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Solution of Gravitational Scalar Potential Exterior and Interior to the Circular-Cylindrical Bodies

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Abstract: In this study, we determine the gravitational scalar potential exterior and interior to the circular-cylindrical bodies by applying the conditions of continuity across all boundaries and normal derivation. Our proposed scalar potential could be used in the limit of weak field to study the energy of system in space along cylindrical coordinate. The reliability of our results still maintains the lead Newton Dynamical Theory of Gravity (NDTG) over all gravitational theory and we hope in the nearest future the practicability of these results shall soon be discovered and applied. This equation contains the density term contribution, which is Inverse Square of the scalar potential which can further be explore. The square density contribution is the contributory factor to the cylindrical satellite's stability in space, which may likely account for the anomalous over stayed of Synchom satellites in its parking orbit. It is therefore recommended that the results from this research work be applied to some of the bodies in the universe that are circular-cylindrical in shape in other to obtain their field equation and hence the planetary parameters such as their orbital eccentricity which can be applied in the study of weather condition.

Keywords: Gravitation, circular-cylindrical, spacetime, soliton, general relativity.

1 Introduction

Initially, the world of Physics was assumed on the fact that, Earth is spherical in nature [1]. Research in recent years have shown that, there are about twenty-six (26) coordinate system in Physics and Mathematics. Such systems include; spherical, spheroidal, oblate, prolate, cylindrical, circular-cylindrical etc. [2].

It is worthy to note that, despite the far-reaching studies of cylindrical space-times in the recent decades, there exist a gap in the literature that led to confusions in the definition of cylindrically symmetric space-times and false claims [3]. For instance, it has been claimed by various authors that the Kompaneets-Jordan-Ehlers-Kundt (KJEK) metric is the most general form that describes cylindrically symmetric space-times [4]. However, it is clear that this metric does not include the rotating cylindrical gravitational wave (GW) space-times studied by Mashhoon and Quevedo [5].

To clarify these issues, it is necessary first to give a rigorous definition of the circular cylindrically symmetric space-times. By rigorously defining cylindrically symmetric space-times [6,7]. clarify various (incorrect) claims existing in the literature, regarding to the generality of such space-times but they were not able to extend it to include other bodies such as circular-cylindrical to study the dynamical theory of gravity. Hence, it is against these frameworks that this research is set to derive the Riemannian field equation for circular-cylindrical coordinates in gravitational field [8,9].
A research was previously reported by [10,11] on the construction of field equations for circular cylindrical bodies, and the research was able to derive the gravitational scalar potential exterior and interior to the circular cylindrical body, but

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unfortunately was unable to solution to the equation of gravitational scalar potential exterior and interior to the circular-cylindrical bodies. Hence, in this study, we determine the equation of gravitational scalar potential exterior and interior to the circular-cylindrical bodies by applying the conditions of continuity across all boundaries and normal derivation.

2 Methodology

Maisalatee *et al*., (2022) previously reported that, by the symmetry of the distribution of density it follows that, the gravitational field depend only on the rho coordinate ρ, then $f(\rho,\phi,z)=f(\rho)$. So that, the equation

$$\nabla_R^2 f(\rho,\varphi,z)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(1+\frac{2f(\rho,\varphi,z)}{c^2}\right)\frac{\partial f(\rho,\varphi,z)}{\partial\rho}\right]+\frac{1}{\rho^2}\frac{\partial^2 f(\rho,\varphi,z)}{\partial\varphi^2}+\frac{\partial^2 f(\rho,\varphi,z)}{\partial z^2}+\frac{1}{c^2}\frac{\partial}{\partial t}\left[-\left(1+\frac{2f(\rho,\varphi,z)}{c^2}\right)^{-1}\frac{\partial f(\rho,\varphi,z)}{\partial t}\right]=\begin{cases}0; & \rho>R\\ 4\pi G\rho_0; & \rho<R\end{cases} \tag{1}$$

Which can be reduces to

$$\nabla_R^2 f(\rho)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(1+\frac{2f(\rho,\varphi,z)}{c^2}\right)\frac{\partial f(\rho)}{\partial\rho}\right]=\begin{cases}0; & \rho>R\\ 4\pi G\rho_0; & \rho<R\end{cases} \tag{2}$$

where,

$\rho=$ radius

∇_R^2=Riemannian Laplacian operator

$f(\rho,\phi,z,x^0)$ equals the complete gravitational scalar potential exterior to the bodies.

