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Error Analysis of the Generalized Jacobi Galerkin Method in Nonlinear Fractional Differential Equations

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Abstract: In this paper, the generalized Jacobi Galerkin (GJG) method is analyzed for the numerical solution of nonlinear multi-order fractional differential equations(FDEs). We consider the generalized Jacobi functions(GJFs) with indexes corresponding to the number of homogeneous initial conditions as natural basis functions for the GJG approximation. The unique solvability of the resulting nonlinear algebraic system, as well as convergence properties of the proposed approach, are discussed. The validity of the method is demonstrated with some illustrative examples.

Keywords: Fractional differential equations (FDEs), Caputo derivative, generalized Jacobi functions(GJFs), Galerkin method, numerical solvability.

1 Introduction

Fractional calculus is the theory of non-integer differential and integral operators, and particularly to differential equations containing such operators. Nowadays, engineers and scientists have developed models involving fractional differential and integral equations. These models have been applied successfully, e.g., in mechanics (theory of viscoelasticity and viscoplasticity), bio-chemistry (modeling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modeling of human tissue under mechanical loads), etc. Some early examples are given in the book of Oldham and Spanier [1], and the classical papers of Bagley and Torvik [2], Caputo [3], and Caputo and Mainardi [4].

It should be mentioned that, from the viewpoint of applications in sciences, the book written by Oldham and Spanier [1], played a notable role in the development of the applied fractional calculus. Later there appeared some fundamental works by Podlubny [5], Kilbas, Srivastava, Trujillo [6], and Diethelm [7] on various aspects of the fractional calculus. So far, several numerical and analytical methods for solving differential equations of fractional order have been developed. These include the Variational Iteration method [8], the Adomian Decomposition method [9], Generalized Differential Transform method [10], etc.

Spectral Galerkin method is one of the weighted residual methods (WRM), in which approximations are defined in terms of truncated series expansions [11,12], such that residual which should be exactly equal to zero, is forced to be zero only in an approximate sense. In this method, the expansion functions satisfy the supplementary conditions. The two main characteristics behind the approach are that firstly this method reduces the given problems to those of solving a system of algebraic equations, thus greatly simplify the problems, and secondly, in general converges exponentially and almost always supplies the tersest representation of a smooth solution [13,14].

In recent years, spectral methods have been studied by many authors to approximate the solutions of FDEs. In [15, 16], spectral Tau method for the numerical solution of such equations is developed. A quadrature Tau method for the numerical solution of FDEs with variable coefficients is investigated in [17]. Moreover, Pedas and Tamme [18] introduced a new efficient spline collocation method for solving such equations.

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As a matter of interest, it is remarked that the basis functions' choice is essential for the superior approximation properties of spectral methods. In this context, Chen et al., in [19] introduced a new family of basis functions called generalized Jacobi functions (GJFs) that are proper choice as basis functions of the Galerkin approximation of initial value problems. These functions are mutually orthogonal concerning the corresponding weight function. They are defined by eliminating the constraint $\alpha,\beta > -1$ in the classical Jacobi polynomials and inherit some important properties. It proved that GJFs with indexes corresponding o the number of homogeneous initial conditions in a given differential equation is the natural basis functions for its Galerkin approximation. Representation of solution as a linear combination of these functions reduces the complexity of the resulting algebraic system due to eliminating the constraint of substituting the approximate solution in the supplementary conditions. These advantages have attracted the attention of many researchers to use these functions in the Galerkin approximation of various functional equations. For example, in [20,21] authors applied GJFs to develop spectral solutions for linear and nonlinear fourth and fifth-order boundary value problems. In [22], the spectral-Petrov-Galerkin methods for the integrated forms of the third- and fifth-order elliptic differential equations are introduced using general parameters GJFs. In [23], authors used GJFs in producing Galerkin solution of nonlinear fractional differential-algebraic equations.

In this article, numerical analysis of nonlinear FDEs

$$L_D(u(x)) = f(x), \quad x \in \Lambda = [0,1], \tag{1}$$

with initial conditions

$$u^{(i)}(0) = 0, \qquad i = 0,1,...,\nu-1, \tag{2}$$

based ongeneralized Jacobi functions(GJFs) is studied. The unique numerical solvability and convergence analysis of the proposed method are also discussed. This discussion is mainly based on Krasnosel'skii [24] and Vainikko [25]. Here, we have

$$L_D(u(x)) = \sum_{r=0}^{N_d} p_r(x)\left(D_C^{\theta_r} u(x)\right)^{\gamma_r}, \qquad \theta_r \in \mathbb{Q}^+ \cup \{0\},\ \gamma_r, N_d \in \mathbb{N}, \tag{3}$$

where $\theta_{N_d} > \theta_{N_d-1} > ... > \theta_0 \geq 0$, with $\nu = \lceil \theta_{N_d} \rceil$, and $\mathbb{Q}^+, \mathbb{N}$ are the collections of all the positive rational and natural numbers respectively. We suppose $p_r(x), f(x)$ are given continuous functions and $u(x)$ is the unknown that is sufficiently smooth.

