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The Analytical and Simplest Resolution of Linear Navier-Stokes Equations

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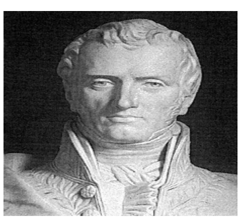
Abstract: The nonlinear Navier-Stokes equations are converted to the linear diffusion equations by Mohammadein (Appl. Math. & Info. Sci. Lett. (2020). The analytical solutions of linear Navier-Stokes equations only are obtained. In this paper, the pressure gradient is redefined by using Bernoulli concept. The peristaltic incompressible viscous Newtonian fluid flow in a horizontal tube is described by Navier-Stokes equations. The stream function described the flow patterns (laminar, transit and turbulent) for different values of wave lengths. The linear and nonlinear Navier-Stokes equations are satisfied by the obtained analytical solutions based on pressure gradient definition.

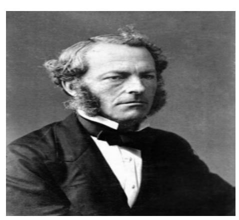
Keywords: Pressure gradient. Nonlinear Navier-Stokes equations. Linear Navier-Stokes equations. Peristaltic flow. Newtonian incompressible fluid. Turbulent flow.

1 Introduction

French physicist engineer Claude-Louis Navier and Anglo-Irish physicist and mathematician George Gabriel Stokes developed the certain nonlinear partial differential equations (**Navier–Stokes equations**) which describes the motion of viscous fluid substances over several decades of progressively building the theories, from 1822 (Navier) to 1842–1850 (Stokes). The solution of nonlinear Navier-Stokes equations is considered as one problem of the "millennium" problems proposed on the Internet at the site http://claymath.org/. and not solved until now.



Claude-Louis Navier



George Stokes

The Navier–Stokes equations may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. Moreover, in their full and simplified forms they help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other things [1-14]. There are many problems can be formulated by the nonlinear partial differential equations, which face some difficulties in the way of analytical solutions and numerical solutions [8,10,11,17] Many physical problems in terms of nonlinear partial differential equations are solved for special cases of fluid and flow properties.

Recently. A finite-difference method for solving the time-dependent Navier-Stokes equations for an incompressible fluid is introduced by Alexandre Chorin [3]. An exact solution of the three-dimensional incompressible Navier-Stokes equations with the continuity equation is produced by Gunawan Nugroho [7]. Mats et al. [10] derived a solution to the Navier–Stokes equation by considering vorticity generated at system boundaries. The transformation of the Navier-Stokes equations to the Schrödinger equation performed by application of the Riccati equation [4]. A particular class of solutions of nonlinear differential equations can be obtained by several procedures [14,17-18] is obtained [14, 18].

Vladimir assumed that fluid velocity vector is a conservative field, and the Cole-Hopf transformation is applicable to the Navier- Stokes equation for an incompressible flow and allows reducing the Navier-Stokes equation to the Schrödinger equation [19]. The peristaltic motion of viscous fluid in different shapes of tubes and plates is obtained for long wavelength and low Reynolds number as given by [5, 6,14].

In the last two years, the nonlinear Partial differential equations are transformed to the linear diffusion ones on the basis of a linear velocity operator concept which was proposed by Mohammadein et al. [15-16]. In fluid mechanics, the fluid state is described by Lagrange and Euler [9] as a particle and point in space, respectively. The fluid state is considered as a particle in the point of view of Lagrange. Moreover, acceleration is defined as a total differentiation of particle velocity like classical mechanics. Euler proposed that acceleration of fluid state consists of local acceleration $\frac{\partial \mathrm{v}}{\partial t}$ and nonlinear convective acceleration

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$(\underline{v}.\underline{\nabla})\underline{v}$; performs a strong obstacle against the analytical solutions of Navier-Stokes equations up to date. The definition of total operator $\frac{D\ldots}{Dt}$ with local and convective terms in fluid mechanics has the form

$$\underbrace{\frac{D\ldots}{Dt}}_{total\ derivative} = \underbrace{\frac{\partial\ldots}{\partial t}}_{local\ derivative} + \underbrace{(\underline{\hat{v}}.\underline{\nabla})\ldots.}_{convective\ ter} \quad (1)$$

The new definition of total operator $\frac{D\ldots}{Dt}$ with local and linear diffusion terms in fluid mechanics assumed by Mohammadein theory [15] in the form

$$\underbrace{\frac{D\ldots}{Dt}}_{total\ derivative} = \underbrace{\frac{\partial\ldots}{\partial t}}_{local\ derivative} + M^* \underbrace{\underline{\nabla}^2\ldots}_{diffusion\ derivative} \quad (2)$$