And by simplification, equation (2) becomes

$$\nabla_R^2 f(\rho)=\frac{1}{\rho}\begin{bmatrix}\left(1+\frac{2f(\rho,\varphi,z)}{c^2}\right)f^I(\rho)+\frac{2\rho}{c^2}(f^I(\rho))^2\\ +\rho\left(1+\frac{2f(\rho,\varphi,z)}{c^2}\right)f^{II}(\rho)\end{bmatrix}=\begin{cases}0; & \rho>R\\ 4\pi G\rho_0; & \rho<R\end{cases} \tag{3}$$

Or explicitly as

$$f^{II}+\frac{1}{\rho}f^I+\frac{2}{c^2}ff^{II}+\frac{2}{c^2}ff^I+\frac{2}{c^2}(f^I)^2=\begin{cases}0; & \rho>R\\ 4\pi G\rho_0; & \rho<R\end{cases} \tag{4}$$

which is the dynamical gravitational field equation for circular-cylindrical bodies.

In this work, we can now extend further by using the dynamical gravitational field equation for circular-cylindrical bodies to seek the solution of the exterior and interior field equation.

Assume the solution for the exterior field equation is given by

$$f(\rho)=\frac{A_1}{\rho}+\frac{A_2}{\rho^2}+\cdots \tag{5}$$

$$f'(\rho)=-\frac{A_1}{\rho^2}-\frac{2A_2}{\rho^3}+\cdots \tag{6}$$

$$f''(\rho)=\frac{2A_1}{\rho^3}+\frac{6A_2}{\rho^4}+\cdots \tag{7}$$

Substituting equations (5) – (7) into left hand side of equation (4) and simplify, we obtained

$$\frac{A_1}{\rho^3}+\frac{4A_2}{\rho^4}+\frac{2{A_1}^2}{c^2\rho^4}+\frac{18A_1A_2}{c^2\rho^5}+\frac{2{A_1}^2}{c^2\rho^5}+\frac{16{A_2}^2}{c^2\rho^6}=\begin{cases}0; & \rho>R\\ 4\pi G\rho_0; & \rho<R\end{cases} \tag{8}$$

where, A_1, and A_2 are arbitrary constants.

Hence, equating the terms of order ρ^{-3}, $\rho^{-4}$and ρ^{-5}, ρ^{-6}we got

$$A_1 = 0 \quad \text{(arbitrary constant)} \tag{9}$$

Similarly, solving equation (8), gives

$$A_2 = -\frac{1}{2}\frac{A_1^2}{C^2} \tag{10}$$

Substituting equations (10) into equation (5), we obtained

$$f^+(\rho) = \frac{A_1}{\rho} - \frac{A_2}{2C^2\rho^2} \tag{11}$$

Equation (11) is the exterior field equation of equation (4)

We assume the complementary and particular solution of equation (4) to be

$$f_G{}^-(\rho) = f_c{}^-(\rho) + f_p{}^-(\rho) \tag{12}$$

where,

$$f_c{}^-(\rho) = B_0 \tag{13}$$

To seek the particular solution,

Let

$$f_p{}^-(\rho) = D_2\rho^2 + D_4\rho^4 \tag{14}$$

where D_2 and D_4 are constants

Differentiating equation (14) with respect to ρ gives (15) and (16)

$$f_p{}'(\rho) = 2D_2\rho + 4D_4\rho^3 + \cdots \tag{15}$$

$$f_p{}''(\rho) = 2D_2 + 12D_4\rho^2 + \cdots \tag{16}$$

where D_2 and D_4 are arbitrary constants

By substituting equation (15) and (16) into equation (8) and simplifying, we got

$$2D_2 + 12D_4\rho^2 + \frac{1}{\rho}[2D_2\rho + 4D_4\rho^2] + 4D_2^2\rho^2 + \frac{28D_2D_4\rho^4}{C^2} + \frac{24D_4^2\rho^6}{C^2} + \frac{4D_2^2\rho^2}{C^2} + \frac{12D_2D_4\rho^4}{C^2} + \frac{8D_4^2\rho^6}{C^2} + \frac{8D_2^2\rho^2}{C^2} + \frac{32D_4^2\rho^6}{C^2} + \frac{32D_2D_4\rho^4}{C^2} = 4\pi G\rho_{0;} \tag{17}$$