Here, the fractional derivative is considered in the Caputo sense

$$D_C^q u(x) = I^{\lceil q \rceil - q} u^{(\lceil q \rceil)}, \tag{4}$$

where $q \in \mathbb{Q}^+$, the symbol $\lceil q \rceil$ is the smallest integer greater than or equal to q and I^μ is the Fractional integral operator, defined by the formula

$$I^\mu u(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-\tau)^{\mu-1} u(\tau) d\tau. \tag{5}$$

$\Gamma(.)$ is the Gamma function, and the properties of operator D_C^q can be found in [7]. We recall one of them as

$$D_C^q\, x^k = \begin{cases} \frac{k!}{\Gamma(k-q+1)}\, x^{k-q}, & k \in \mathbb{N} \text{ and } k \geq \lceil q \rceil \text{ or } k \notin \mathbb{N} \text{ and } k > \lceil q \rceil, \\ 0, & k \in \mathbb{N} \text{ and } k < \lceil q \rceil. \end{cases} \tag{6}$$

This article is organized as follows: In Section 2, we introduce some properties of GJFs required for our subsequent development. Section 3, presents the Galerkin approximation using GJFs to solve (1 - 2). In Section 4, the numerical solvability of the resulting nonlinear algebraic system, as well as error analysis of the proposed approach, are discussed. In Section 5, the proposed method is applied to several numerical examples to clarify the efficiency of the method.

2 Basic Properties of Generalized Jacobi Functions

In this section, firstly we give some definitions related to the L^2 space, and afterward, we recall the GJFs from [19], and investigate their basic properties, which will be needed in the sequel.

Let $w^{(\alpha,\beta)}(x) = (2-2x)^\alpha (2x)^\beta$, with parameters α, β, be the shifted Jacobi weight function on Λ. The weighted L^2-norm of a function u over Λ is given by

$$\|u\|_{L^2_{\alpha,\beta}(\Lambda)} = \left(\int_\Lambda |u(x)|^2 w^{(\alpha,\beta)}(x) dx\right)^{\frac{1}{2}},$$

and the space of all continuous functions on Λ is denoted by $C(\Lambda)$. In addition, the inner product formula is defined by

$$\left(u,v\right)_{\alpha,\beta}=\int_{\Lambda}u(x)v(x)w^{(\alpha,\beta)}(x)dx.$$

The shifted GJFs is defined as follows

$$\mathbf{J}_n^{(\alpha,-\beta)}(x)=(2x)^{\beta}J_n^{(\alpha,\beta)}(x),\ \ x\in\Lambda,\ \alpha,\ \beta>-1,\ n\geq 0, \tag{7}$$

where $J_n^{(\alpha,\beta)}(x)$ is the classical shifted Jacobi polynomials on Λ [13].

From the relation (3.13) in [19], it can be obtained

$$D_C^q\mathbf{J}_n^{(\alpha,-\beta)}(x)=\frac{2^q\Gamma(n+\beta+1)}{\Gamma(n+\beta-q+1)}\mathbf{J}_n^{(\alpha+q,-\beta+q)}(x),\ \ \beta>q-1. \tag{8}$$

An important property of the shifted GJFs on Λ is that

$$\partial_x^i\mathbf{J}_n^{(0,-\nu)}(0)=0,\quad i=0,1,...,\nu-1. \tag{9}$$

Hence we can consider $\{\mathbf{J}_n^{(0,-\nu)},\ n\geq 0\}$ as suitable basis functions in Galerkin solution of differential equations with ν homogeneous initial conditions on Λ.

An important fact is that the shifted GJFs $\{\mathbf{J}_n^{(0,-\nu)};n\geq 0\}$ form a complete orthogonal system in $L^2_{0,-\nu}(\Lambda)$ [19]. So we define

$$P_N^{0,-\nu}=\mathrm{Span}\{\mathbf{J}_0^{(0,-\nu)},\mathbf{J}_1^{(0,-\nu)},...,\mathbf{J}_{N-\nu}^{(0,-\nu)}\},$$

and consider the orthogonal projection $\Pi_N^{0,-\nu}:L^2_{0,-\nu}(\Lambda)\rightarrow P_N^{0,-\nu}$ defined by

$$\left(u-\Pi_N^{0,-\nu}u,v_N\right)_{0,-\nu}=0,\qquad \forall v_N\in P_N^{0,-\nu}.$$

3 The Generalized Jacobi Galerkin Method

In this section, we are concerned with the formulation of GJG method for solving the nonlinear FDEs (1 - 2). In the Galerkin method we seek a polynomial solution $u_N(x)$ of the form

$$u_N(x)=\sum_{i=0}^{N-\nu}a_i\mathbf{J}_i^{(0,-\nu)}(x), \tag{10}$$

where we have $u_N^{(i)}(0)=0$ for $i=0,1,...,\nu-1$.

The $N-\nu+1$ equations for the unknown expansion coefficients $\{a_i\}_{i=0}^{N-\nu}$, are determined from $(1-3)$, by requiring that, the residual

$$R_N(x)=L_D(u_N(x))-f(x), \tag{11}$$

to be orthogonal to$P_N^{0,-\nu}$. In other words, the Galerkin formulation of (1-3) is finding $u_N(x)\in P_N^{0,-\nu}$, such that

$$\left(R_N(x),\mathbf{J}_s^{(0,-\nu)}(x)\right)_{0,-\nu}=0,\quad 0\leq s\leq N-\nu,$$

or

$$\int_{\Lambda}\Big(L_D(u_N(x))-f(x)\Big)\mathbf{J}_s^{(0,-\nu)}(x)w^{(0,-\nu)}(x)dx=0,\quad 0\leq s\leq N-\nu. \tag{12}$$

Since $\mathbf{J}_s^{(0,-\nu)}(x)w^{(0,-\nu)}(x)=J_s^{(0,\nu)}(x)$, the relations (3) and (12), concludes

$$\int_{\Lambda}\Big(\sum_{r=0}^{N_d}p_r(x)\Big(D_C^{\theta_r}u_N(x)\Big)^{\gamma_r}-f(x)\Big)J_s^{(0,\nu)}(x)dx=0,\quad 0\leq s\leq N-\nu.$$

