Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170203

Some Differential Identities for Rings Involving Prime Ideals

J. J. Jaraden[1,], B. N. Al-Hasanat[1], L. Oukhtite[2] and I. A. Al-lawama[1]*

[1]Department of Mathematics, Faculty of Science, Al-Hussein Bin Talal University, Ma'an - Jordan
[2]Department of Mathematics S. M. ben Abdellah University of Fez Faculty of Science and Technology, Fez, Morocco

Received: 22 Nov. 2022, Revised: 2 Jan. 2023, Accepted: 27 Jan. 2023
Published online: 1 Mar. 2023

Abstract: In many studies on ring structures, derivations are found to play a significant role in determining commutativity. In this paper we show the commutativity of a prime ring using certain identities on the generalized semi-derivation of such ring. The considered derivation maps will be either centralizing or commuting. We intend to propose such results as a new method to demonstrate prime ring commutativity.

Keywords: Rings, prime ring, derivations, generalized derivations, semi-derivation, generalized semi-derivation.

1 Introduction

Several papers in the literature indicate that additive mappings defined on prime rings are tightly related to ring structures. Many of the obtained results go beyond the ones previously obtained for some mapping acting on the whole ring .

The second theorem of Posner in [1], demonstrated the commutativity of a prime ring R that admits a nonzero centralizing derivation. Although, it is not completely clear why Posner's second theorem has been conjectured and for what motivation it has been proven, but this theorem introduced some new notions and realized the commutativity of rings using different perspective.

Nowadays, there is a considerable interest in investigating the commutativity of rings, mostly of prime rings admitting the derivation mappings which are commuting on some convenient subsets of R. Moreover, the use of derivation maps, will be an iterative method that shows the commutativity of a given ring instead of the usual methods. Recently, the commutativity of prime and semi-prime rings admitting derivations and generalized derivations has been addressed extensively in literature, as shown in [2,3,4,5].

To extend the previous work on prime ring commutativity, this study establishes new identities of ring derivation (the generalized semi-derivation) on prime rings that indicate their commutativity.

2 Preliminaries

Recall that, an additive map ϕ on a ring R is a function that preserves the addition operation of R. That is, if $\phi : R \to R$ is an additive map, then $\phi(x+y) = \phi(x) + \phi(y)$ for all $x, y \in R$. On the other hand, ϕ does not necessarily preserve the product operation of R. Many ways have been considered to study the ring properties involving some additive mappings, as the derivation maps. Where a map $d : R \to R$ is a derivation of a ring R if d is additive and satisfies the Leibnitz' rule; $d(ab) = d(a)b + ad(b)$, for all $a, b \in R$.

Let R be a ring. The trivial derivation map of R can be defined as $d : R \to R$ by $d(x) = 0$ for all $x \in R$. Also, the well-known derivation on $\mathbb{R}[X]$ is the map $d : \mathbb{R}[X] \to \mathbb{R}[X]$ defined by $d(P) = P'$ for all $P \in \mathbb{R}[X]$ (usual derivation). Certainly, not every additive map on R is a derivation map. For example the additive map $d : R \to R$ defined by $d(x) = x$ does not satisfy the Leibniz's rule, so it is not a derivation of R.

The following is an additive map that associated with a given derivation map.

Definition 2.1.[6] An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

It is possible to construct many additive maps as a generalized derivation on a ring R, this can be usually established using the trivial derivation

* Corresponding author e-mail: jjaradeen@ahu.edu.jo

$d : R \to R$, $d(x) = 0$. As an example, for a ring R, Define the map $F : R \to R$ by $F(x) = 2x$ and $d : R \to R$ by $d(x) = 0$. Then

$$2xy = F(xy) = F(x)y + xd(y) = 2xy + x(0) = 2xy$$

This implies that F is a generalized derivation on R.

Using the next definition one can infer a derivation map associated with a given ring function.

Definition 2.2.[7] For a ring R, an additive function $f : R \to R$ is called a semi-derivation associated with a function $g : R \to R$ (or simply a semi-derivation of a ring R) if:

1. $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$, for all $x, y \in R$.
2. $d(g(x)) = g(d(x))$, for all $x \in R$

Remark 2.1. Every derivation is a semi-derivation, but the converse is not true. As shown on the next example.

