Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170311

Some Properties of Incomplete First Appell Hypergeometric Matrix Functions

Ayed Al e'damat[1,*], *Ashish Verma* [2], *Jihad Younis* [3], *and Hassen Aydi* [4]

[1]Department of Mathematics, Faculty of Science, Al-Hussein Bin Talal University, Ma'an - Jordan
[2]Department of Mathematics, V. B. S. Purvanchal University, Jaunpur (U.P.)- 222003, India
[3]Department of Mathematics, Aden University, Aden, P.O.Box 6014, Yemen
[4]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

Received: 23 Feb. 2023, Revised: 2 Apr. 2023, Accepted: 22 Apr. 2023
Published online: 1 May 2023

Abstract: The aim of this paper to introduce two incomplete first Appell hypergeometric matrix functions (IFAHMFs) γ_1 and Γ_1 by means of the incomplete Pochhammer matrix symbols. Furthermore, there is a derivation of some results such as integral formula, recursion formula, differentiation formula and finite summation formula of the IFAHMFs γ_1 and Γ_1.

Keywords: Gamma matrix function, incomplete Pochhammer symbols, hypergeometric matrix function, Bessel matrix function.

1 Introduction

In 2012, Srivastava *et al.* [1] introduced new incomplete Pochhammer symbols and discussed many related applications. Recently, Bansal *et al.* [2] established certain incomplete ℵ- functions and investigated some properties of them. Several properties of the incomplete multivariable hypergeometric functions have been investigated in the recent papers [3,4,5,6,7,8].

The matrix theory is appearing in the field of mathematical, physical and engineering. In recent years, many researchers have introduced and investigated several kind of special matrix functions [9,10,11,12,13]. Matrix analogue of the two variable Appell hypergeometric functions are defined in [14,15,16]. The incomplete multivariable hypergeometric matrix functions have been studied by many authors (see, e.g., [17,18,19]). Recursion formula, infinite summation formula for the Srivastava's triple hypergeometric matrix functions $H_{\mathscr{A}}$, $H_{\mathscr{B}}$ and $H_{\mathscr{C}}$ are presented in [20]. Verma *et al.* [21] have obtained some results of the Kampé de Feŕiet hypergeometric matrix function.

Throughout in this paper, let $\mathbb{C}^{s\times s}$ be the complex space of complex matrices of common order s. For any matrix $E \in \mathbb{C}^{s\times s}$, its spectrum $\nu(E)$ is the family of eigenvalues of E. Suppose that $f_1(z)$ and $f_2(z)$ are holomorphic functions in Θ an open set of the complex plane and $E \in \mathbb{C}^{s\times s}$ with $\nu(E) \subset \Theta$, then by means of the properties of the matrix functional calculus [22], we get $f_1(E)f_2(E) = f_2(E)f_1(E)$. Moreover, let F be a matrix in $\mathbb{C}^{s\times s}$ for which $\nu(F) \subset \Theta$, then $f_1(E)f_2(F) = f_2(F)f_1(E)$. A matrix $E \in \mathbb{C}^{s\times s}$ is called positive stable (In short, PS) if $Re(\tau) > 0$ for all $\tau \in \sigma(E)$.

The Gamma matrix function $\Gamma(E)$ is given by [23]

$$\Gamma(E) = \int_0^\infty e^{-t} t^{E-I} dt; \; t^{E-I} = \exp((E-I)\ln t), \quad (1)$$

where E is a PS matrix in $\mathbb{C}^{s\times s}$.
In addition, if $E + dI$ is invertible for each integer $d \geq 0$, hence the reciprocal gamma function [23] is stated as:

$$\Gamma^{-1}(E) = (E)_d \, \Gamma^{-1}(E + dI).$$

Here, $(E)_d$ denotes the shifted factorial matrix function for $E \in \mathbb{C}^{s\times s}$ stated as ([24]):

$$(E)_d = \begin{cases} E(E+I)\cdots(E+(d-1)I), & d \geq 1 \\ I, & d = 0. \end{cases} \quad (2)$$

I denotes the identity matrix in $\mathbb{C}^{s\times s}$. If the matrix $E \in \mathbb{C}^{s\times s}$ is PS and $d \geq 1$, so by [23], one has $\Gamma(E) = \lim_{d\to\infty}(d-1)!(E)_d^{-1} d^E$.