From (10) we have

$$\int_{\Lambda}\Big(\sum_{r=0}^{N_d} p_r(x)\Big(\sum_{i=0}^{N-\nu} a_i D_C^{\theta_r}\mathbf{J}_i^{(0,-\nu)}(x)\Big)^{\gamma_r}\Big)J_s^{(0,\nu)}(x)dx=\int_{\Lambda} f(x)J_s^{(0,\nu)}(x)dx,\quad 0\le s\le N-\nu. \tag{13}$$

The relations (8), and (7) yield

$$D_C^{\theta_r}\mathbf{J}_i^{(0,-\nu)}(x)=\frac{2^{\nu}\Gamma(i+\nu+1)}{\Gamma(i+\nu-\theta_r+1)}x^{\nu-\theta_r}J_i^{(\theta_r,\nu-\theta_r)}(x)=:\Theta_{i,\nu,r}(x). \tag{14}$$

Inserting (14) into (13) gives

$$\int_{\Lambda}\Big(\sum_{r=0}^{N_d} p_r(x)\Big(\sum_{i=0}^{N-\nu} a_i\Theta_{i,\nu,r}(x)\Big)^{\gamma_r}\Big)J_s^{(0,\nu)}(x)dx=\int_{\Lambda} f(x)J_s^{(0,\nu)}(x)dx,\quad 0\le s\le N-\nu. \tag{15}$$

Using a $(N+1)$-point Gauss-Lobatto Legendre quadrature formula, the integration terms in (15) can be approximated as

$$\sum_{j=0}^{N}\Big(F(x_j)-f(x_j)\Big)J_s^{(0,\nu)}(x_j)w_j=0,\quad 0\le s\le N-\nu, \tag{16}$$

where

$$F(x)=\sum_{r=0}^{N_d} p_r(x)\Big(\sum_{i=0}^{N-\nu} a_i\Theta_{i,\nu,r}(x)\Big)^{\gamma_r},$$

and the set $\{x_i,w_i\}_{i=0}^{N}$ coincides with the Gauss-Lobatto Legendre collocation points and corresponding weights on Λ, respectively. In this position, we have a nonlinear algebraic system, which when solved gives us unknown coefficients $\{a_i\}_{i=0}^{N-\nu}$ in (10).

4 Existence, Uniqueness and Convergence Results

In this section, the unique numerical solvability, and convergence properties of GJG approach for a special case of (1-2) with $\theta_{N_d}=\nu,\gamma_{N_d}=1$ and $p_{N_d}(x)=1$, as

$$\begin{cases} u^{(\nu)}(x)+\sum_{r=0}^{N_d-1} p_r(x)\Big(D_C^{\theta_r}u(x)\Big)^{\gamma_r}=f(x)\\ u^{(i)}(0)=0,\quad i=0,1,...,\nu-1,\end{cases} \tag{17}$$

are investigated.

We recall some fundamental definitions and theorems which are used later. Let E and F be two normed spaces. The operator $T:(E,\|.\|_E)\to(F,\|.\|_F)$ is called continuous if it transforms every sequence of elements which is convergent with respect to the norm $\|.\|_E$ into a sequence of elements which is convergent concerning the norm $\|.\|_F$. The operator T is called bounded if it transforms every bounded set of elements in E into a bounded set in F. The operator T, is called compact if it transforms every bounded set into a compact set and it is completely continuous if it is continuous and compact. This operator is differentiable in the Fréchet sense at the point $v\in E$, if there exists a bounded linear operator $T':(E,\|.\|_E)\to(F,\|.\|_F)$ such that

$$Tw-Tv=T'(v)(w-v)+\omega(w,v),$$

where

$$\frac{\|\omega(w,v)\|_F}{\|w-v\|_E}\to 0\quad\text{as}\quad w\to v,$$

and the operator $T'(v)\in\mathscr{L}(E,F)$ is called the Fréchet derivative of the operator T, at v. Here, $\mathscr{L}(E,F)$ is the Banach space of bounded linear operators from E to F. It can be shown that, the Fréchet derivative of a completely continuous operator is a completely continuous linear operator [24,25].

We shall say that the operator T is continuously differentiable at v_0, if it is differentiable at each point of some neighborhoods of the point v_0, and

$$\|T'(v)-T'(v_0)\|_F\to 0,\quad\text{as}\quad\|v-v_0\|_E\to 0.$$

Now we assume that $\mathscr{T}$ and $\mathscr{T}_N$ are presented in the form $\mathscr{T}=PT$, $\mathscr{T}_N=P_NT$, where T is a nonlinear completely continuous operator, and $P,P_N:F\to E$ are linear continuous operators. The sequence of operators P_N is assumed to converge strongly to the operator P, i.e., for any $w\in F$ we have

$$P_Nw\to Pw, \quad \text{as} \quad N\to\infty.$$

The following Theorem is valid([24,25]):

Theorem 1 *1. Let equation*

$$v=\mathscr{T}v, \tag{18}$$

has a unique solution v_0 with non-zero index in the ball $\|v-v_0\|_E\le\delta$. Then the equation

$$v=\mathscr{T}_Nv, \tag{19}$$

has a solution $v_N(\|v_N-v_0\|_E\le\delta)$, for sufficiently large values of N, and $v_N\to v_0$ as $N\to\infty$.

2. Let the operator T is differentiable at v_0, and the homogeneous equation $v-PT'(v_0)v=0$ has only a trivial solution $v=0$, then following inequality holds

$$\|v_N-v_0\|_E\le M\|(P_N-P)Tv_0\|_E.$$

3. Assume the operator T be differentiable at v_0 continuously, then the solution v_N is unique in the ball $\|v_N-v_0\|_E\le\delta$, for sufficiently large N.

Note that if v_0 is a solution of (18), and the operator T is differentiable at this point, where the homogeneous equation $v-PT'(v_0)v=0$ has only a trivial solution $v=0$, then v_0 is an isolated solution of equation (18) with non-zero index [24, 25]. Thus we can accumulate the conditions 1 and 2 in Theorem 1, and give the following simpler form.