Example 2.1. Consider the ring $(R,+,\cdot)$, where $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c \in \mathbb{Z} \right\}$ and define $d : R \longrightarrow R$ by $d \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ along with $g : R \longrightarrow R$ by $g \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. We claim that d is a semi-derivation associated with the additive mapping g, indeed for $A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix}$ in R, one has:

$$\begin{aligned}
d(AB) &= d\left[\begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & a_1b_2 + a_2b_4 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix} \\
&= d\begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix} + g\begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix} d\begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix} \\
&= d(A)B + g(A)d(B).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d(g(A)) &= d\left(g\begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix} \right) \\
&= d\begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix} = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix} \\
&= g\begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} = g\left(d\begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix} \right) \\
&= g(d(A)).
\end{aligned}$$

One can easily show that d does not satisfy Leibnitz' rule. Therefore, d is a semi-derivation ring of R associated with g, but d is not a derivation ring of R. This shows that not every semi-derivation is a derivation.

Combining the two previous definitions, one can be able to present the next derivation map.

Definition 2.3.[8] The additive map F on a ring R is a generalized semi-derivation of R associated with a function $g : R \to R$ and a derivation $d : R \to R$ if $F(xy) = F(x)y + g(x)d(y) = F(x)g(y) + xd(y)$ and $F(g(x)) = g(F(x))$, for all $x, y \in R$.

Remark 2.2. Every semi-derivation is a generalized semi-derivation, but the converse is not true. See the next example.

Example 2.2. Let $M_2(\mathbb{Z}_2)$ be the ring of all 2×2 matrices over $\mathbb{Z}_2$. Define the additive map $F : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$, by $F\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, the derivation $d : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$, by $d\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ and $g : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$, by $g\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ for all $a,b,c,d \in \mathbb{Z}_2$. Therefore, if $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in M_2(\mathbb{Z}_2)$, then

1. F is generalized semi-derivation of $M_2(\mathbb{Z}_2)$, because

$$F(AB) = \begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & 0 \end{bmatrix}$$

and

$$\begin{aligned}
F(A)B + g(A)d(B) &= \begin{bmatrix} a_1 & a_2 \\ a_3 & 0 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} 0 & -b_2 \\ b_3 & 0 \end{bmatrix} \\
&= \begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 & a_3b_2 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ a_4b_3 & -a_3b_2 \end{bmatrix} \\
&= \begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & 0 \end{bmatrix}.
\end{aligned}$$

Further,

$$F(g(A)) = F\left(\begin{bmatrix} 0 & 0 \\ c_1 & d_1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ c_1 & 0 \end{bmatrix} = g(F(A))$$

2. F is not a semi-derivation ring of R. Since

$$F(AB) = \begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & 0 \end{bmatrix},$$

and

$$\begin{aligned} F(A)B+g(A)F(B) &= \begin{bmatrix} a_1 & a_2 \\ a_3 & 0 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1b_1+a_2b_3 & a_1b_2+a_2b_4 \\ a_3b_1 & a_3b_2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ a_3b_1+a_4b_3 & a_3b_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1b_1+a_2b_3 & a_1b_2+a_2b_4 \\ a_4b_3 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $F(AB) \neq F(A)B+g(A)F(B)$. This show that not every generalized semi-derivation is a semi-derivation.

3 Main results

Throughout this paper R represents an associative ring, the center of R is $Z(R)$ and the commutator of $a,b \in R$ is $[a,b] = ab-ba$. Recall that R is a prime ring if $\forall a,b \in R$, then $arb = 0$ for all r in R implies that either $a = 0$ or $b = 0$. Which indicates that, a commutative ring is a prime ring if and only if it is an integral domain. A generalized semi-derivation F of a given ring associated with an additive map f and a derivation d will be written as F associated with (f,d).

In this section, we show the commutativity of prime ring R using certain identities on its semi-generalized derivations.

Theorem 3.1. For a prime ring R. Let F be a nonzero generalized semi-derivation of R that associated with (f,d) and $F(xy) \in Z(R)$ for all $x,y \in R$. Then R is commutative.