Equating the constant term gives

$$D_2 = \pi G\rho \tag{18}$$

but

$$\rho = \frac{m}{V} \tag{19}$$

and

Therefore, equation (18) becomes

$$D_2 = \frac{Gm}{r^2 h} \tag{20}$$

where,

ρ is the density

m equals the mass of the body

v equals the volume of the circular cylindrical

$k = Gm$

Solving for D_4 in terms of D_2, we obtained

$$D_4 = -\frac{k^2}{4r^4\,h^2} - \frac{3k^2}{4r^4\,h^2\,c^2} \tag{21}$$

Using the well- known physical relationship between mass and density for a cylinder of radius r and by substituting equations (20) and (21) into (14) yielded

$$f^- = B_0 + \frac{k\rho^2}{r^2 h} - \frac{k^2\rho^4}{4r^4\,h^2} - \frac{3k^2\rho^4}{4r^4\,h^2\,c^2} \tag{22}$$

By the condition of continuity of gravitational scalar potential across boundaries,

$$B_0 + D_2\rho^2 + D_4\rho^4 + \cdots = \frac{A_1}{\rho} + \frac{A_2}{\rho^2} + \frac{A_3}{\rho^3} + \cdots \tag{23}$$

By the condition of continuity of normal derivatives across all boundaries

$$\left(\frac{\partial f^+}{\partial \rho}\right)_{\rho=R} = \left(\frac{\partial f^-}{\partial \rho}\right)_{\rho=R}$$

Hence by differentiating equation (23) w. r. t. ρ and simplifying, we obtained

$$2D_2\rho + 4D_4\rho^3 = -\frac{A_1}{\rho^2} - \frac{A_1^2}{C^2\rho^3} \tag{24}$$

Multiplying through by $c^2\rho^3$ gives

$$2c^2\rho^4 D_2 + 4c^2\rho^6 D_4 + c^2\rho A_1 + A_1^2 = 0 \tag{25}$$

Equation (25) can be written as

$$\alpha_1 A_1^2 + \alpha_2 A_1 + \alpha_3 = 0 \tag{26}$$

where,

$$\alpha_1 = 1 \tag{27}$$

$$\alpha_2 = c^2\rho \tag{28}$$

$$\alpha_3 = 2c^2\rho^4 D_2 + 4c^2\rho^6 D_4 \tag{29}$$

Solving equation (26) quadratically we obtained

$$A_1 = \frac{-c^2\rho}{2} \pm \frac{c^2\rho}{2}\left[1 + \frac{4D_2\rho^2}{c^2} + \frac{8D_4\rho^4}{c^2}\right] \tag{30}$$

Equation (30) reduces to

$$A_1 = 2D_2\rho^3 + 4D_4\rho^5 \tag{31}$$

or

$$A_1 = -c^2\rho - 2D_2\rho^3 + 8D_4\rho^5 \tag{32}$$

Substituting Equation (31) into (10) we obtained

$$A_2 = -\frac{2D_2^2\rho^6}{c^2} - \frac{8D_2D_4\rho^8}{c^2} - \frac{16D_4^2\rho^{10}}{c^2} \tag{33}$$

Substituting equation (20) and (21) into (31) we obtained

$$A_1 = \frac{2Gm}{r^2h}\left[\frac{m^3}{\pi^3 r^6 h^3}\right] + 4\left[-\frac{G^2m^2}{4r^4h^2} - \frac{G^2m^2}{4r^4h^2C^2}\right]\left[\frac{m^5}{\pi^5 r^{10} h^5}\right] \tag{34}$$

Let $k = Gm$, so that

$$A_1 = \frac{2k}{r^2h}\rho^3 - \frac{k^2}{r^4h^2}\rho^5 - \frac{3k^2}{r^4h^2C^2}\rho^5 \tag{35}$$

Substituting for A_2 we obtained

$$A_2 = -\frac{2}{c^2}\frac{k^2}{r^2h}\rho^6 - \frac{8}{c^2}\left[\frac{k}{r^2h}\right]\left[-\frac{k^2}{4r^4h^2} - \frac{3k^2}{4r^4h^2c^2}\right]\rho^{10} \tag{36}$$