Theorem 2 *Assume the equation (18) has a solution v_0, the operator T is differentiable at v_0, and the homogeneous equation $v-PT'(v_0)v=0$ has only a trivial solution $v=0$. Then the equation (19) has a solution v_N ($\|v_N-v_0\|_E\le\delta$), for sufficiently large N, and $v_N\to v_0$ as $N\to\infty$.*

Also the speed of convergence can be evaluated by the inequality

$$\|v_N-v_0\|_E\le M\|(P_N-P)Tv_0\|_E.$$

In addition, if the operator T be differentiable at v_0 continuously, then the solution v_N is unique in the ball $\|v_N-v_0\|_E\le\delta$, for sufficiently large N.

In the sequel, the existence, uniqueness, and convergence results of GJG approach for solving (17) is justified. Our strategy is based on Theorem 2. To this end, we first show that the equation (17) can be written as (18) with a completely continuous operator $\mathscr{T}$, and also its GJG method can be represented as (19) with a suitable projection operator P_N. Next, we apply Theorem 2 to ensure the desired results.

As we know, the integral operators with weakly singular kernels are compact operators on $\left(L^2(\Lambda),\|.\|_{L^2(\Lambda)}\right)$ [26]. Since we have $\left(C(\Lambda),\|.\|_{L^2(\Lambda)}\right)\subseteq\left(L^2(\Lambda),\|.\|_{L^2(\Lambda)}\right)$, so these operators can be considered as compact operators from $\left(L^2(\Lambda),\|.\|_{L^2(\Lambda)}\right)$ to $\left(C(\Lambda),\|.\|_{L^2(\Lambda)}\right)$. Consequently, the fractional integral operator I^μ is a compact operator from $\left(L^2(\Lambda),\|.\|_{L^2(\Lambda)}\right)$ to $\left(C(\Lambda),\|.\|_{L^2(\Lambda)}\right)$.

In this position, we first represent the equation (17) as (18). For this purpose, let $u_0(x)$ be solution of (17), and $v_0(x)=u_0^{(\nu)}(x)\in L^2(\Lambda)$. Thus we have $D_C^{\theta_i}u_0(x)=I^{\nu-\theta_i}v_0,\ \ i=0,1,...,N_d-1$, and (17) can be written as follows

$$v_0(x)=f(x)-\sum_{r=0}^{N_d-1}p_r(x)\left(I^{\nu-\theta_r}v_0(x)\right)^{\gamma_r}. \tag{20}$$

We define $T:\left(L^2(\Lambda),\|.\|_{L^2(\Lambda)}\right)\to\left(C(\Lambda),\|.\|_{L^2(\Lambda)}\right)$ by

$$\left(Tv\right)(x)=f(x)-\sum_{r=0}^{N_d-1}p_r(x)\left(I^{\nu-\theta_r}v(x)\right)^{\gamma_r},\ \ \|v-v_0\|_{L^2(\Lambda)}\le\delta. \tag{21}$$

Due to compactness of $I^{\nu-\theta_i}$, $\ i=0,1,...,N_d-1$, the above operator is completely continuous on the ball $\|v-v_0\|_{L^2(\Lambda)}\le\delta$.

The operator $\mathscr{I}$ is considered for embedding the space $C(\Lambda)$ in the space $L^2(\Lambda)$. Clearly, $\mathscr{I}$ is a linear continuous operator, and (17) is equivalent with the following equation

$$v = \mathscr{I}Tv, \tag{22}$$

which is in form (18) with $\mathscr{T} = \mathscr{I}T$, such that theirs solutions are equivalent with $v_0(x) = u_0^{(v)}(x)$.

Now, we rewrite the GJG approach for (17) as the operational form (19). For this purpose, we define

$$R_N(x) = u_N^{(v)}(x) + \sum_{r=0}^{N_d-1} p_r(x)\left(D_C^{\theta_r} u_N(x)\right)^{\gamma_r} - f(x), \tag{23}$$

where $u_N(x)$ is given by (10). Following the relation

$$\mathbf{J}_s^{(0,-\nu)}(x)w^{(0,-\nu)}(x) = J_s^{(0,\nu)}(x), \ \ 0 \le s \le N-\nu,$$

and according to the proposed method (11)-(16), we have

$$\sum_{i=0}^{N} R_N(x_i)J_s^{(0,\nu)}(x_i)w_i = 0, \quad 0 \le s \le N-\nu, \ \ i = 0,1,...,N, \tag{24}$$

where $\{x_i, w_i\}_{i=0}^{N}$ are the Gauss-Lobatto Legendre points and the corresponding weights respectively [27]. Since $R_N(x_i) = I_N(R_N(x))|_{x=x_i}$, where $I_N u$ be the Lagrange interpolation polynomial approximation of u associated with the shifted Gauss-Lobatto Legendre points, we may write

$$\sum_{i=0}^{N} I_N(R_N(x))|_{x=x_i} J_s^{(0,\nu)}(x_i)w_i = 0, \quad 0 \le s \le N-\nu.$$

According to the Gauss-Lobatto integration formula we obtain

$$\sum_{i=0}^{N} I_N(R_N(x))|_{x=x_i} J_s^{(0,\nu)}(x_i)w_i = \int_\Lambda I_N(R_N(x))J_s^{(0,\nu)}(x)dx = 0, \quad 0 \le s \le N-\nu. \tag{25}$$

