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Ola Distribution: A New One Parameter Model with Applications to Engineering and Covid-19 Data

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Abstract: We employed the notion of mixture distributions to suggest a new one parameter continuous distribution for modeling real lifetime data called Ola Distribution. Its properties are explored including moments and related measures, moment generating function, reliability analysis functions, order statistics, Bonferroni and Lorenz curves, stochastic ordering, Rényi entropy and mean deviations. The maximum likelihood method is adapted to estimate the parameter of the distribution. Applications to engineering and COVID-19 data sets are presented to illustrate the usefulness of the suggested distribution. The applications showed that Ola distribution outperforms some competitive distributions and can be considered as a useful tool for modeling such real data.

Keywords: Mixture distributions, Reliability, Mean residual life function, Moments, Generating function, Mean deviations, Maximum likelihood, Rényi entropy, Order statistics.

1 Introduction

The motivation of exploring new models comes from the variety of real life data that can't be properly fitted by available distributions. Therefore, many approaches are adopted by researchers to suggest new distributions or improve the flexibility of the existing distributions. Some of them improved the flexibility of the available models by increasing the number of parameters such as: power Lindley distribution [1] , Topp-Leone Mukherjee-Islam distribution [2], two-parameter power Rama distribution [3], transmuted Ishita distribution [4], Darna distribution [5], power Prakaamy distribution [6], power Ishita distribution [7], generalization of Sujatha distribution [8], weighted exponential distribution [9], transmuted Lindley distribution and transmuted Rayleigh distribution [10,11], transmuted Aradhana distribution [12], Loai distribution [13], Sameera distribution [14], transmuted Shanker distribution [15], among others. However, this technique of adding more parameters to base model may increase parameters estimation complexity.

Some authors suggested mixing distributions or weighted distributions, without adding extra parameters, to introduce new distributions such as: Lindely distribution, Ishita distribution [16], weighted Gharaibeh distribution [17], size-biased Ishita distribution [18], Shanker distribution [19], Aradhana distribution [20], Akash distribution [21], Epanechnikov-Weibull distribution [22], Rama distribution [23], Suja distribution [24], Gharaibeh distribution [25], Karam distribution [26]. In this paper, we propose a novel one parameter distribution called Ola distribution using the concept of mixing distributions.

The rest of the paper is organized as follows. The probability density function (pdf) of Ola distribution and their corresponding cumulative distribution function (cdf) are defined in Section 2. In Section 3, moments and associated measures along with the moment generating function (mgf) are derived. In Section 4, reliability analysis functions and their corresponding plots are obtained. In Section 5, order statistics distributions are investigated. Section 6 provides the Bonferroni and Lorenz Curves as well as the Gini Index. In Section 7, the stochastic ordering is investigated. In Section 8, the Rényi entropy is derived. In Section 9, mean deviations about median and mean are obtained. Maximum likelihood estimation (MLE) of the distribution parameter is achieved in Section 10. Applications to engineering and Covid-19 data sets are presented in Section 11. Section 12 summarizes the conclusion.

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2 Proposed Model

The proposed model is a mixture of the following three distributions $Exp(\beta)$, $Gamma(4,\beta)$ and $Gamma(8,\beta)$ with probability density functions $g_1(x) = \beta \mathrm{e}^{-\beta x}$, $g_2(x) = \frac{\beta^4}{3!}x^3\mathrm{e}^{-\beta x}$ and $g_3(x) = \frac{\beta^8}{7!}x^7\mathrm{e}^{-\beta x}$ and mixing proportions $p_1 = \frac{\beta^7}{\beta^7+6\beta^4+5040}$, $p_2 = \frac{6\beta^4}{\beta^7+6\beta^4+5040}$ and $p_3 = \frac{5040}{\beta^7+6\beta^4+5040}$, respectively. Therefore, the (pdf) and (cdf) of the proposed Ola distribution are defined, respectively, as:

$$f(x) = \frac{\beta^8 \cdot \left(x^7+x^3+1\right)\mathrm{e}^{-\beta x}}{\beta^7+6\beta^4+5040}; \quad x,\beta > 0, \tag{1}$$

and

$$F(x) = 1 - \frac{\begin{pmatrix} \beta^7x^7+7\beta^6x^6+42\beta^5x^5 \\ +210\beta^4x^4+\left(\beta^7+840\beta^3\right)x^3 \\ +\left(3\beta^6+2520\beta^2\right)x^2 \\ +\left(6\beta^5+5040\beta\right)x \\ +\beta^7+6\beta^4+5040 \end{pmatrix}\mathrm{e}^{-\beta x}}{\beta^7+6\beta^4+5040}; x,\beta > 0. \tag{2}$$

Figure 1 displays the pdf and cdf plots of the Ola distribution for various distribution parameter values. As can be observed, the Ola distribution is decreasing and right skewed. Additionally, the cdf is an increasing function of β.