* Corresponding author e-mail: ayed.h.aledamat@ahu.edu.jo

The Gauss hypergeometric matrix function [24] is stated as

$$ {}_2F_1(E,F;G;z_1) = \sum_{d=0}^{\infty} \frac{(E)_d(F)_d(G)_d^{-1}}{d!} z_1^d, \tag{3} $$

for matrices E, F and G in $\mathbb{C}^{s\times s}$ so that $G+dI$ is invertible for each integer $d \geq 0$ and $|z_1| \leq 1$.

The incomplete gamma matrix functions $\gamma(E,x)$ and $\Gamma(E,x)$ are respectively given as (see [17])

$$ \gamma(E,x) = \int_0^x e^{-t} t^{E-I} dt \tag{4} $$

and

$$ \Gamma(E,x) = \int_x^{\infty} e^{-t} t^{E-I} dt. \tag{5} $$

The next decomposition identity

$$ \gamma(E,x) + \Gamma(E,x) = \Gamma(E), \tag{6} $$

is fulfilled. The incomplete Pochhammer matrix symbols $(E;x)_d$ and $[E;x]_d$ are defined by (see [17])

$$ (E;x)_d = \gamma(E+dI,x)\Gamma^{-1}(E) \tag{7} $$

and

$$ [E;x]_d = \Gamma(E+dI,x)\Gamma^{-1}(E), \tag{8} $$

where E and x denote the PS matrix and positive real number, respectively. By using (6), we get the following decomposition formula:

$$ (E;x)_d + [E;x]_d = (E)_d, \tag{9} $$

where $(E)_d$ is the Pochhammer matrix symbol introduced in (2).
The incomplete Gauss hypergeometric matrix functions are given as (see [17])

$$ {}_2\gamma_1\Big[(E;x),F;G;z_1\Big] = \sum_{m=0}^{\infty} (E;x)_m(F)_m(G)_m^{-1} \frac{z_1^m}{m!} \tag{10} $$

and

$$ {}_2\Gamma_1\Big[[E;x],F;G;z_1\Big] = \sum_{m=0}^{\infty} [E;x]_n(F)_n(G)_n^{-1} \frac{z_1^m}{m!}, \tag{11} $$

where E, F and G are matrices in $\mathbb{C}^{s\times s}$ such that $G+k_1I$ is invertible for each integer $k_1 \geq 0$.
Furthermore, the integral representation of the incomplete Gauss hypergeometric matrix function ${}_2\Gamma_1$ is stated as:

$$ {}_2\Gamma_1\left[[E;x],F;G;z_1\right] = \left(\int_0^1 {}_1\Gamma_0\left[[E;x];-;z_1t\right] t^{F-I}(1-t)^{G-F-I} dt\right) \times \Gamma^{-1}(F)\Gamma^{-1}(G-F)\Gamma(G),\ |z_1|<1, \tag{12} $$

where G, F and $G-F$ are PS, $GF = FG$, and ${}_1\Gamma_0\Big[[E;x];-;z_1t\Big]$ is the incomplete Gauss hypergeometric matrix function of one numerator.

The Bessel matrix function (see, e.g.,[25,26,27]) is stated as:

$$ J_E(z) = \sum_{m\geq 0} \frac{(-1)^m\,\Gamma^{-1}((m+1)I+E)}{m!} \left(\frac{z_1}{2}\right)^{2mI+E}, \tag{13} $$

where k_1I+E is invertible for all integers $k_1 \geq 0$. Also, the modified Bessel matrix functions are defined as follows (see[27]):

$$ \begin{aligned} I_E &= e^{\frac{-Ei\pi}{2}} J_E(z_1 e^{\frac{i\pi}{2}});\ -\pi < arg(z_1) < \frac{\pi}{2}, \\ I_E &= e^{\frac{Ei\pi}{2}} J_E(z_1 e^{\frac{-i\pi}{2}});\ -\frac{\pi}{2} < arg(z_1) < \pi. \end{aligned} \tag{14} $$

2 Main Results

This section deals with the IFAHMFs γ_1 and Γ_1 as follows:

$$ \gamma_1\Big[(E;x),F,F';G;z_1,w_1\Big] = \sum_{m_1,m_2\geq 0} \frac{(E;x)_{m_1+m_2}(F)_{m_1}(F')_{m_2}(G)_{m_1+m_2}^{-1}}{m_1!m_2!} z_1^{m_1} w_1^{m_2}, \tag{15} $$

$$ \Gamma_1\Big[[E;x],F,F';G;z_1,w_1\Big] = \sum_{m_1,m_2\geq 0} \frac{[E;x]_{m_1+m_2}(F)_{m_1}(F')_{m_2}(G)_{m_1+m_2}^{-1}}{m_1!m_2!} z_1^{m_1} w_1^{m_2}, \tag{16} $$

where E, F, F', G are PS matrices in $\mathbb{C}^{s\times s}$ such that $G+k_1I$ is invertible for every integer $k_1 \geq 0$ and z_1, w_1 are complex variables.
From (9), we get the following decomposition formula

$$ \gamma_1\Big[(E;x),F,F';G;z_1,w_1\Big] + \Gamma_1\Big[[E;x],F,F';G;z_1,w_1\Big] = F_1\Big[E,F,F';G;z_1,w_1\Big], \tag{17} $$

where $F_1\Big[E,F,F';G;z_1,w_1\Big]$ is the first Appell hypergeometric matrix function [16].