Since $I_N(R_N(x))$ is a polynomial, it can be represented by a linear orthogonal polynomial expansion as

$$I_N(R_N(x)) = \sum_{s=0}^{N-\nu} \frac{\left(I_N(R_N(x)), \mathbf{J}_s^{0,-\nu}(x)\right)_{0,-\nu}}{\|\mathbf{J}_s^{0,-\nu}\|^2_{L^2_{0,-\nu}(\Lambda)}} \mathbf{J}_s^{0,-\nu}(x).$$

Using the relation (25) we have $I_N(R_N(x)) = 0$, and thereby the relation (23) yields

$$u_N^{(v)}(x) = I_N\left(f(x) - \sum_{r=0}^{N_d-1} p_r(x)\left(D_C^{\theta_r} u_N(x)\right)^{\gamma_r}\right). \tag{26}$$

Assuming $v_N(x) = u_N^{(v)}(x)$, we obtain $D_C^{\theta_i} u_N(x) = I^{\nu-\theta_i} v_N, \ \ i = 0,1,...,N_d-1$, and then the equation (26) can be written as

$$v_N = I_N T v_N, \tag{27}$$

which is an operational equation on $L^2(\Lambda)$, if we consider I_N as an operator from $C(\Lambda)$ into $L^2(\Lambda)$. Consequently, the above equation is in form (19) with $\mathscr{T}_N = I_N T$, and the solutions of (26), and (27) are equivalent by $v_N(x) = u_N^{(v)}(x)$.

To ensure existence, uniqueness, and convergence results for (26) we apply Theorem 2. To this end, we need to prove the following statements:

- The sequence of operators I_N converge strongly to the operator $\mathscr{I}$,
- The operator T defined by (21) is differentiable and the homogeneous equation $v - \mathscr{I}T'(v_0)v = 0$ has only a trivial solution $v = 0$,
- The operator T is differentiable at the point v_0 continuously.

The first condition is deduced by Edros-Turan Theorem [28], which indicates that the interpolation polynomial of any continuous function associated with the Gauss-Lobatto Legendre points approached in mean square, to the given function, i.e.,

$$\|I_N v - v\|_{L^2(\Lambda)} \to 0, \;\; v \in C(\Lambda). \tag{28}$$

The remaining conditions are proved in the next theorem.

Theorem 3 *Assume that*

1.Initial value problem (17) has a unique smooth solution $u_0(x)$.
2.The linear homogeneous initial value problem

$$\begin{cases} u^{(\nu)}(x) = \sum_{r=0}^{N_d-1}\left(\gamma_r p_r(x)\left(D_C^{\theta_i} u_0(x)\right)^{\gamma_r-1} D_C^{\theta_r} u(x)\right), \;\; x\in\Lambda, \\ u^{(i)}(0)=0, \quad i=0,1,...,\nu-1, \end{cases} \tag{29}$$

has only the trivial solution $u(x)\equiv 0$.

Then there exists a number $\delta>0$, such that the equation (26) has a unique GJG solution $u_N(x)$ defined by (10) in the ball $\|u^{(\nu)} - u_0^{(\nu)}\|_{L^2(\Lambda)} \le \delta$, for sufficiently large N.
The rate of convergence is given by

$$\|u_N^{(\nu)} - u_0^{(\nu)}\|_{L^2(\Lambda)} \le C \max_{x\in\Lambda} |v_0 - p_N|,$$

where C is a constant and p_N is any polynomial of degree not exceeding from N.

*Proof.*Consider $E=(L^2(\Lambda), \|.\|_{L^2(\Lambda)})$, and $F=(C(\Lambda),\|.\|_{L^2(\Lambda)})$. First, we show that the operator T is differentiable. To this end, we define

$$\mathscr{F}\left(x, I^{\nu-\theta_0}v(x), I^{\nu-\theta_1}v(x),..., I^{\nu-\theta_{N_d-1}}v(x)\right) = f(x) - \sum_{r=0}^{N_d-1} p_r(x)\left(I^{\nu-\theta_r}v(x)\right)^{\gamma_r}. \tag{30}$$

From Taylor formula we can write

$$(Tv)(x)-(Tw)(x) = \sum_{i=0}^{N_d-1} \frac{\partial\mathscr{F}\left(x, z_0^w, z_1^w, ..., z_{N_d-1}^w\right)}{\partial z_i^v} z_i^{v-w} + \frac{1}{2!}\sum_{i=0}^{N_d-1}\sum_{j=0}^{N_d-1} \frac{\partial^2\mathscr{F}\left(x, z_0^w, z_1^w, ..., z_{N_d-1}^w\right)}{\partial z_i^v \partial z_j^v} z_i^{v-w} z_j^{v-w} + ...,$$

where v and w are included in the ball $\|v-v_0\|_{L^2(\Lambda)}\le\delta$, and $z_i^w = I^{\nu-\theta_i}w(x)$. Following the relation (30), and definition of Fréchet derivative, the above relation can be rewrite as follows

$$\begin{aligned}(Tv)(x)-(Tw)(x) &= \sum_{i=0}^{N_d-1}\gamma_r p_r(x)(z_i^w)^{\gamma_r-1} z_i^{v-w} + \frac{1}{2!}\sum_{i=0}^{N_d-1}\sum_{j=0}^{N_d-1} \frac{\partial^2\mathscr{F}\left(x, z_0^w, z_1^w, ..., z_{N_d-1}^w\right)}{\partial z_i^v \partial z_j^v} z_i^{v-w} z_j^{v-w} + ... \\ &= (T'(w))(v-w) + \omega(v,w),\end{aligned}$$

where

$$(T'(w))(v) = \sum_{i=0}^{N_d-1}\gamma_r p_r(x)(I^{\nu-\theta_i}w)^{\gamma_r-1} I^{\nu-\theta_i}v, \tag{31}$$

and

$$\omega(v,w) = \frac{1}{2!}\sum_{i=0}^{N_d-1}\sum_{j=0}^{N_d-1} \frac{\partial^2\mathscr{F}\left(x, z_0^w, z_1^w, ..., z_{N_d-1}^w\right)}{\partial z_i^v \partial z_j^v} z_i^{v-w} z_j^{v-w} + ...,$$

such that

$$\frac{\|\omega(v,w)\|_{L^2(\Lambda)}}{\|v-w\|_{L^2(\Lambda)}} \to 0, \text{ as } v \to w.$$