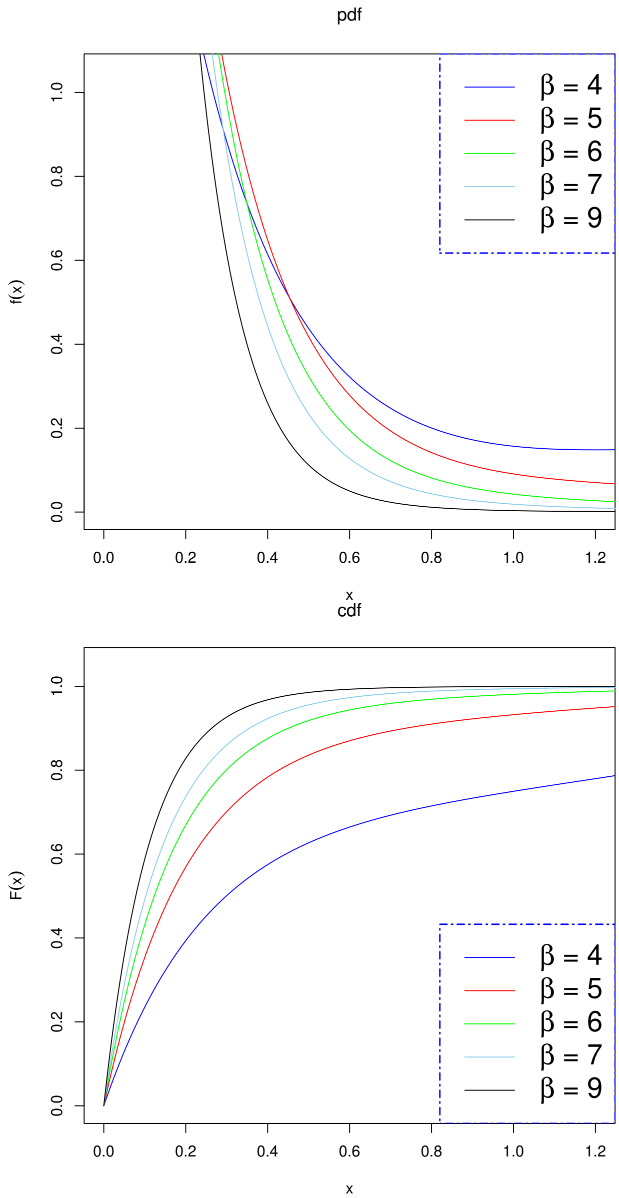

Fig. 1: The pdf and cdf plots of Ola distribution with different β values

3 Moments and associated measures

The mgf and r^{th} moment along with moments related measures for the Ola distribution are explored in this section. Some numerical values of these measures with different distribution parameter values are provided as well.

Theorem 1. *For Ola distribution random variable, the mgf and r^{th} moment about the origin are, respectively, given by*

$$M_X(t) = \frac{\beta^8 \cdot \left((\beta-t)^4\left((\beta-t)^3+6\right)+5040\right)}{(\beta-t)^8\left(\beta^7+6\beta^4+5040\right)} \tag{3}$$

$$E(X^r) = \frac{\beta^7 r! + \beta^4(r+3)! + (r+7)!}{\beta^r\left(\beta^7+6\beta^4+5040\right)}; \; r = 1,2,3,\ldots \tag{4}$$

Proof. Using the Ola distribution's pdf in (1), the mgf in (3) can be proved as

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx \\ &= \int_0^\infty e^{tx} \frac{\beta^8 \cdot \left(1+x^3+x^7\right)\mathrm{e}^{-\beta x}}{\beta^7+6\beta^4+5040}dx \\ &= \int_0^\infty \frac{\beta^8 \cdot \left(1+x^3+x^7\right)\mathrm{e}^{-(\beta-t)x}}{\beta^7+6\beta^4+5040}dx \\ &= \frac{\beta^8}{\beta^7+6\beta^4+5040}\left(\int_0^\infty \left(1+x^3+x^7\right)\mathrm{e}^{-(\beta-t)x}dx\right) \end{aligned}$$

Using the gamma function, we have $\int_0^\infty x^{\alpha-1}\mathrm{e}^{-\beta x}dx = \frac{(\alpha-1)!}{\beta^\alpha}$. Thus,

$$M_X(t) = \frac{\beta^8}{\beta^7+6\beta^4+5040}\left(\frac{1}{(\beta-t)}+\frac{6}{(\beta-t)^4}+\frac{5040}{(\beta-t)^8}\right)$$
$$= \frac{\beta^8\cdot\left((\beta-t)^4\left((\beta-t)^3+6\right)+5040\right)}{(\beta-t)^8\left(\beta^7+6\beta^4+5040\right)}.$$

Similarly, the r^{th} moment in (4) can be proved as

$$E(X^r) = \int_0^\infty x^r f(x)dx$$
$$= \int_0^\infty x^r \frac{\beta^8\cdot\left(1+x^3+x^7\right)\mathrm{e}^{-\beta x}}{\beta^7+6\beta^4+5040}dx$$
$$= \frac{\beta^8}{\beta^7+6\beta^4+5040}\int_0^\infty\left(x^r+x^{r+3}+x^{r+7}\right)\mathrm{e}^{-\beta x}dx$$
$$= \frac{\beta^8}{(\beta^7+6\beta^4+5040)}\left[\frac{r!}{\beta^{r+1}}+\frac{(r+3)!}{\beta^{r+4}}+\frac{(r+7)!}{\beta^{r+8}}\right]$$
$$= \frac{\beta^7 r!+\beta^4(r+3)!+(r+7)!}{\beta^r\left(\beta^7+6\beta^4+5040\right)}$$