*Remark.*If we set $z_1 = 0$ or $w_1 = 0$ in (15) and (16), we obtain the classical incomplete families of Gauss hypergeometric matrix functions [17].

By means of the properties of $\gamma_1\Big[(E;x),F,F';G;z_1,w_1\Big]$, we can determine the properties of $\Gamma_1\Big[[E;x],F,F';G;z_1,w_1\Big]$ using the decomposition formula (17).

Theorem 1. *Let E, F, F' and G be matrices in $\mathbb{C}^{s\times s}$ such that $FG = GF, FF' = F'F$ and $F'G = GF'$. Then the following function:*

$$\mathscr{S} = \mathscr{S}(z_1,w_1) = \gamma_1\left[(E;x),F,F';G;z_1,w_1\right] + \Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]$$

meets the system of partial differential equations:

$$z_1(1-z_1)\frac{\partial^2\mathscr{T}}{\partial z_1^2} + (1-z_1)w_1\frac{\partial^2\mathscr{T}}{\partial z_1\partial w_1} - z_1(E+I)\frac{\partial\mathscr{T}}{\partial z_1} - z_1\frac{\partial\mathscr{T}}{\partial z_1}F - w_1\frac{\partial\mathscr{T}}{\partial w_1}F + \frac{\partial\mathscr{T}}{\partial z_1}G - E\mathscr{T}F = O, \tag{18}$$

$$w_1(1-w_1)\frac{\partial^2\mathscr{T}}{\partial w_1^2} + (1-w_1)z_1\frac{\partial^2\mathscr{T}}{\partial z_1\partial w_1} - w_1(E+I)\frac{\partial\mathscr{T}}{\partial w_1} - w_1\frac{\partial\mathscr{T}}{\partial w_1}F' - z_1\frac{\partial\mathscr{T}}{\partial z_1}F' + \frac{\partial\mathscr{T}}{\partial w_1}G - E\mathscr{T}F' = O. \tag{19}$$

Proof. The relation (17) succeeds into the following proof conjoined with $F_1\left[E,F,F';G;z_1,w_1\right]$ which adequately fulfil the matrix differential equations given in [14,15].

Theorem 2. *Let E, F, F' and G be non commuting matrices in $\mathbb{C}^{s\times s}$ so that E and G are PS, then we have the following integral representation.*

$$\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] = \Gamma^{-1}(E)\left[\int_x^\infty e^{-t}t^{E-I}\Phi_2(F,F';G;z_1t,w_1t)dt\right], \tag{20}$$

where Φ_2 is Humbert's hypergeometric matrix function given by (see [28])

$$\Phi_2(F,F';G;z_1,w_1) = \sum_{m_1,m_2\geq 0}\frac{(F)_{m_1}(F')_{m_2}(G)^{-1}_{m_1+m_2}}{m_1!m_2!}z_1^{m_1}w_1^{m_2}. \tag{21}$$

Proof. By substituting $[E;x]_{m_1+m_2}$ in (5) and (8) by its integral representation in (16), we have

$$\begin{aligned}\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] &= \Gamma^{-1}(E)\sum_{m_1,m_2\geq 0}\left(\int_x^\infty e^{-t}t^{E+(m_1+m_2-1)I}dt\right)\\ &\quad\times(F)_{m_1}(F')_{m_2}(G)^{-1}_{m_1+m_2}\frac{z^{m_1}w^{m_2}}{m_1!m_2!},\\ &= \Gamma^{-1}(E)\sum_{m_1,m_2\geq 0}\left(\int_x^\infty e^{-t}t^{E-I}(F)_{m_1}(F')_{m_2}(G)^{-1}_{m_1+m_2}\right.\\ &\quad\left.\times\frac{(z_1t)^{m_1}(w_1t)^{m_2}}{m_1!m_2!}dt\right).\end{aligned} \tag{22}$$

Hence, the proof is completed.