Then the operator $(T'(w))(v)$ defined in (31) is the Fréchet derivative of the completely continuous operator (21). Since the linear operator $(T'(w))(v)$ is continuous, then the nonlinear operator T in (21) is continuously differentiable at v_0. Thus, regarding the assumption 2, all requirements of Theorem 2 are satisfied. Based on the Theorem 2, the equation (27) has a unique solution v_N in the ball $\|v_N - v_0\|_{L^2(\Lambda)} \leq \delta$, for sufficiently large N. Since $v_N(x) = u_N^{(v)}(x), v_0(x) = u_0^{(v)}(x)$, then the equation (26) has a unique solution $u_N(x)$ in the ball $\|u_N^{(v)} - u_0^{(v)}\|_{L^2(\Lambda)} \leq \delta$.

Also, we have

$$\|v_N - v_0\|_{L^2(\Lambda)} \leq M\|(I_N - \mathscr{I})T v_0\|_{L^2(\Lambda)} = M\|I_N \mathscr{I}^{-1} v_0 - v_0\|_{L^2(\Lambda)},$$

which Edros-Turan Theorem [28](boundedness of operator I_N) yields

$$\begin{aligned}\|v_N - v_0\|_{L^2(\Lambda)} &\leq M\Big(\|v_0 - p_N\|_{L^2(\Lambda)} + \|I_N \mathscr{I}^{-1}(v_0 - p_N)\|_{L^2(\Lambda)}\Big) \\ &\leq (M+C)\max_{x\in\Lambda}|v_0 - p_N|,\end{aligned}$$

where p_N is any polynomial of degree not exceeding from N and $I_N \mathscr{I}^{-1} p_N = p_N$.

5 Numerical Results

The main purpose of this section is to describe some numerical experiments performed with the GJG method. The reported GJG approximation errors are computed in L^2-norm and the CPU time of Non-Linear Solver are reported.All of the calculations were performed using Mathematica software v 12.1, running in an Intel (R) Core (TM) i7-7700 CPU@ 3.60 GHz. As we develop in Section 3, the approximate solution (10) can be characterized by solving the nonlinear algebraic system (16) for the unknowns $\{a_i\}_{i=0}^{N-v}$. Clearly, if we use the classical Newton's methods for solving (16), the function and its Jacobian evaluations are needed which can be a computationally expensive process. This can be influenced the accuracy of the obtained approximations. Due to the high complexity of (16), we apply a suitable root finding approach which controls the computational costs by reducing the Jacobian evaluations. To this end, we solve all of the computed nonlinear algebraic systems by Affine-Covariant Newton's method [29]. This algorithm, is an implementation of the error oriented exact global Newton approach, which significantly reduces the number of Jacobian calculations. This property decrease evaluation costs, and increase computational speed in the root finding process. The Affine-Covariant Newton's method is available as a method option to "FindRoot" in Mathematica 12.1.

Example 1*([30]) Consider the Riccati nonlinear FDE*

$$L_D(u(x)) = u^{(3)}(x) + D_C^{(2.5)}(u(x)) + u^2(x) = x^4, \quad x \in \Lambda,$$

with initial conditions $u(0) = u'(0) = 0, u''(0) = 2$, *and the exact solution* $u(x) = x^2$.

As we pointed out in the previous sections, GJG scheme is easily constructed when considering homogeneous initial conditions. If the given initial conditions are not homogeneous, they can always be made with the adding an appropriate function to the unknown solution with modification of the associated equation. So by applying a function transformation method we have

$$\begin{cases}\bar{L}_D(U(x)) = f, \\ U^{(i)}(0) = 0, \quad i = 0,1,2,\end{cases}$$

where

$$U(x) = u(x) - \sum_{i=0}^{2} \tfrac{d_i}{i!}x^i = u(x) - \tfrac{d_2}{2!}x^2 = u(x) - x^2,$$

$$\bar{L}_D(U(x)) = L_D(U(x)) + 2U(x)x^2, \quad f(x) = 0.$$

We define approximate solution $U_N(x) = \sum_{i=3}^{4} a_i \mathbf{J}_i^{(0,-3)}(x)$. Using GJG method we have

$$\sum_{i=3}^{4} a_i M_{ki} = \bar{f}_k, \quad k = 3,4,$$

where $\bar{f} = [0,0]^T$ and

$$M = \begin{pmatrix} 28.8131 + 4.24811a_3 & 192.858a_4 + 3.5341a_3 + 1.87434a_4 \\ -19.5698 + 1.66705a_3 & -13.0513 + 3.74868a_3 + 0.908717a_4 \end{pmatrix},$$

which when solved, gives us $a_3 = a_4 = 0$. Then $U_N(x) = 0$ and so $u_N(x) = U_N(x) + x^2 = x^2$, which is the exact solution.

Example 2*Consider the FDE*

$$x^2 D_C^{\frac{3}{2}} u(x) + \sqrt{x+x^2} u^2(x) = f(x), \quad x \in \Lambda,$$

with the initial conditions $u(0) = 0, u'(0) = 1$, *where*

$$f(x) = \sqrt{x+x^2}\Big((Ln(1+x))^2 - \frac{x^2}{\sqrt{\pi}(1+x)^{\frac{3}{2}}}\Big) - \frac{x^2 Arcsinh(\sqrt{x})}{\sqrt{\pi}(1+x)^{\frac{3}{2}}}.$$

The exact solution is $u(x) = Ln(1+x)$.