The linear velocity operator [15] is modified in terms of the physical parameter M* as follows

$$\underbrace{\underline{\hat{v}}}_{linear\ velocity\ operator} = -M^*\underline{\nabla}, \quad (3)$$

where M^* is called Mohammadein parameter. The nonlinear acceleration of fluid in the point of view of Euler has the form

$$\underbrace{\frac{D\underline{v}}{Dt}}_{total\ acceleration} = \underbrace{\frac{\partial\underline{v}}{\partial t}}_{local\ acceleration} + \underbrace{(\underline{\hat{v}}.\underline{\nabla})\underline{v}}_{convective\ acceleration} \quad (4)$$

is converted to the linear acceleration as follows

$$\frac{D\underline{v}}{Dt} = \frac{\partial\underline{v}}{\partial t} - \nu\,\underline{\nabla}^2\underline{v}. \quad (5)$$

In this paper, the pressure gradient is performed in a physical form on the basis of Bernoulli equation. The continuity and linear Navier-Stokes equations reformulated in the mathematical model, which are valid for many physical models of fluid flow.

The analytical solution of continuity and linear Navier-Stokes equations in terms of stream function and fluid velocity components are obtained. As a physical application, the present mathematical model is used for studying the peristaltic flow of an incompressible Newtonian viscous fluid in a horizontal tube. Moreover, the analytical solutions of continuity and linear Navier-Stokes equations are obtained for different values of Reynolds number, wave lengths λ and flow patterns (laminar, transit and turbulent flow). The linear and nonlinear Navier- Stokes equations are satisfied by the obtained analytical solutions in this work.

In sec,2,1, the pressure gradient concept on the basis of Bernoulli by using Mohammadein theory [15] is reformulated. The Navier-Stokes equations formulated in the vector form under the effect of surface and body forces in sec, 2,2. Moreover, the incompressible and viscous Newtonian fluid motion in 2D cartesian coordinates is formulated by continuity and Navier-Stokes equations and the analytical solution is obtained. The discussion of analytical solution and conclusions of the proposed model are introduced in section 2.3. In section 3. For a first time, the unsteady incompressible and viscous Newtonian fluid flow in a horizontal tube for different wave lengths ($\lambda \neq 0$ and $\delta \neq 0$) are described by linear Navier-Stokes equations. The results and graphs are discussed in detail. Finally, in section 4, the concluded remarks are tabulated.

2 Analysis

In this section, the pressure gradient in Navier-Stokes equations is adjusted. In current work, the mathematical model is formulated in two dimensional cartesian coordinates (x, y).

2.1. Pressure gradient concept

The pressure gradient represents a surface force, which is a dominant parameter for the fluid flow. The gradient of pressure on the basis of Bernoulli equation has the form

$$\underline{\nabla}P = -\rho(\underline{v}.\underline{\nabla})\underline{v} - \rho g\,\underline{\hat{n}} \quad (6)$$

The above formula of pressure gradient based on the theory [15] which becomes

$$\underline{\nabla}P = \eta\underline{\nabla}^2\underline{V} - \rho\,g\,\underline{\hat{n}}. \quad (7)$$

2.2. Linear Navier-Stokes Equations

Consider an incompressible viscous fluid flow under the effect of surface and body forces, which are described by continuity and nonlinear Navier-Stokes equations in the vector form

$$\underline{\nabla}.\underline{v} = 0, \quad (8)$$

$$\rho(\frac{\partial V}{\partial t} + (\underline{v}.\underline{\nabla})\underline{v}) = -\underline{\nabla}P + \underline{\nabla}.\tau_{ij}, \quad (9)$$

where $\underline{\nabla}P$ is the gradient of pressure, and τ_{ij} is the shearing stress for two different kinds of fluids (Newtonian and non-Newtonian fluids).

where

$$\underline{\nabla}.\tau_{ij} = \begin{cases} \eta\,\underline{\nabla}^2\underline{v} & for\ \text{Newtonian fluids} \\ \underline{\nabla}.\tau_{ij} & for\ \text{Non Newtonian fluids} \end{cases}.$$

On the basis of the above equation (7) and the linear acceleration form (5), the vector Navier- Stokes equation (9) becomes

$$\frac{\partial V}{\partial t} = \nu\underline{\nabla}^2\underline{v} + g\,\underline{\hat{n}}. \quad (10)$$

The Navier- Stokes equations in two dimensional cartesian coordinates has the form

$$u_x + v_y = 0\,, \quad (11)$$

$$\frac{\partial u}{\partial t} = \nu(u_{xx} + u_{yy}) + g_x, \quad (12)$$

$$\frac{\partial v}{\partial t} = \nu(v_{xx} + v_{yy}) + g_y, \quad (13)$$

The above linear system called linear Navier-Stokes equations and can be solved by Picard method [16] as an analytical way under the proposed physical initial and boundary conditions.