Theorem 3. *For matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that $FG = GF$, $FF' = F'F$ and $F'G = GF'$, and F, F', G are PS, we have the following integral representation:*

$$\begin{aligned}&\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\ &= \left[\int_0^\infty\int_0^\infty e^{-t_1-t_2}\,{}_1\Gamma_1\left[[E;x];G;z_1t_1+w_1t_2\right]t_1^{F-I}t_2^{F'-I}dt_1dt_2\right]\\ &\quad\times\Gamma^{-1}(F)\Gamma^{-1}(F').\end{aligned} \tag{23}$$

Proof. By using the integral representation of the Pochhammer matrix symbols $(F)_m$, $(F')_n$ in the definition of (16), we get

$$\begin{aligned}&\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\ &= \sum_{m_1,m_2\geq 0}\left[\int_0^\infty\int_0^\infty e^{-t_1-t_2}[E;x]_{m_1+m_2}\right.\\ &\quad\left.\times t_1^{F-I}t_2^{F'-I}(G)^{-1}_{m_1+m_2}\frac{(z_1t)^{m_1}(w_1t)^{m_2}}{m_1!m_2!}dt_1dt_2\right]\Gamma^{-1}(F)\Gamma^{-1}(F').\end{aligned} \tag{24}$$

With the help of the summation formula [29]

$$\sum_{M\geq 0}f(M)\frac{(z+w)^M}{M!} = \sum_{m_1,m_2\geq 0}f(m_1+m_2)\frac{z^{m_1}w^{m_2}}{m_1!m_2!}, \tag{25}$$

we get (23).

Theorem 4. *For matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that $FG = GF$, $FF' = F'F$ and $F'G = GF'$, and E, F, F', G are PS, the following integral representation holds true:*

$$\begin{aligned}&\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\ &= \Gamma^{-1}(E)\left[\int_0^\infty\int_0^\infty\int_x^\infty e^{-s-t_1-t_2}s^{E-I}\right.\\ &\quad\left.\times t_1^{F-I}t_2^{F'-I}{}_0F_1(-;G;z_1t_1s+w_1t_2s)dt_1dt_2ds\right]\\ &\quad\Gamma^{-1}(F)\Gamma^{-1}(F').\end{aligned} \tag{26}$$

Proof. By substituting $[E;x]_{m+n}$ in (5) and (8) by its integral representation in (23), we are led to the desired result (26).

Corollary 1. *We have*

$$\begin{aligned}&\Gamma_1\left[[E;x],F,F';G+I;-z_1,-w_1\right]\\ &= \Gamma^{-1}(E)\left[\int_0^\infty\int_0^\infty\int_x^\infty e^{-s-t_1-t_2}s^{E-\frac{G}{2}-I}t_1^{F-I}t_2^{F'-I}\right.\\ &\quad\left.\times(z_1t_1+w_1t_2)^{-\frac{G}{2}}J_G(2\sqrt{z_1t_1s+w_1t_2s})dt_1dt_2ds\right]\\ &\quad\Gamma^{-1}(F)\Gamma^{-1}(F')\Gamma(G+I)\end{aligned} \tag{27}$$

$$\begin{aligned}&\Gamma_1\left[[E;x],F,F';G+I;z_1,w_1\right]\\ &= \Gamma^{-1}(E)\left[\int_0^\infty\int_0^\infty\int_x^\infty e^{-s-t_1-t_2}s^{E-\frac{G}{2}-I}t_1^{F-I}t_2^{F'-I}\right.\\ &\quad\left.\times(z_1t_1+w_1t_2)^{-\frac{G}{2}}I_G(2\sqrt{z_1t_1s+w_1t_2s})dt_1dt_2ds\right]\\ &\quad\Gamma^{-1}(F)\Gamma^{-1}(F')\Gamma(G+I),\end{aligned} \tag{28}$$

Theorem 5. *For non commuting matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that E and G are PS, we have the following recursion relation:*

$$\Gamma_1\left[[E;x],F+sI,F';G;z_1,w_1\right] = \Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] + z_1E\left[\sum_{k=1}^{n}\Gamma_1\left[[E+I;x],F+kI,F';G+I;z_1,w_1\right]\right]G^{-1}. \quad (29)$$

Also, if $F-kI$ is invertible for every integer $k\le n$ where n is a non-negative integer, then

$$\Gamma_1\left[[E;x],F-sI,F';G;z_1,w_1\right] = \Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] - z_1E\left[\sum_{k=0}^{n-1}\Gamma_1\left[[E+I;x],F-kI,F';G;z_1,w_1\right]\right]G^{-1}. \quad (30)$$