From (4), we have

$$\mu = E(X) = \frac{\beta^7+24\beta^4+40320}{\beta\cdot(\beta^7+6\beta^4+5040)}, \quad (5)$$
$$E(X^2) = \frac{2\beta^7+120\beta^4+362880}{\beta^2\cdot(\beta^7+6\beta^4+5040)},$$
$$E(X^3) = \frac{6\beta^7+720\beta^4+3628800}{\beta^3\cdot(\beta^7+6\beta^4+5040)},$$
$$E(X^4) = \frac{24\beta^7+5040\beta^4+39916800}{\beta^4\cdot(\beta^7+6\beta^4+5040)}.$$

Therefore, the variance (σ^2), skewness (SK), kurtosis (KUR) and coefficient of variation (CV) of Ola distribution are, respectively, defined as:

$$\sigma^2 = E(X^2)-\mu^2 = \frac{\left[\begin{array}{l}\beta^{14}+84\beta^{11}+144\beta^8+292320\beta^7\\ +846720\beta^4+203212800\end{array}\right]}{\beta^2\cdot\left(\beta^7+6\beta^4+5040\right)^2}$$

$$SK = \frac{E(X^3)-3\mu E(X^2)+2\mu^3}{\sigma^3}$$
$$= \frac{\left[\begin{array}{l}2\beta^{21}+396\beta^{18}+648\beta^{15}+2570400\beta^{14}\\ +1728\beta^{12}+11612160\beta^{11}+26127360\beta^8\\ -4115059200\beta^7+3657830400\beta^4\\ +2048385024000\end{array}\right]}{\left(\begin{array}{l}\beta^{14}+84\beta^{11}+144\beta^8+292320\beta^7\\ +846720\beta^4+203212800\end{array}\right)^{\frac{3}{2}}}$$

Table 1: Mean, variance, coefficient of variation, skewness and kurtosis for Ola distribution

β	μ	σ^2	CV	SK	KUR
0.5	15.999	32.004	0.354	0.707	3.750
1.0	7.994	8.023	0.354	0.701	3.749
1.5	5.302	3.649	0.360	0.645	3.752
2.0	3.878	2.295	0.391	0.407	3.708
2.5	2.846	1.928	0.488	0.113	3.188
3.0	1.921	1.740	0.687	0.283	2.472
3.5	1.163	1.246	0.960	0.862	2.886
4.0	0.684	0.692	1.216	1.597	4.996
4.5	0.434	0.348	1.361	2.343	8.837
5.0	0.307	0.179	1.378	2.962	13.749
5.5	0.240	0.100	1.322	3.335	18.236
6.0	0.200	0.062	1.248	3.429	20.848
6.5	0.174	0.042	1.182	3.317	21.226
7.0	0.156	0.031	1.132	3.108	20.044
7.5	0.142	0.024	1.096	2.886	18.214
8.0	0.131	0.020	1.071	2.691	16.359

$$KUR = \frac{E(X^4)-4\mu E(X^3)+6\mu^2E(X^2)-3\mu^4}{\sigma^4}$$
$$= \frac{9\left[\begin{array}{l}\beta^{28}+312\beta^{25}+2304\beta^{22}+3030720\beta^{21}\\ +10368\beta^{19}+32094720\beta^{18}+10368\beta^{16}\\ +148538880\beta^{15}+5012582400\beta^{14}\\ +194503680\beta^{12}+110547763200\beta^{11}\\ +212154163200\beta^8+54282203136000\beta^7\\ +133145026560000\beta^4\\ +17206434201600000\end{array}\right]}{\left(\begin{array}{l}\beta^{14}+84\beta^{11}+144\beta^8+292320\beta^7\\ +846720\beta^4+203212800\end{array}\right)^2}$$

$$CV = \frac{\sigma}{\mu} = \frac{\left(\begin{array}{l}\beta^{14}+84\beta^{11}+144\beta^8+292320\beta^7\\ +846720\beta^4+203212800\end{array}\right)^{1/2}}{\beta^7+24\beta^4+40320}$$

Numerical values of these measures with different values of the parameter β are provided in Table 1. The table illustrates that the mean and variance values decrease as the parameter value increases. Positive skewness values suggest that the Ola distribution is right skewed, as seen in Figure 1. Furthermore, there is no fixed pattern to the skewness and kurtosis values, which increase and decrease depending on the parameter values.