The numerical results are presented in Table 1, and Figure 1, for various values of N. Clearly, the approximate solutions are in a good agreement with the exact ones.

Table 1: The GJG approximation errors of example 2.

N	GJG Errors	CPU Time(s)
3	2.88×10^{-3}	0.00
5	1.04×10^{-4}	0.01
7	3.92×10^{-6}	0.07
9	1.45×10^{-7}	0.12
11	5.25×10^{-9}	0.23
13	1.87×10^{-10}	0.50
15	6.54×10^{-12}	0.84
17	2.26×10^{-13}	0.87
19	7.13×10^{-15}	1.32
21	1.39×10^{-15}	1.87

Example 3*Consider the following nonlinear FDE*

$$\Big(D^{\frac{3}{2}} u(x)\Big)^3 + x^{\frac{m}{2}} u(x) = f(x), \quad m > 1,$$

with $u(0) = u'(0) = 0$,

$$f(x) = x^{4+\frac{m}{2}}(1 - 3x^{\frac{m}{2}}) - \frac{1}{64} x^{\frac{15}{2}} \Big(- \frac{256}{5\sqrt{\pi}} + \frac{12x^{\frac{m}{2}} \Gamma(5+\frac{m}{2})}{\Gamma(\frac{7+m}{2})} \Big)^3,$$

and the exact solution $u(x) = x^4 \Big(1 - 3x^{\frac{m}{2}}\Big)$.

This example has considered with $m = 1.3, 1.5, 1.7,\ 1.9$. Since for $m = 2$ we have smooth solution, so we can expect an exponential rate of convergence. In Table $2-3$, and Figure 2, numerical errors are reported for several values of N. As expected, the rate of convergence is increased when m tends to 2. Numerical results for $m = 2$ has not presented since the numerical solution is very close to the exact solution.

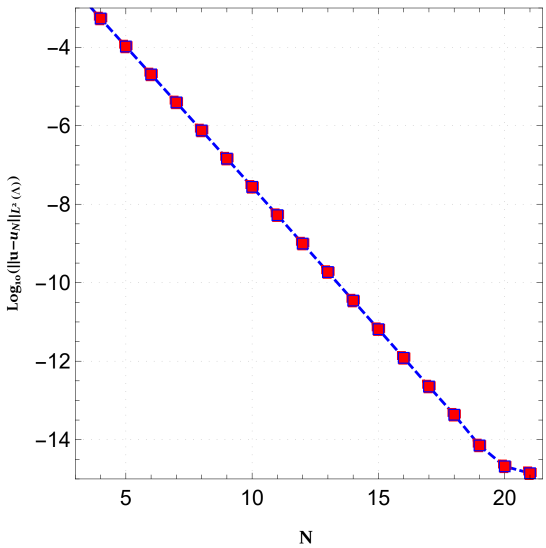

Fig. 1: The GJG approximation errors of example 2 with various values of N.

Table 2: The GJG approximation errors of example 3 with $m = 1.3$.

N	GJG Errors	CPU Time(s)
3	6.58×10^{-1}	0.01
5	5.48×10^{-3}	0.07
7	1.63×10^{-4}	0.26
9	7.86×10^{-6}	0.96
11	2.57×10^{-7}	2.15
13	1.54×10^{-7}	4.18
15	1.83×10^{-8}	7.28
17	8.54×10^{-9}	11.73
19	2.11×10^{-9}	20.04
21	1.88×10^{-9}	37.53

Table 3: The GJG approximation errors of example 3 with $N = 10$.

m	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
1.3	1.83×10^{-6}	2.2×10^{-6}	1.65×10^{-6}	3.39×10^{-6}	1.51×10^{-6}
1.5	9.71×10^{-7}	5.53×10^{-7}	5.52×10^{-7}	5.64×10^{-7}	3.09×10^{-7}
1.7	3.25×10^{-7}	2.83×10^{-7}	5.05×10^{-7}	8.02×10^{-7}	3.09×10^{-7}
1.9	4.91×10^{-8}	2.37×10^{-7}	5.25×10^{-7}	4.49×10^{-7}	2.26×10^{-7}

Example 4*Consider the following nonlinear FDE*

$$\begin{cases} D^{\frac{1}{2}}u(x) = \frac{u^2(x)}{x} + f(x), & x \in \Lambda, \\ u(0) = 0, \end{cases}$$

where $f(x) = e^{-\frac{1}{x}}x^{\frac{-3}{2}}\left(1 - e^{-\frac{1}{x}}x^{\frac{-1}{2}}\right)$, *and the exact solution of the problem is* $u(x) = \frac{1}{\sqrt{x}}e^{-\frac{1}{x}}$.

The numerical results are presented in Table 4 and Figure 3. Indeed, due to continuity and non-smooth behavior of solution we have a linear error variations, which can be approved by the reported results.

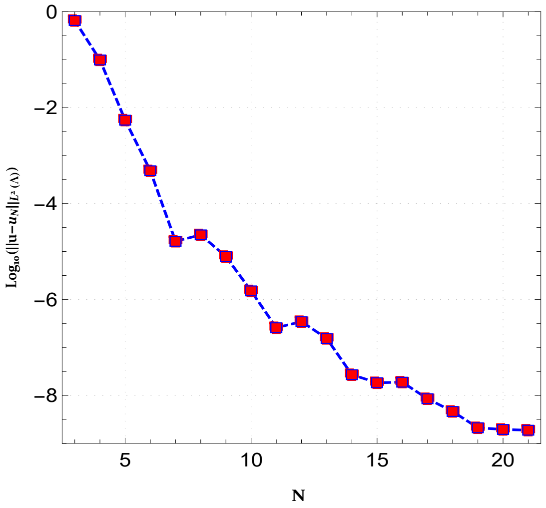

Fig. 2: The GJG approximation errors of example 3 with various values of N.