Proof. By using (20) and the following formula:

$$(F+I)_m = F^{-1}(F)_m(F+mI),$$

we have

$$\Gamma_1\left[[E;x],F+I,F';G;z_1,w_1\right] = \Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] + z_1E\left[\Gamma_1[(E+I;x],F+I,F';G+I;z_1,w_1]\right]G^{-1}. \quad (31)$$

Now, applying (31) to the matrix function Γ_1 with the matrix parameter $F+2I$, we find that

$$\Gamma_1\left[[E;x],F+2I,F';G;z_1,w_1\right] = \Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] + z_1E\left[\sum_{k=1}^{2}\Gamma_1\left[[E+I;x],F+kI,F';G+I;z_1,w_1\right]\right]G^{-1}. \quad (32)$$

Recursion formula (29) follows by repeating n-times the process of result (31).
Again, replace F with $F-I$ in (31) to get

$$\Gamma_1\left[[E;x],F-I,F';G;z_1,w_1\right] = \Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] - z_1E\left[\Gamma_1\left[[E+I;x],F,F';G+I;z_1,w_1\right]\right]G^{-1}. \quad (33)$$

Iteratively, we obtain (30).

By using the relations (31) and (33), we have another form of recursion formulas for Γ_1.

Theorem 6. *For non commuting matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that E and G are PS, we have the following recursion relation:*

$$\Gamma_1\left[[E;x],F+nI,F';G;z_1,w_1\right] = \sum_{k_1\le n}\binom{n}{k_1}(E)_{k_1}z_1^{k_1} \times\left[\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right]\right](G)_{k_1}^{-1}. \quad (34)$$

Also, if $F-kI$ is invertible for every integer $k\le n$ (where n is a non-negative integer), then

$$\Gamma_1\left[[E;x],F-nI,F';G;z_1,w_1\right] = \sum_{k_1\le n}\binom{n}{k_1}(E)_{k_1}(-z_1)^{k_1} \times\left[\Gamma_1[[E+k_1I;x],F,F';G+k_1I;z_1,w_1]\right](G)_{k_1}^{-1}. \quad (35)$$

Proof. To prove the result (34), it suffices to apply the induction on $n\in\mathbb{N}$. For $n=1$, (34) holds. Suppose (34) is true for $n=t$, i.e.,

$$\Gamma_1\left[[E;x],F+tI,F';G;z_1,w_1\right] = \sum_{k_1\le t}\binom{t}{k_1}(E)_{k_1}z_1^{k_1}\left[\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right]\right](G)_{k_1}^{-1}. \quad (36)$$

Replacing F with $F+I$ in (36) and using (31), we get

$$\Gamma_1\left[[E;x],F+(t+1)I,F';G;z_1,w_1\right] = \sum_{k_1\le t}\binom{t}{k_1}(E)_{k_1}z_1^{k_1}\left[\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right]\right. + z_1(E+k_1I)\Gamma_1\left[[E+(k_1+1)I;x],F+(k_1+1)I,F';G+(k_1+1)I;z_1,w_1\right] \left.(G+k_1I)^{-1}\right]\times(G)_{k_1}^{-1}. \quad (37)$$

After some simplification, (37) takes the form

$$\Gamma_1\left[[E;x],F+(t+1)I,F';G;z_1,w_1\right] = \sum_{k_1\le t}\binom{t}{k_1}(E)_{k_1}z_1^{k_1}\,\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right](G)_{k_1}^{-1} + \sum_{k_1\le t+1}\binom{t}{k_1-1}(G)_{k_1}z_1^{k_1}\,\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right](G)_{k_1}^{-1}. \quad (38)$$

By applying Pascal's formulas (38), we obtain

$$\Gamma_1\left[[E;x],F+(t+1)I,F';G;z_1,w_1\right] = \sum_{k_1\le t+1}\binom{t+1}{k_1}(E)_{k_1}z_1^{k_1}\,\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right](G)_{k_1}^{-1}. \quad (39)$$

We get the desired formula (34) for $n=t+1$. Hence, through induction, the relation (34) stands true for all values of n. A similar argument will establish the formula (35).

The recursion formulas for $\Gamma_1\left[(E;x),F,F'\pm nI;G;z_1,w_1\right]$ are obtained by replacing $F\leftrightarrow F'$ and $z_1\leftrightarrow w_1$ in Theorems 5 – 6, respectively.