4 Reliability analysis

Based on the pdf and cdf of Ola distribution given in (1) and (2), the survival, $S(x)$, hazard, $h(x)$, cumulative hazard, $H(x)$, reversed hazard, $rh(x)$, odds, $O(x)$, and mean residual life, $MRL(x)$, functions are defined, respectively, as

$$S(x) = 1 - F(x) = \frac{\begin{pmatrix} \beta^7x^7 + 7\beta^6x^6 + 42\beta^5x^5 \\ +210\beta^4x^4 + (\beta^7 + 840\beta^3)\,x^3 \\ + \left(3\beta^6 + 2520\beta^2\right)x^2 \\ + \left(6\beta^5 + 5040\beta\right)x + \beta^7 + 6\beta^4 + 5040 \end{pmatrix} e^{-\beta x}}{\beta^7 + 6\beta^4 + 5040}$$

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\beta^8 \cdot (x^7 + x^3 + 1)}{\begin{pmatrix} \beta^7x^7 + 7\beta^6x^6 + 42\beta^5x^5 + 210\beta^4x^4 \\ + (\beta^7 + 840\beta^3)\,x^3 + \left(3\beta^6 + 2520\beta^2\right)x^2 \\ + \left(6\beta^5 + 5040\beta\right)x \\ + \beta^7 + 6\beta^4 + 5040 \end{pmatrix}}$$

$$H(x) = -ln(1 - F(x)) = \beta x - ln \frac{\begin{pmatrix} \beta^7x^7 + 7\beta^6x^6 + 42\beta^5x^5 \\ +210\beta^4x^4 + (\beta^7 + 840\beta^3)\,x^3 \\ + \left(3\beta^6 + 2520\beta^2\right)x^2 \\ + \left(6\beta^5 + 5040\beta\right)x \\ + \beta^7 + 6\beta^4 + 5040 \end{pmatrix}}{\beta^7 + 6\beta^4 + 5040}$$

$$rh(x) = \frac{f(x)}{F(x)} = \frac{\beta^8 \cdot (x^7 + x^3 + 1)}{\begin{pmatrix} (e^{\beta x} - 1)(\beta^7 + 6\beta^4 + 5040) \\ -\beta^7x^7 - 7\beta^6x^6 - 42\beta^5x^5 - 210\beta^4x^4 \\ -(\beta^7 + 840\beta^3)\,x^3 - \left(3\beta^6 + 2520\beta^2\right)x^2 \\ - \left(6\beta^5 + 5040\beta\right)x \end{pmatrix}}$$

and

$$O(x) = \frac{F(x)}{1-F(x)} = \frac{e^{\beta x}(\beta^7 + 6\beta^4 + 5040)}{\begin{pmatrix} \beta^7x^7 + 7\beta^6x^6 + 42\beta^5x^5 + 210\beta^4x^4 \\ + (\beta^7 + 840\beta^3)\,x^3 + \left(3\beta^6 + 2520\beta^2\right)x^2 \\ + \left(6\beta^5 + 5040\beta\right)x + \beta^7 + 6\beta^4 + 5040 \end{pmatrix}} - 1$$

$$MRL(x) = E(X - x|X > x) = \frac{1}{1-F(x)} \int_x^{\infty} (1 - F(y))\,dy = \frac{\begin{pmatrix} \beta^7x^7 + 14\beta^6x^6 + 126\beta^5x^5 \\ +840\beta^4x^4 + (\beta^7 + 4200\beta^3)\,x^3 \\ + \left(6\beta^6 + 15120\beta^2\right)x^2 \\ + \left(18\beta^5 + 35280\beta\right)x \\ + \beta^7 + 24\beta^4 + 40320 \end{pmatrix}}{\beta \cdot \begin{pmatrix} \beta^7x^7 + 7\beta^6x^6 + 42\beta^5x^5 \\ +210\beta^4x^4 + (\beta^7 + 840\beta^3)\,x^3 \\ + \left(3\beta^6 + 2520\beta^2\right)x^2 \\ + \left(6\beta^5 + 5040\beta\right)x \\ + \beta^7 + 6\beta^4 + 5040 \end{pmatrix}} \quad (6)$$

From (6), $MRL(0) = \frac{\beta^7 + 24\beta^4 + 40320}{\beta \cdot (\beta^7 + 6\beta^4 + 5040)} = E(X)$ given in (5).

The graphs of these aforementioned functions for variant β values are given in Figures 2 and 3. As can be observed, the values of the odds, cumulative hazard and hazard functions increase as β value increases, whereas the values of the survival, reversed hazard, and mean residual life functions decrease.

5 Order Statistics

For a random sample $X_1, X_2, \ldots X_n$ from Ola distribution with pdf in (1) and cdf in (2), assume that $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ represents the random sample order statistics. [27] defined the pdf of the i^{th} order statistics, $X_{(i)}$, as

$$f_{(i)}(x) = n\binom{n-1}{i-1} f(x)(F(x))^{i-1}(1-F(x))^{n-i}, i = 1, 2, .., n \quad (7)$$