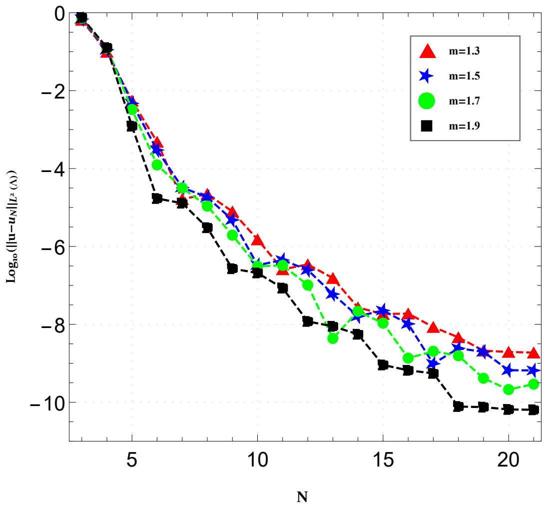

Fig. 3: The GJG approximation errors of example 3 with various values of m.

Table 4: The GJG approximation errors of example 4.

N	GJG Errors	CPU Time(s)
3	1.03×10^{-2}	0.01
5	2.71×10^{-3}	0.02
7	4.41×10^{-4}	0.05
9	1.92×10^{-4}	0.12
11	5.05×10^{-5}	0.12
13	3.05×10^{-5}	0.34
15	1.39×10^{-5}	0.50
17	6.05×10^{-6}	0.67
19	1.47×10^{-6}	0.97
21	1.05×10^{-6}	0.98
23	7.98×10^{-7}	1.01

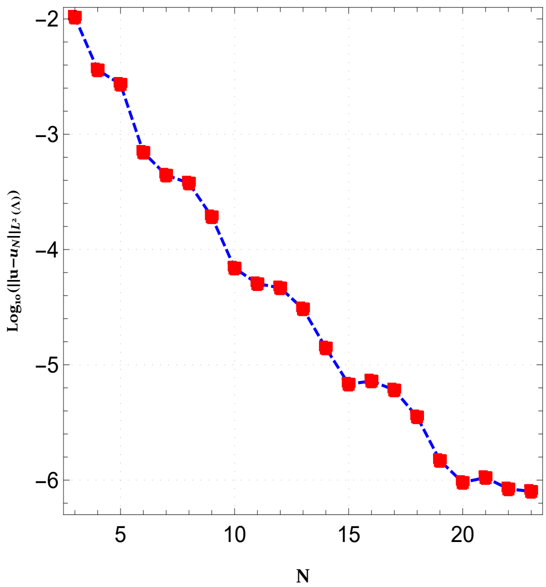

Fig. 4: The GJG approximation errors of example 4 with various values of N.

Example 5(*[31]*) *In this example, we analyze Micro-electro-mechanical(MEMS) instrument that has been designed to measure the viscosity of fluids which are encountered "downhole" during the process of oil well logging showing that, in one mode of operation, its motion is governed by following non-linear FDE*

$$u''(x)+\frac{\sqrt{\pi}}{5}D_C^{\alpha}u(x)+u(x)+u^3(x)=0, \quad x\in[0,10] \tag{32}$$

with initial conditions $u(0)=1, u'(0)=0$.

Here, we do not access to any closed-form solution. To illustrate the efficiency of the proposed method, we solve (32) with $\alpha=1.5,1.7,1.9,2$, and report the obtained results for $N=16$ in Table 5 and Figure 4. As we can see, when α tends 2, the numerical solutions converge to the corresponding solution with $\alpha=2$.

Table 5: The GJG solutions for example 5 with various α.

x	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$	$\alpha = 2$
2	-0.54354	-0.56576	-0.59635	-0.61599
4	-0.20500	-0.22078	-0.20179	-0.17404
6	0.57891	0.70879	0.83008	0.87154
8	-0.24021	-0.39530	-0.70678	-0.92195
10	-0.26390	-0.27135	-0.01539	0.30925

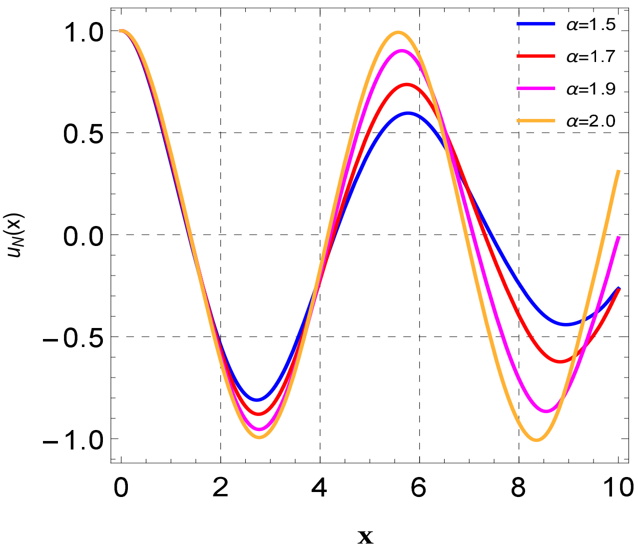

Fig. 5: The GJG solutions of example 5 with various values of N and α.

6 Conclusion

This work has been concerned with the GJG analysis of the nonlinear multi order FDEs. The unique solvability of the obtained nonlinear algebraic system discussed. We proved the convergence of the proposed method and obtained the error estimate in the weighted L^2-norm of the solution. This result confirmed by some numerical examples.

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