Proof. Assume that

$$F(xy) \in Z(R) \quad \text{for all } x,y \in R. \tag{1}$$

in (1), replace y by yr, it follows

$$F(xy)f(r)+xyf(r) \in Z(R) \quad \forall r,x,y \in R. \tag{2}$$

Commuting equation (2) with $f(r)$, implies that

$$[F(xy),f(r)]f(r)+[xyf(r),f(r)] = 0$$

and since $F(xy) \in Z(R)$, then $[F(xy),f(r)]f(r) = 0$, and so

$$[xyf(r),f(r)] = 0 \quad \forall r,x,y \in R.$$

Therefore, $xyf(r)f(r)-f(r)xyf(r) = 0$, which indicates that $f(r) \in Z(R)$ for all $r \in R$. That is to say that, if $F(xy) \in Z(R)$, then the associated additive map f maps R to $Z(R)$. The last equation formulated as

$$xy[d(r),f(r)]+[x,f(r)]yf(r)+x[y,f(r)]d(r) = 0, \tag{3}$$

for all $r,x,y \in R$.
Replacing x by tx in (3), implies

$$[t,f(r)]xyf(r) = 0 \quad \forall r,t,x,y \in R. \tag{4}$$

Writing $d(r)y$ for y in (4), it follows that $[t,f(r)]xf(r)yf(r) = 0$, so that

$$[t,f(r)]Rf(r)Rf(r) = 0 \quad \forall r,t \in R. \tag{5}$$

Since R is prime, then for each $r \in R$, either $[t,f(r)] = 0$ for all $t \in R$ or $d(r) = 0$.
Since $f(R) \subset Z(R)$, then using (2), one finds that

$$xy[d(r),t]+[x,t]yf(r)+x[y,t]d(r) = 0 \quad \forall r,t,x,y \in R. \tag{6}$$

Putting sx for x in (6), implies

$$[x,t]yRf(r) = 0 \quad \text{for all } r,t,x,y \in R. \tag{7}$$

Since R is proper, so R is commutative or $d(R) = 0$.
Now, if $d(R) = 0$, then our assumption becomes

$$F(x)[y,r]+[F(x),r]y = 0 \quad \forall r,x,y \in R. \tag{8}$$

Writing ys instead of y in (8), one can easily verify that

$$F(x)R[s,r] = 0 \quad \forall r,s,x \in R.$$

Therefore, $F = 0$ or R is commutative, which contradicts our hypothesis. □

Certainly, any prime ring admits a generalized semi-derivation with the same identities in Theorem 3 is an integral domain.

If we set $F \pm G$ instead of F in Theorem 3, then we get the next results.

Corollary 3.1. For a prime ring R. Let F and G be two generalized semi-derivations of R associated with (f,d_1) and (g,d_2) respectively. If $F \neq G$ and $F(xy)-G(xy) \in Z(R)$, for all $x,y \in R$, then R is commutative.

Corollary 3.2. For a prime ring R. Let F and G be two generalized semi-derivations of R associated with (f,d_1) and (g,d_2) respectively. If $F \neq -G$ and $F(xy)+G(xy) \in Z(R)$, for all $x,y \in R$, then R is commutative.

Theorem 3.2. Let R be a prime ring and F is a nonzero generalized semi-derivation of R associated with (f,d) . Then the following are equivalent:

1.$F(xy)-yx \in Z(R)$ for all $x,y \in R$.
2.R is commutative.

Proof. Suppose that

$$F(xy)-yx \in Z(R) \quad \forall x,y \in R. \tag{9}$$

Analogously, substituting yr for y in (9), we get

$$F(xy)f(r)+xyf(r)-yrx \in Z(R) \quad \forall r,x,y \in R. \tag{10}$$

From this relation it follows that

$$[xyf(r),f(r)]-[yrx,f(r)]+[yx,f(r)]f(r)=0 \quad \forall r,x,y\in R. \tag{11}$$

Replacing y by $f(r)y$ in (11), implies that

$$\big[[x,f(r)]yf(r),f(r)\big]=0 \quad \forall r,x,y\in R. \tag{12}$$

Write xt for x in (12), we obtain

$$\big[x[t,f(r)]yf(r),f(r)\big]=0 \quad \forall r,t,x,y\in R$$

and so

$$[x,f(r)][t,f(r)]yf(r)=0 \quad \forall r,t,x,y\in R. \tag{13}$$

Replacing x by tx in (13) and combining it with the above relation, one finds that

$$[t,f(r)]R[t,f(r)]Rf(r)=0 \quad \forall r,t,y\in R. \tag{14}$$

By the primeness of R, we get $f(R)\subset Z(R)$ or $d(R)=0$. If $f(R)\subset Z(R)$, then equation (10) reduces to