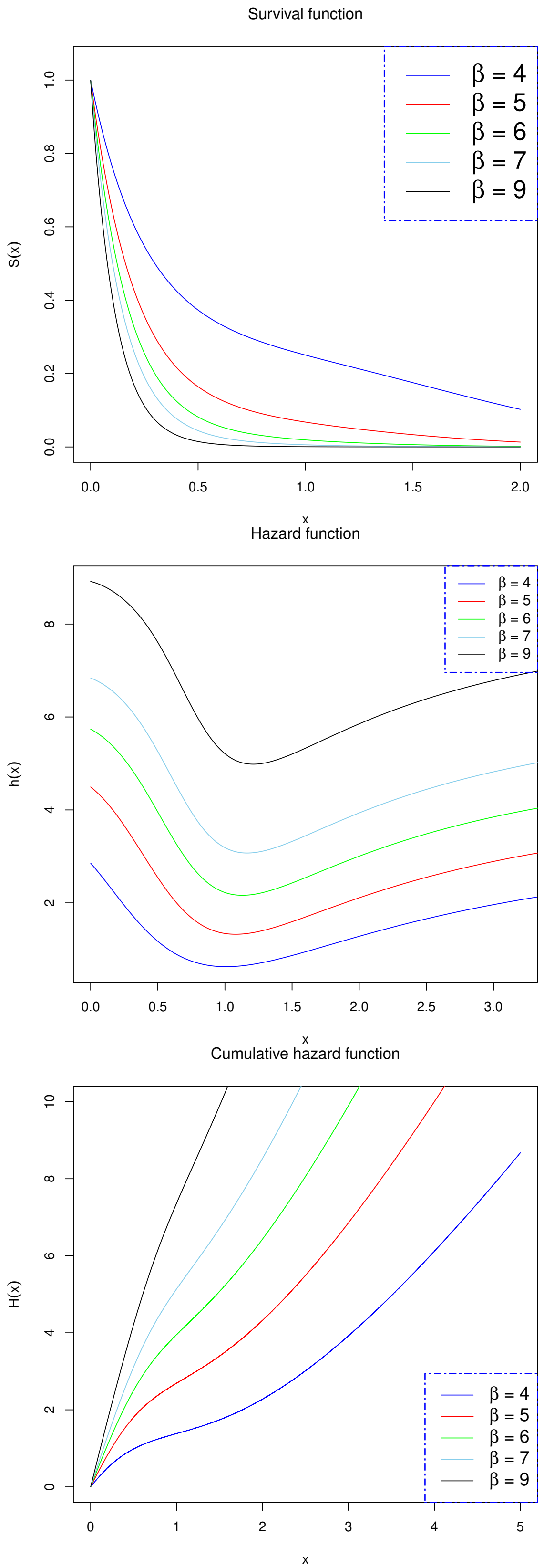

Fig. 2: Ola distribution's survival, hazard, and cumulative hazard functions

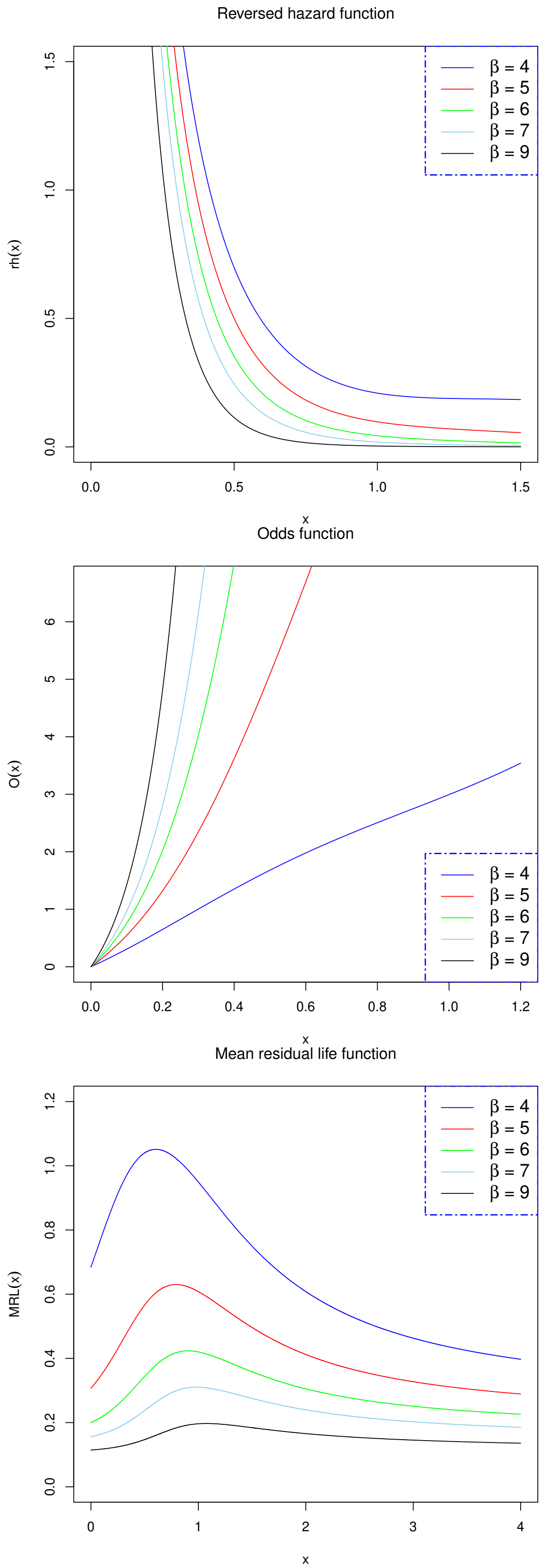

Fig. 3: Ola distribution's reversed hazard, odds, and mean residual life functions