By simplification, equation (36) become

$$A_2 = -\frac{2k^2\rho^6}{r^4h^2c^2} + \frac{2k^3\rho^{10}}{r^6h^3c^2} \tag{37}$$

Substituting equations (35) and (37) into (5) yielded

$$f = \frac{2k\rho^2}{r^2h} - \frac{k^2\rho^4}{r^4h^2} - \frac{3k^2\rho^4}{r^4h^2c^2} + \frac{2k^3\rho^8}{r^6h^3c^2} - \frac{2k^2\rho^4}{r^4h^2c^2} \tag{38}$$

Equation (38) is the exterior Scalar Potential for circular cylindrical bodies. This equation can be used to study motion of particles in circular-cylindrical coordinate.

Recall that, equation (23) is given as,

$$B_0 + D_2\rho^2 + D_4\rho^4 + \cdots = \frac{A_1}{\rho} + \frac{A_2}{\rho^2} + \frac{A_3}{\rho^3} + \cdots$$

Making B_0 the subject of the relation gives

$$B_0 = \frac{A_1}{\rho} + \frac{A_2}{\rho^2} - D_2\rho^2 + D_4\rho^4 \tag{39}$$

Substituting equations (20), (21), (35) and (37) into (39) yielded

$$B_0 = \frac{2k}{r^2h}\rho^2 - \frac{k}{r^2h}\rho^2 - \frac{k^2}{r^4h^2}\rho^4 - \frac{5k^2}{r^4h^2}\rho^4 - \frac{k^2}{4r^4h^2}\rho^4 - \frac{3k^2}{4r^4h^2c^2}\rho^4 + \frac{2k^3}{4r^6h^2c^2}\rho^8 \tag{40}$$

By simplifying the like terms, we obtained

$$B_0 = \frac{k}{r^2h}\rho^2 - \frac{5k^2}{4r^4h^2}\rho^4 - \frac{23k^2}{4r^4h^2c^2}\rho^4 + \frac{2k^3}{r^6h^3}\rho^8 \tag{41}$$

Substituting equation (41) into (32) yielded

$$f^- = \frac{k\rho^2}{r^2h} + \frac{k\rho^2}{r^2h} - \frac{3k^2\rho^4}{2r^4\,h^2} - \frac{13k^2\rho^4}{2r^4\,h^2\,c^2} + \frac{3k^3\rho^8}{r^6\,h^3\,c^2} \tag{42}$$

Equation (42) is the interior Scalar Potential for circular-cylindrical bodies. This equation can be used to study motion of particles in circular-cylindrical coordinate.

3 Conclusions

Equation (38) is the post Riemannian-Newton dynamical Scalar Potential exterior for circular cylindrical coordinate while Equation (42) is the interior Scalar Potential for circular cylindrical bodies. This equation can be used to study motion of particles in circular cylindrical coordinate. Equation (38) is indeed a profound discovery most especially in this age of high space satellite technology with circular cylindrical coordinate such as Mari-sat and Synchom satellite. It could be observed that, our obtained result in equation (38) reduces to the limit of weak field to the Newton Dynamical Scalar Potential (NDSP) which indeed a profound discovery, thus equivalent principle of physics is fully established with the new proposed Great Metric Tensor (GMT). Our method is thus reliable and accurate. We are theoretically satisfied with the formulation of natural laws with this proposed GMT. Our proposed scalar potential could be used in the limit of weak field to study the energy of system in space along cylindrical coordinate. The reliability of our results still maintains the lead Newton Dynamical Theory of Gravity (NDTG) over all gravitational theory and we hope in the nearest future the practicability of these results shall soon be discovered and applied. This equation contains the density term contribution, which is inverse square of the scalar potential which can further be explore. The square density contribution is the contributory factor to the cylindrical satellite's stability in space, which may likely account for the anomalous over stayed of Synchom satellites in its parking orbit.

It is therefore recommended that the results from this research work be applied to some of the bodies in the universe that are circular-cylindrical in shape in other to obtain their field equation and hence the planetary parameters such as their orbital eccentricity which can be applied in the study of weather condition.

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