$$xyf(r)-yxr\in Z(R) \quad \forall r,x,y\in R. \tag{15}$$

Substituting sx for x in (15), we obviously get

$$\big[[y,s]xr,s\big]=0 \quad \forall r,s,x,y\in R \tag{16}$$

Replacing r by rt in (16), we find that

$$[y,s]Rx[r,s]=0 \quad \forall r,s,x,y\in R.$$

That is, R commutative.
Now if $d(R)=0$, then our assumption becomes

$$[F(x)y,r]-[yx,r]=0 \quad \forall r,x,y\in R. \tag{17}$$

Substituting yr for y in (17), we acquire

$$\big[y[x,r],r\big]=0 \quad \forall r,x,y\in R. \tag{18}$$

Writing ry for y in (18) and using the primeness of R, we get the commutativity of R. $\square$

4 Conclusion

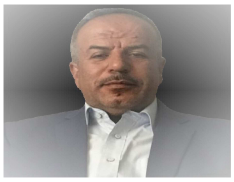
In this paper, we demonstrate the commutativity of prime rings using some identities on their derivations, the used derivation on this issue is the generalized semi-derivation, where the main assumptions is to use a centralizing generalized semi-derivation, or two commuting generalized semi-derivations. The obtained results show that the prime ring derivations associate some of the ring properties such as the commutativity of such rings.

Acknowledgments

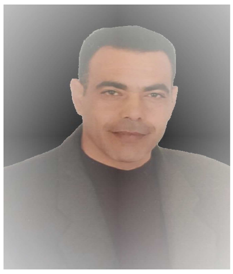
The authors are grateful to Al-Hussein Bin Talal University, for supporting this research under Research Grant No. 401/2020 and to the referees for their valuable remarks.

References

[1] E. Posner , Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8**, 1093-1100 (1957).
[2] S. Ali snd N. A. Dar, On $*$-centralizing mapping in rings with involution. *Georgian Math. J.*, **21**(1):25-28 (2014).
[3] S. Ali, N. A.Dar and M. Asci, On derivations and commutativity of prime rings with involution. *Georgian Math. J.* , **23**(1):9-14 (2016).
[4] J. J. Jaradeen and A. Momouni, Semi-derivations acting on Jordan ideals of ring with involution, *Journal of Taibah University for Science*, **9** (3), 403–406 (2015).
[5] A. Firat, Some results for semi-derivations of prime rings, *Int. J. of Pure and Applied Math.*, **28**(3), 363-368 (2006).
[6] S. Ali, On generalized $*$-derivations in $*$-rings, *Palestine Journal of Mathematics*, **1**, 32-37 (2012).
[7] J. Chang, On Semi-derivations of Prime Rings, *Chinese Journal of Mathematics*, **12**, 255-262 (1984).
[8] C. Haetinger and A. Mamouni, Generalized semi-derivations and generalized left semi-derivations of prime rings, *Palestine Journal of Mathematics*, **7**(1), 28-35 (2018).

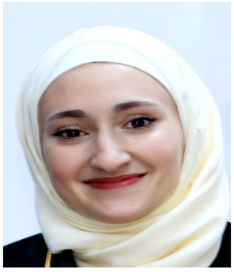


J. J. Jaraden is a professor in pure mathematics. He holds PhD degree in Mathematics from Gomel state University. His current research interests include Group theory and generalizations. He is a full professor in Department of Mathematics, Al Hussein Bin Talal University, Ma'an, Jordan.



B. N. AL-Hasanat completed his PhD in Abstract Algebra, Universiti Sains Malaysia, Penang, Malaysia in 2015. His current research interests include Group theory and generalizations, Geometry, Computing, Graph theory, Field theory and polynomials, He has many publications in these topics. He is an associate professor in Department of Mathematics, Al Hussein Bin Talal University, Ma'an, Jordan.

L. Oukhtite is a professor in pure mathematics. He holds PhD degree in Mathematics from University of Fez, Morocco. He is a full professor in Department of Mathematics, S. M. ben Abdellah University of Fez Faculty of Science and Technology, Fez, Morocco.



I. Al-lawama is a master student. Department of Mathematics, Al Hussein Bin Talal University, Ma'an, Jordan.