By plugging (1) and (2) in (7) with using binomial series, we have

$$f_{(i)}(x) = n\binom{n-1}{i-1} f(x)\left[\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j(1-F(x))^j\right] \times [1-F(x)]^{n-i}$$

$$= n\binom{n-1}{i-1}\frac{\beta^8\cdot\left(x^7+x^3+1\right)e^{-\beta x}}{\beta^7+6\beta^4+5040} \times \sum_{j=0}^{i-1}(-1)^j e^{-\beta x(n+j-i)} \times \left[\frac{\left(\begin{array}{l}\beta^7x^7+7\beta^6x^6+42\beta^5x^5+210\beta^4x^4+\\ \left(\beta^7+840\beta^3\right)x^3+\left(3\beta^6+2520\beta^2\right)x^2\\ +\left(6\beta^5+5040\beta\right)x+\beta^7+6\beta^4+5040\end{array}\right)}{\beta^7+6\beta^4+5040}\right]^{n+j-i}$$

$$= n\binom{n-1}{i-1}\beta^8\cdot\left(x^7+x^3+1\right)\sum_{j=0}^{i-1}(-1)^j e^{-\beta x(n+j-i+1)} \times \frac{\left(\begin{array}{l}\beta^7x^7+7\beta^6x^6+42\beta^5x^5+210\beta^4x^4+\\ \left(\beta^7+840\beta^3\right)x^3+\left(3\beta^6+2520\beta^2\right)x^2\\ +\left(6\beta^5+5040\beta\right)x+\beta^7+6\beta^4+5040\end{array}\right)^{n+j-i}}{\left(\beta^7+6\beta^4+5040\right)^{n+j-i+1}}$$

6 Bonferroni and Lorenz Curves and Gini index

Bonferroni and Lorenz curves, as well as Gini index, have several applications in various fields and they are, respectively, defined as

$$B(p) = \frac{1}{\mu p}\int_0^q xf(x)dx, \qquad L(p) = \frac{1}{\mu}\int_0^q xf(x)dx, \quad (8)$$

$$G = 1-2\int_0^1 L(p)dp, \quad (9)$$

where $q = F^{-1}(p); 0 < p \leq 1$.

For Ola distribution, substituting (1) and (5) in (8) and (9), we have

$$B(p) = \frac{1}{p} - \frac{\left[\begin{array}{l}\left(q^8+q^4+q\right)\beta^8+\left(8q^7+4q^3+1\right)\beta^7+\\ \left(56q^6+12q^2\right)\beta^6+\left(336q^5+24q\right)\beta^5\\ +\left(1680q^4+24\right)\beta^4+6720q^3\beta^3\\ +20160q^2\beta^2+40320q\beta+40320\end{array}\right]e^{-q\beta}}{p\cdot\left(\beta^7+24\beta^4+40320\right)}$$

$$L(p) = 1 - \frac{\left[\begin{array}{l}\left(q^8+q^4+q\right)\beta^8+\left(8q^7+4q^3+1\right)\beta^7+\\ \left(56q^6+12q^2\right)\beta^6+\left(336q^5+24q\right)\beta^5\\ +\left(1680q^4+24\right)\beta^4+6720q^3\beta^3\\ +20160q^2\beta^2+40320q\beta+40320\end{array}\right]e^{-q\beta}}{\left(\beta^7+24\beta^4+40320\right)}$$

$$G = \frac{2\left[\begin{array}{l}\left(q^8+q^4+q\right)\beta^8+\left(8q^7+4q^3+1\right)\beta^7+\\ \left(56q^6+12q^2\right)\beta^6+\left(336q^5+24q\right)\beta^5\\ +\left(1680q^4+24\right)\beta^4+6720q^3\beta^3\\ +20160q^2\beta^2+40320q\beta+40320\end{array}\right]e^{-q\beta}}{\left(\beta^7+24\beta^4+40320\right)} - 1$$

7 Stochastic Ordering

If X and W are two random variables, then X is smaller than W in

- –Stochastic order $(X \leq_{STO} W)$ if $F_X(x) \geq F_W(x); \forall x$.
- –Mean residual life order $(X \leq_{MRLO} W)$ if $MRL_X(x) \leq MRL_W(x); \forall x$.
- –Likelihood ratio order $(X \leq_{LRO} W)$ if $\frac{f_X(x)}{f_W(x)}$ decreases in x.
- –Hazard rate order $(X \leq_{HRO} W)$ if $h_X(x) \geq h_W(x); \forall x$.

The authors have shown in [28] that

$$X \leq_{LRO} W \Rightarrow X \leq_{HRO} W \Rightarrow X \leq_{STO} W \text{ and } X \leq_{MRLO} W \quad (10)$$

Theorem 2. *Assume that $X \sim Ola(\beta_1)$ and $W \sim Ola(\beta_2)$. For $\beta_1 > \beta_2$, we have $X \leq_{LRO} W$ and hence $X \leq_{HRO} W$, $X \leq_{STO} W$ and $X \leq_{MRLO} W$.*

Proof. Based on Ola distribution pdf in (1), we have

$$\frac{f_X(x;\beta_1)}{f_W(x;\beta_2)} = \frac{\frac{\beta_1^8\cdot\left(x^7+x^3+1\right)e^{-\beta_1 x}}{\beta_1^7+6\beta_1^4+5040}}{\frac{\beta_2^8\cdot\left(x^7+x^3+1\right)e^{-\beta_2 x}}{\beta_2^7+6\beta_2^4+5040}} = \frac{\beta_1^8\left(\beta_2^7+6\beta_2^4+5040\right)}{\beta_2^8\left(\beta_1^7+6\beta_1^4+5040\right)}e^{-x(\beta_1-\beta_2)}$$