Theorem 7. *Given the matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ so that $EF=FE$, $F'G=GF'$, and E, G are PS, then we have the following recursion relation:*

$$\begin{aligned}
&\Gamma_1\left[(E;x),F,F';G-mI;z_1,w_1\right]\\
&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\
&\quad+z_1EF\left[\sum_{l=1}^{m}\Gamma_1\left[[E+I;x],F+I,F';G+(2-l)I;z_1,w_1\right]\right.\\
&\quad\left.\times(G-lI)^{-1}(G-(l-1)I)^{-1}\right]\\
&\quad+w_1E\left[\sum_{l=1}^{m}\Gamma_1\left[[E+I;x],F,F'+I;G+(2-l)I;z_1,w_1\right]\right.\\
&\quad\left.\times(G-lI)^{-1}(G-(l-1)I)^{-1}\right]F'. \qquad (40)
\end{aligned}$$

Proof. Applying the integral formula (20) of Γ_1 and the following transformation:

$$(G-I)^{-1}_{n_1+n_2}=(G)^{-1}_{n_1+n_2}\left[I+n_1(G-I)^{-1}+n_2(G-I)^{-1}\right],$$

we obtain the contiguous matrix relation

$$\begin{aligned}
&\Gamma_1\left[[E;x],F,F';G-I;z_1,w_1\right]\\
&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\
&\quad+z_1EF\left[\Gamma_1\left[[E+I;x],F+I,F';G+I;z_1,w_1\right](G-I)^{-1}(G)^{-1}\right]\\
&\quad+w_1E\left[\Gamma_1\left[[E+I;x],F,F'+I;G+I;z_1,w_1\right](G-I)^{-1}(G)^{-1}\right]F'. \qquad (41)
\end{aligned}$$

Replacing G with $G-I$ in (41), we arrive at

$$\begin{aligned}
&\Gamma_1\left[[E;x],F,F';G-2I;z_1,w_1\right]\\
&=\Gamma_1\left[[E;x],F,F';z_1,w_1\right]\\
&\quad+z_1EF\left[\sum_{l=1}^{2}\Gamma_1\left[[E+I;x],F+I,F';G+(2-l)I;z_1,w_1\right]\right.\\
&\quad\left.\times(G-lI)^{-1}(G-(l-1)I)^{-1}\right]\\
&\quad+w_1E\left[\sum_{l=1}^{2}\Gamma_1\left[[E+I;x],F,F'+I;G+(2-l)I;z_1,w_1\right]\right.\\
&\quad\left.\times(G-lI)^{-1}(G-(l-1))^{-1}\right]F'. \qquad (42)
\end{aligned}$$

Repeating this relation s-times on $\Gamma_1\left[[E;x],F,F';G-mI;z_1,w_1\right]$, we get (40).

Theorem 8. *Given the matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ so that E and G are PS, then we have the following derivative formulas:*

$$\begin{aligned}
&D^{k_1}_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\right]\\
&=(E)_{k_1}\left[\Gamma_1\left[[E+k_1I;x],F,F'+k_1I;G+k_1I;z_1,w_1\right]\right](F')_{k_1}(G)^{-1}_{k_1},F'G=GF'; \qquad (43)\\
&D^{k_1}_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]w_1^{F'+(k_1-1)I}\right]\\
&=\left[\Gamma_1\left[[E;x],F,F'+k_1I;G;z_1,w_1\right]\right]w_1^{F'-I}(F')_{k_1},F'G=GF'; \qquad (44)\\
&D^{k_1}_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1w_1,w_1\right]w_1^{G-I}\right]\\
&=\left[\Gamma_1\left[[E;x],F,F';G-k_1I;z_1w_1,w_1\right]\right](-1)^{k_1}(I-G)_{k_1}w_1^{G-(k_1+1)I}, \qquad (45)
\end{aligned}$$

where $D_{w_1}f=\frac{df}{dw_1}$ and $G-I$ is an invertible matrix for (45).

Proof. By differentiating (20) with respect to w, we get

$$\begin{aligned}
&\frac{d}{dw_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\right]=E\Gamma^{-1}(E+I)\\
&\times\left[\int_x^\infty e^{-t}t^{(E+I)-I}\Phi_2(E,E'+I;G+I;z_1t,w_1t)dt\right]F'G^{-1}. \qquad (46)
\end{aligned}$$

From the relations (20) and (46), we find that

$$\begin{aligned}
&\frac{d}{dw_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\right]\\
&=E\left[\Gamma_1\left[[E+I;x],F,F'+I;G+I;z_1,w_1\right]\right]F'G^{-1}. \qquad (47)
\end{aligned}$$

Hence, (43) is true for $k_1=1$. The significant formula comes by the principle of induction on k_1. Thus, we obtain (43). Formulas (44) and (45) can be established in a similar way.

