0)}-\partial_i\frac{\partial h}{\partial(\partial_i\psi_l)}+\partial_0\partial_r\frac{\partial h}{\partial(\partial_0^3\partial_r\psi_l)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_0^2\partial_i\partial_r\psi_0)} \\ +\partial_i\partial_r\frac{\partial h}{\partial(\partial_0^2\partial_i\partial_r\psi_l)}+\partial_0\partial_r\frac{\partial h}{\partial(\partial_0^3\partial_r\psi_l)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_0^2\partial_i\partial_r\psi_0)}+\partial_i\partial_r\frac{\partial h}{\partial(\partial_0^2\partial_i\partial_r\psi_l)} \\ -\partial_0\partial_0\partial_f\frac{\partial h}{\partial\left(\partial_0^4\partial_f\psi_0\right)}-\partial_0\partial_0\partial_f\frac{\partial h}{\partial\left(\partial_0^4\partial_f\psi_l\right)}-\partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0^4\partial_r\psi_0)} \\ -\partial_0\partial_r\partial_0\frac{\partial h}{\partial(\partial_0^4\partial_r\psi_l)}-\partial_0\partial_r\partial_f\frac{\partial h}{\partial\left(\partial_0\partial_r\partial_f\psi_0\right)}-\partial_0\partial_r\partial_f\frac{\partial h}{\partial\left(\partial_0\partial_r\partial_f\psi_l\right)} \\ -\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0\partial_0\psi_0)}-\partial_i\partial_0\partial_0\frac{\partial h}{\partial(\partial_i\partial_0^4\psi_l)}-\partial_i\partial_0\partial_f\frac{\partial h}{\partial\left(\partial_i\partial_0^3\partial_f\psi_0\right)}- \\ \partial_i\partial_0\partial_f\frac{\partial h}{\partial\left(\partial_i\partial_0^3\partial_f\psi_l\right)}-\partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0^3\psi_0)}-\partial_i\partial_r\partial_0\frac{\partial h}{\partial(\partial_i\partial_r\partial_0^2\psi_l)} \\ -\partial_i\partial_r\partial_f\frac{\partial h}{\partial\left(\partial_0^2\partial_i\partial_r\partial_f\psi_0\right)}-\partial_i\partial_r\partial_f\frac{\partial h}{\partial\left(\partial_0^2\partial_i\partial_r\partial_f\psi_l\right)} \end{bmatrix}=\dddot{\psi}_\rho\frac{\partial g}{\partial\left(\partial_0^2\psi_\rho\right)} \qquad (C6)$$