Therefore,

$$log\frac{f_X(x;\beta_1)}{f_W(x;\beta_2)} = log\left(\frac{\beta_1^8\left(\beta_2^7+6\beta_2^4+5040\right)}{\beta_2^8\left(\beta_1^7+6\beta_1^4+5040\right)}\right)+x(\beta_2-\beta_1)$$

and

$$\frac{\partial}{\partial x}\left[log\frac{f_X(x;\beta_1)}{f_W(x;\beta_2)}\right] = (\beta_2-\beta_1) < 0; \quad for \quad \beta_1 > \beta_2.$$

Thus, $X \leq_{LRO} W$. Consequently and based on (10), we have $X \leq_{HRO} W$, $X \leq_{MRLO} W$ and $X \leq_{STO} W$.

8 Rényi Entropy

The Rényi entropy [29] has many applications in various fields. It is defined as

$$E_R(\gamma) = \frac{1}{1-\gamma}log\int_0^\infty [f(x)]^\gamma dx; \quad \gamma \neq 1 \text{ and } \gamma > 0. \quad (11)$$

Theorem 3. *For Ola distribution, the Rényi entropy is given as*

$$E_R(\gamma) = \frac{1}{1-\gamma}\left[\gamma\, log\left[\frac{\beta^8}{\beta^7+6\beta^4+5040}\right] + log\left[\sum_{i=1}^{\gamma}\sum_{j=1}^{i}\binom{\gamma}{i}\binom{i}{j}\frac{(3i+4j)!}{(\beta\gamma)^{3i+4j+1}}\right]\right]$$

Proof. By plugging (1) in (11) and using the binomial series, we have

$$\begin{aligned}
E_R(\gamma) &= \frac{1}{1-\gamma} log \int_0^\infty \left[\frac{\beta^8\cdot\left(x^7+x^3+1\right)e^{-\beta x}}{\beta^7+6\beta^4+5040}\right]^\gamma dx \\
&= \frac{1}{1-\gamma} log\left[\left(\frac{\beta^8}{\beta^7+6\beta^4+5040}\right)^\gamma \times \int_0^\infty \left(x^7+x^3+1\right)^\gamma e^{-\beta\gamma x}dx\right] \\
&= \frac{1}{1-\gamma} log\left[\left(\frac{\beta^8}{\beta^7+6\beta^4+5040}\right)^\gamma \times \int_0^\infty \left(x^3(x^4+1)+1\right)^\gamma e^{-\beta\gamma x}dx\right] \\
&= \frac{1}{1-\gamma} log\left[\left(\frac{\beta^8}{\beta^7+6\beta^4+5040}\right)^\gamma \times \int_0^\infty \sum_{i=1}^{\gamma}\binom{\gamma}{i}(x^3(x^4+1))^i e^{-\beta\gamma x}dx\right] \\
&= \frac{1}{1-\gamma} log\left[\left(\frac{\beta^8}{\beta^7+6\beta^4+5040}\right)^\gamma \times \int_0^\infty \sum_{i=1}^{\gamma}\binom{\gamma}{i}x^{3i}\sum_{j=1}^{i}\binom{i}{j}x^{4j}e^{-\beta\gamma x}dx\right] \\
&= \frac{1}{1-\gamma}\left[\gamma\, log\left(\frac{\beta^8}{\beta^7+6\beta^4+5040}\right) + log\left(\sum_{i=1}^{\gamma}\sum_{j=1}^{i}\binom{\gamma}{i}\binom{i}{j}\int_0^\infty x^{3i+4j}e^{-\beta\gamma x}dx\right)\right] \\
&= \frac{1}{1-\gamma}\left[\gamma\, log\left(\frac{\beta^8}{\beta^7+6\beta^4+5040}\right) + log\left(\sum_{i=1}^{\gamma}\sum_{j=1}^{i}\binom{\gamma}{i}\binom{i}{j}\frac{(3i+4j)!}{(\beta\gamma)^{3i+4j+1}}\right)\right]
\end{aligned}$$

9 Mean Deviations about median and mean

Mean deviations about the median $MD(\omega)$ and mean $MD(\mu)$ are , respectively, defined as

$$MD(\omega) = \int_0^\infty |x-\omega| f(x)dx = \mu - 2\int_0^\omega xf(x)dx, \quad (12)$$

$$MD(\mu) = \int_0^\infty |x-\mu| f(x)dx = 2\mu F(\mu) - 2\int_0^\mu xf(x)dx, \quad (13)$$

where ω and μ are the median and mean, respectively. Using the pdf of Ola distribution in (1), we have

$$\int_0^k xf(x)dx = \mu - \frac{\begin{pmatrix}\left(k^8+k^4+k\right)\beta^8+\left(8k^7+4k^3+1\right)\beta^7+\\ \left(56k^6+12k^2\right)\beta^6+\left(336k^5+24k\right)\beta^5\\ +\left(1680k^4+24\right)\beta^4+6720k^3\beta^3\\ +20160k^2\beta^2+40320k\beta+40320\end{pmatrix}e^{-k\beta}}{\beta\cdot\left(\beta^7+6\beta^4+5040\right)} \quad (14)$$