Theorem 9. *For matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that $F'G=GF'$ and E, G are PS, the following summation formula holds true:*

$$\begin{aligned}
&\sum_{l=0}^{k_1}\binom{k_1}{l}(E)_l w_1^l\Gamma_1\left[[E+lI;x],F,F'+lI;G+lI;z_1,w_1\right](G)^{-1}_l\\
&=\Gamma_1\left[[E;x],F,F'+lI;G;z_1,w_1\right]. \qquad (48)
\end{aligned}$$

Proof. From definition of incomplete matrix function Γ_1 and the generalized Leibnitz formula for differentiation of a product of two functions, we have

$$\begin{aligned}
&D^{k_1}_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]w_1^{F'+(k_1-1)I}\right]\\
&=\sum_{l=0}^{k_1}\binom{k_1}{l}D^l_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\right]D^{k_1-l}_{w_1}\left[w_1^{F'+(k_1-1)I}\right]\\
&=\sum_{l=0}^{k_1}\binom{k_1}{l}(E)_l\left[\Gamma_1\left[[E+lI;x],F,F'+lI;G+lI;z_1,w_1\right]\right]\\
&\quad(F')_l(G)^{-1}_l w_1^{F'+(l-1)I}. \qquad (49)
\end{aligned}$$

We used (43) and some simplification in the second equality. From (44) and (49), we get (48).

*Remark.*The first Appell hypergeometric matrix function F_1 will be obtained if we assume $x=0$ in the IFAHMF Γ_1. Hence, taking $x=0$, the obtained formulas for Γ_1 convert to the formulas for the Appell hypergeometric matrix function F_1.

3 Conclusion

In this paper, we studied the IFAHMFs Γ_1 and γ_1. We obtained some integral formula, recursion formula, differentiation formula and finite summation formula of the IFAHMFs Γ_1 and γ_1. The particular case of our results coincides with the results obtained in [4] when taking matrices from $\mathbb{C}^{1\times 1}$.

References

[1] H.M. Srivastava, Chaudhary MA, Agarwal RP. The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, *Integral Transforms Spec. Funct.* **23** , 659-683, 2012.

[2] M. Bansal, D. Kumar, K.S. Nisar, J. Singh, Certain fractional calculus and integral transform results of incomplete ℵ- functions with applications, *Math. Math. Appl. Sci.*, Accepted (2020) DOI: 10.1002/mma.6299.

[3] A. Cetinkaya, The incomplete second Appell hypergeometric functions, *Appl. Math. Comput.*, **219**, 8332-8337, 2013.

[4] J. Choi, R. Parmar, P. Chopra, The Incomplete Lauricella and First Appell Functions and Associated Properties, *Honam Math. J.*, **36**, 531-542, 2014.

[5] J. Choi, R. Parmar, P. Chopra, The Incomplete Lauricella and Fourth Appell Functions, *Far East J. Math. Sci.*, **96** , 315-328, 2015.

[6] J. Choi, R. Parmar, P. Chopra, The incomplete Srivastava's triple hypergeometrics γ_B^H and Γ_B^H, *Filomat*, **30**, 1779-1787, 2016.

[7] J. Choi, R. Parmar, P. Chopra, The incomplete Srivastava's triple hypergeometrics γ_A^H and Γ_A^H, *Miskolc Mathematical Notes*, **19** , 191-200, 2018.

[8] H.M. Srivastava, R.K. Saxena, R.K. Parmar, Some Families of the Incomplete H-Functions and the Incomplete $\overline{H}$- Functions and Associated Integral Transforms and Operators of Fractional Calculus with Applications, *Russ. J. Math. Phys.*, **25**, 116-138, 2018.

[9] A.G. Constantine, RJ. Muirhead, Partial differential equations for hypergeometric functions of two argument matrix, *J. Multivariate Anal.*, **3**, 332-338, 1972.

[10] AT. James, Special functions of matrix and single argument in statistics, in: Theory and Applications of Special functions, R.A. Askey (Ed.), Academic Press, New York, 497-520 (1975),.

[11] L. Jódar, R. Company, E. Navarro, Laguerre matrix polynomials and systems of second order differential equations, *Appl. Numer. Math.*, **15** , 53-63, 1994.

[12] L. Jódar, R. Company, E. Ponsoda, Orthogonal matrix polynomials and systems of second order differential equations, *Diff. Equations Dynam. Systems*, **3**, 269-288, 1995.

[13] L. Jódar, R. Company, Hermite matrix polynomials and second order differential equations, *Approx. Theory. Appl.*, **12** , 20-30, 1996.