The stream function $\Psi(x, y, t)$ is obtained by using both relations $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$.

when g_x and g_y are considered. The linear stream function equation becomes

$$\psi_t = \nu(\psi_{xx} + \psi_{yy}) \quad (14)$$

And the solution of above equation (14) becomes

$$\Psi(x, y, t) = -\frac{A_1}{c_2} e^{\nu({c_1}^2+{c_2}^2)t-(c_1 x + c_2 y)} - gt, \quad (15)$$

where c_1,c_2, and $A_1$are constants. The obtained analytical solution (15) in terms of stream function and fluid velocity components are satisfied by continuity and linear Navier-Stokes equations (11-13).

2.3 Discussion of analytical solution of linear system of Navier-Stokes equations

On the basis of pressure gradient concept (7), the nonlinear system of Navier-Stokes equations for Newtonian fluid is transformed to the linear diffusion equations (11-13) on the basis of New treatment theory [15]. The analytical solutions (15) are satisfied the continuity and linear system of Navier-Stokes equations in case of two-dimensional flow.

The discussion of results concluded the following points:

1.The analytical solution of continuity and linear Navier-Stokes equations is obtained in a simple way.

2.The parameter M^*represents the kinematic viscosity ν of nanofluid state in case of Navier-Stokes equations.

3. When fluid acceleration is equal to zero, the fluid velocity has a constant value in the point of view of Lagrange and Euler description. On contrary, in this treatment [16], the fluid flow velocity still existed in unsteady states, in both cases of motion and rest.

4.The fluid velocity components and stream function perform the same order of magnitude in plane (x, y) with constant difference between their values.

In the next section, the problem of peristaltic flow of an unsteady incompressible and viscous Newtonian fluid flow in a horizontal tube is described by continuity and linear Navier-Stokes equations with analytical solution as an application of Mohammadein theory [15].

3 Peristaltic flow of unsteady incompressible and viscous Newtonian fluid flow in a Horizontal Tube for different values of wave lengths ($\lambda \neq 0$ and $\delta \neq 0$)

3.1 Introduction

Most of the previous problems are described by the nonlinear Navier-Stokes equations, which are approximately solved for long wavelength $\delta = 0$ and low Reynolds number [1-14]. In the present application, the proposed problem is solved analytically. Moreover, the stream function and fluid velocity components are obtained for different values of wave lengths λ and Reynolds number values.

3.2 The Physical and Mathematical Description

The peristaltic motion of fluid flow is described by many authors [2, 14] in case of long wave lengths. In the follows, we consider the peristaltic flow of an incompressible Newtonian viscous fluid in a horizontal tube (see Fig. 1). The flow is caused by infinite sinusoidal wave train moving ahead with constant velocity c along the walls of the tube. The gravity force is ignored in our case. The peristaltic boundary condition has the form

$$H = a + b \sin\left(\frac{2\pi}{\lambda}(x - ct)\right), \quad (16)$$

where a is the tube half width, b is the wave amplitude, λ is the wave length and t is the time.

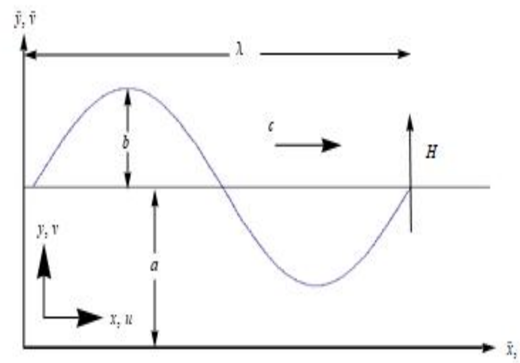


Fig. 1. Sketch of the problem.

Method of solution

The mathematical model for the fluid flow can be written in the form

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (17)$$

Navier-Stokes equations

$$x: \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad (18)$$

$$y: \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \quad (19)$$

where $\nabla P = -\rho(\underline{\hat{v}}.\underline{\nabla})\underline{v} - \rho g\, \underline{\hat{n}}$.