[14] A. Altin, B. Cekim, R. Sahin, On the matrix versions of Appell hypergeometric functions, *Quaest. Math.*, **37**, 31-38, 2014.

[15] R. Dwivedi , V. Sahai, On the hypergeometric matrix functions of two variables, *Linear Multilinear Algebra*, **66** , 1819-1837, 2017.

[16] R. Dwivedi , V. Sahai, A note on the Appell matrix functions, *Quaest. Math.*, **43**, 321-334 , 2019.

[17] M. Abdalla, On the incomplete hypergeometric matrix functions, *Ramanujan J.* 43, 663-678 (2017).

[18] A. Verma, On the incomplete Srivastava's triple hypergeometric matrix functions, *Quaest. Math.*, (2020), DOI: 10.2989/16073606.2020.1753123.

[19] A. Verma, S. Yadav, On the incomplete second Appell hypergeometric matrix functions, *Linear Multilinear Algebra*, **69** , 1747-1760, 2021.

[20] A. Verma, Some results on the Srivastava's triple hypergeometric matrix functions, *Asian-Eur. J. Math.* , **14** (4), 2150056, 2021.

[21] A. Verma, J. Younis , H. Aydi, On the Kampé de Fériet hypergeometric matrix function, *Mathematical Problems in Engineering*, Volume 2021, Article ID 9926176, 11 pages, 2021.

[22] N. Dunford, J.Schwartz, Linear Operators, Vol. I, Interscience, New York, 1957.

[23] L. Jódar, JC.Cortés, Some properties of gamma and beta matrix functions, *Appl. Math. Lett.*, **11** , 89-93, 1998.

[24] L. Jódar, JC.Cortés, On the hypergeometric matrix function, *J. Comput. Appl. Math.*, **99** , 205-217, 1998.

[25] L. Jódar, R. Company, E. Navarro Bessel matrix functions: explicit solution of coupled Bessel type equations, *Util. Math.*, **46** , 129-141, 1994.

[26] L. Jódar, Sastre J. On the Laguerre matrix polynomials, *Util. Math.*, **53**, 37-48, 1998.

[27] J. Sastre, L.Jódar, Asymptotics of the modified Bessel and incomplete gamma matrix functions, *Appl. Math. Lett.*, **16** , 815-820, 2003.

[28] S.Z. Rida, M. Abul-Dahab, MA. Saleem, On Humbert matrix function $\Psi_1(A,B;C,C';z,w)$ of two complex variables under differential operator, *Int. J Ind Math.*, **32** , 167-179, 2010.

[29] H.M. Srivastava, H.L.Manocha , *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

Ayed Al e'damat He received his PhD of Mathematics from Damascus University(Syria) 2013. He is Associate Professor in Department of mathematics in Al Hussein Bin Talal University(Jordan) from 2014 until Now . He is the author of more than 14 research papers indexed in Scopus. He has many research projects with different international Universities. His research interests include Mathematical Analysis; Fractional Calculus, Special Functions, Differential Equations, Statistics.

Ashish Verma He is presently working as Assistant Professor at the Department of Mathematics, Prof. Rajendra Singh (Rajju Bhaiya), Institute of Physical Sciences for Study and Research, V.B.S. Purvanchal University, Jaunpur (U.P.)-222003, India. He secured first position in order of merit in M.Sc. (Mathematics) examination of 2009 from Lucknow University and qualified CSIR (NET) and GATE examination in 2012. He obtained his Ph.D. degree in Mathematics from Lucknow University, Lucknow in the year 2016. His research interests are Special Functions. He has published more than 30 research papers in national and international Journals of high repute and delivered invited talks/ presented research papers in 10 scientific conferences

Jihad Ahmed Younis He is a Lecturer, in Department of Mathematics, Aden University, Yemen. He has received his Bachelor's, Master's and Pre-PhD degrees from the University of Aden, Department of Mathematics, Yemen. His main research interests include Special Functions and Fractional calculus. He has co-authored over 70 scientific research papers in high-ranked International Journals

Hassen Aydi He received his M.S degree from University of Paris 6 (Pierre et Marie Curie, France) and his PhD from University of Paris 12 (Val de Marne, France), in 2001 and 2004, respectively. He is actually associate Professor in University of Sousse (Tunisia). He is the author of more than 370 research papers indexed in WOS and Scopus. He has many research projects with different international Universities. His research interests include Ginzburg-Landau model, Nonlinear Analysis, Magnetic vorticity, Fixed point theory, Best proximity point theory. He is Editor in many international journals. He was Highly cited Researcher from 2015 till 2018 in Clarivate (WOS) database.