Applying the new treatment theory [15] for the above system (17-19), in the frame $(\overline{x}, \overline{y})$, then

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0, \quad (20)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \nu\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right), \tag{21}$$

$$\frac{\partial \bar{v}}{\partial \bar{t}} = \nu\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right), \tag{22}$$

where $\bar{u} = \frac{\partial \bar{\psi}}{\partial \bar{y}}$ and $\bar{v} = -\frac{\partial \bar{\psi}}{\partial \bar{x}}$.

The nondimensional parameters in terms of dimensional ones have the form

$$\bar{x} = \lambda x,\ \bar{y} = ay,\ \bar{u} = cu,\ \ \bar{v} = c\delta v,\ \ \delta = \frac{a}{\lambda},\ \bar{t} = \frac{\lambda}{c}t,$$

$$\bar{\psi} = a\, c\psi\ , e = \frac{b}{a},\ \text{ and }\ \ h = \frac{H}{a} \tag{23}$$

The above equations (20-22) by using the above transformations (23) in frame (x, y) introduces a linear partial differential equation in terms of stream function ψ in the form

$$R_e\ \delta\ \psi_t = (\delta^2 \psi_{xx} + \psi_{yy}). \tag{24}$$

The analytical solution by using Picard method [16] of above linear partial differential equation (24) has the form

$$\psi(x, y, t) = A_1 e^{\frac{t}{R_e\,\delta}(c_1^2\,\delta^2 + c_2^2) - (c_1 x + c_2 y)}. \tag{25}$$

under the effect of initial and boundary conditions

$$\psi(x, y, 0) = f(x, y) = e^{-(c_1\, x + \, c_2\, y)}$$

$$\psi(0, y, t) = 3$$

$$\psi(L_1, y, t) = 1$$

$$\psi(x, 0, t) = 5, \qquad \psi(x, h_1, t) = 1, \tag{26}$$

where c_1, c_2 and A_1 are constants can be estimated from the initial and boundary conditions (26) as follows:

$$c_1 = \frac{1}{L_1}\ln 3,\quad c_2 = \frac{1}{h}\ln 5, A_1 = 1,$$

$$h = 1 + e\,\sin\,(2\pi(x - t)), L_1 = 5 \tag{27}$$

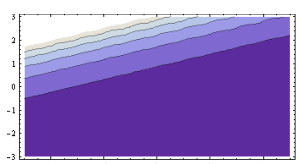
3.3 Discussion of Results

The peristaltic flow of an incompressible Newtonian fluid in a horizontal tube is described by continuity and linear Navier-stokes equations on basis of pressure gradient definition (7). The system of linear partial differential equations (20-22) is transformed to the non-dimensional linear equation (24). The analytical solution is obtained by Picard method [16] in terms of stream function. The obtained solution in terms of wave lengths and Reynolds number. The stream function (25) is obtained graphically for three different values of time and wave lengths as a function of the physical parameters.

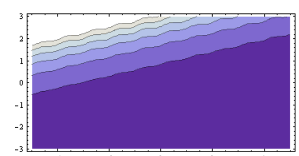
In the follows, numerical values, which are used in calculations of solutions and graphs are considered for flow patterns at Tube radius a=10, b=0.1, e=0.01, $R_e = 7$.as shown in Figs. 2-3 such that, each group of alphabetically lettered figures are put in one row so that all parameters are fixed except one parameter.

In Figs. 2a-f, the streamlines are plotted when $\delta = 0.1$ for time intervals t=0.0,1,5,10,15,20 respectively. In Fig. 2a, the streamlines form straight lines and uniform at time t=0.0. It means that, the streamlines represent a laminar flow. By increasing the time t=1 in Fig. 2b, the streamlines behave a weak peristaltic laminar flow. In Fig. 2c streamlines represent a transitional flow at time t=5. In Fig. 2d, the streamlines transformed to a weak turbulent flow at t=10.In Figs. 2e and 2f, the streamlines transformed to a turbulent flow at time t=15 and t=30. Moreover, the trapped bolus appears for a large value of time where the formation of internally circulating bolus of fluid by the closed streamlines is known as trapping. It is noted that, the transformation of flow patterns from laminar, transit to turbulent flow is proportional directly with increasing time at constant value of parameter $\delta = 0.1$.

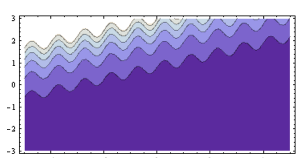
In Figs. 3, the streamlines are plotted at time t=5 for different wave lengths values of parameter $\delta = 0.5, 0.3, 0.1, 0.05, 0.03, 0.01$ respectively. In the Figs. 3a-b, the streamlines are straight and uniform at $\delta = 0.5$, and $\delta = 0.3$;which the laminar and weak laminar flow is observed respectively. In the Figs. 3c-d, the streamlines represent the transitional and weak turbulent flow at $\delta = 0.1$, and $\delta = 0.05$ is observed respectively. In the Figs. 3e-f, the streamlines represent the turbulent and strong turbulent flow at $\delta = .003$ and $\delta = 0.01$ is observed respectively. It is noted that, the transformation of flow patterns from laminar, transit to turbulent flow is inversely proportional with parameter δ at constant time t=5s.



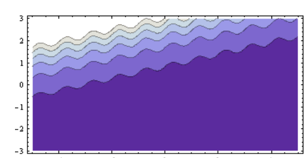
(2a) stream lines at t=0



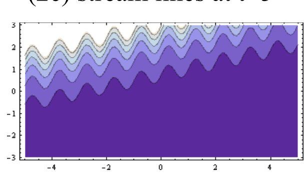
(2b) stream lines at t=1



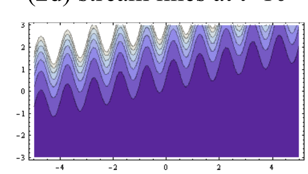
(2c) stream lines at t=5



(2d) stream lines at t=10

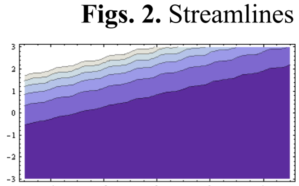


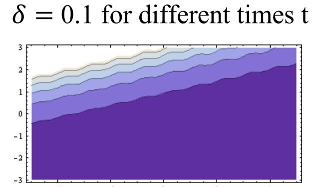
(2e) streamlines at t=15



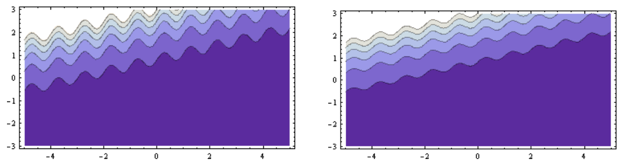
(2f) stream lines at t=30

Figs. 2. Streamlines at $\delta = 0.1$ for different times t

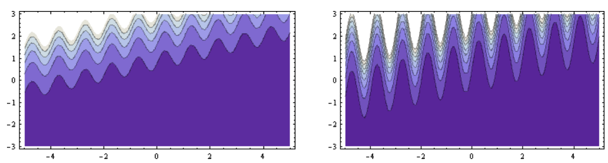




(3a) stream lines at δ = 0.5 (3b) stream lines at δ = 0.3



(3c) streamlines at δ = 0.1 (3d) stream lines at δ= 0.05



(3e) stream lines at $\delta = 0.03$ (3f) stream lines at $\delta = 0.01$.

Figs. 3. Streamlines at time t=5 for different values of δ

4 Conclusions

The peristaltic flow of an incompressible and Newtonian viscous fluid in a horizontal tube is studied as application of linear Navier-Stokes equations. The linear system of Navier-Stokes equations (20-22) is obtained based on New treatment theory [15]. The stream function ψ and fluid velocity components u and v are obtained as an analytical solution of equation (25). The discussion of results and figures concluded the following remarks:

1. The peristaltic motion of Newtonian fluid flow in horizontal tube is studied.

2. The analytical solution in terms of stream function and velocity components is obtained for laminar, transit and turbulent flows in terms of parameter δ.

3. The stream function and fluid velocity components are obtained for different values of wave lengths λ and Reynolds number Re.

4. The time of transformation of flow patterns stages (laminar, transit and turbulent) is proportional directly with the different values of parameter δ.

5. The fluid velocity has a constant value in the point of view of Lagrange when fluid acceleration equal to zero. On contrary, in this new treatment [16], the fluid flow velocity still existed in an unsteady state in both cases of motion and rest.

6. The fluid velocity components are similar to the stream function in plane (x, y) with small difference between them in calculation values.

7. The fluid flow takes more time to transform from laminar to turbulent flow when parameter δ increases.

8. The equation (5) represents the third formulation of fluid mechanics in a linear acceleration diffusion form.

9. The linear Navier-Stokes equations can be solved for different cases of fluid and flow as a future prospect.

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Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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