By using (2) and (14) in (13) and (12), the mean deviations for Ola distribution are obtained as

$$MD(\omega) = \frac{2e^{-\omega\beta}\begin{pmatrix}\left(\omega^8+\omega^4+\omega\right)\beta^8+\left(8\omega^7+4\omega^3+1\right)\beta^7+\\ \left(56\omega^6+12\omega^2\right)\beta^6+\left(336\omega^5+24\omega\right)\beta^5\\ +\left(1680\omega^4+24\right)\beta^4+6720\omega^3\beta^3\\ +20160\omega^2\beta^2+40320\omega\beta+40320\end{pmatrix}}{\beta\cdot\left(\beta^7+6\beta^4+5040\right)} - \mu$$

and

$$MD(\mu) = \frac{2e^{-\mu\beta}\begin{pmatrix}\left(\mu^7+\mu^3+1\right)\beta^7+\left(14\mu^6+6\mu^2\right)\beta^6+\\ \left(126\mu^5+18\mu\right)\beta^5+\left(840\mu^4+24\right)\beta^4\\ +4200\mu^3\beta^3+15120\mu^2\beta^2\\ +35280\mu\beta+40320\end{pmatrix}}{\beta\cdot\left(\beta^7+6\beta^4+5040\right)}.$$

10 Maximum Likelihood Estimation

Consider $X_1, X_2, \ldots, X_n$ to be a random sample from Ola distribution with parameter β and pdf in (1). Then, the Ola distribution likelihood function can be written as

$$\begin{aligned}
L(\beta|x_1,x_2,\ldots,x_n) &= \prod_{i=1}^{n}\frac{\beta^8\cdot\left(x_i^7+x_i^3+1\right)e^{-\beta x_i}}{\beta^7+6\beta^4+5040} \\
&= \left(\frac{\beta^8}{\beta^7+6\beta^4+5040}\right)^n \times \prod_{i=1}^{n}\left(x_i^7+x_i^3+1\right)e^{-\beta\sum_{i=1}^{n}x_i}.
\end{aligned}$$

Thus, the log-likelihood function is

$$\begin{aligned}
L^* &= lnL(\beta|x_1,x_2,\ldots,x_n) \\
&= nln\left(\frac{\beta^8}{\beta^7+6\beta^4+5040}\right) + \sum_{i=1}^{n} ln\left(x_i^7+x_i^3+1\right) - \beta\sum_{i=1}^{n}x_i \\
&= 8n\,ln(\beta) - n\,ln\left(\beta^7+6\beta^4+5040\right) + \sum_{i=1}^{n} ln\left(x_i^7+x_i^3+1\right) - \beta\sum_{i=1}^{n}x_i
\end{aligned}$$

Table 2: Data 1: ball bearings data set

51.84	51.96	54.12	68.88	55.56	67.80
68.44	68.64	84.12	98.64	105.12	93.12
105.84	127.92	128.04	173.40		

Table 3: Data 2: The number of new deaths caused by COVID-19 in Jordan from January 31, 2021 to February 28, 2021

12	10	8	10	10	8	7	10	6	10
16	10	12	11	11	18	18	12	9	16
15	11	16	19	22	16	23	25	26	

Table 4: Numerical description (mean and five-number summary) for both data sets

	Min.	Q_1	*Median*	*Mean*	Q_3	*Max.*
Data 1	51.84	64.74	76.5	87.72	105.3	173.4
Data 2	6.00	10.00	12.0	13.69	16.0	26.0

Therefore,

$$\frac{\partial L^*}{\partial \beta} = \frac{8n}{\beta} - \frac{n(7\beta^6+24\beta^3)}{\beta^7+6\beta^4+5040} - \sum_{i=1}^{n} x_i$$

The MLE of β is the solution of $\frac{\partial L^*}{\partial \beta} = 0$. This can be found by solving the equation below

$$(\beta^7+24\beta^4+40320) - \bar{x}\beta\left(\beta^7+6\beta^4+5040\right) = 0.$$

11 Applications to Real Data

This section illustrates the applicability of Ola distribution to two real data sets. Table 2 shows the first data set, which indicates revolutions-to-failure (in millions) for ball bearings in a fatigue test which is studied by [18]. Table 3 contains the second set of data, which reflects the number of new deaths caused by COVID-19 in Jordan from January 31, 2021 to February 28, 2021. This data can be found at the website of World Health Organization (WHO). Table 4 provides a numerical description for both data sets.

These two considered data sets are fitted using the proposed Ola distribution and its goodness of fit is compared with the following one parameter distributions:

–Exponential: $f(x) = \theta e^{-x\theta}$; $x,\theta > 0$.
–Lindley: $f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-x\theta}$; $x,\theta > 0$.
–Ishita [16]: $f(x) = \frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-x\theta}$; $x,\theta > 0$.
–Shanker [19]: $f(x) = \frac{\theta^2}{\theta^2+1}(\theta+x)e^{-x\theta}$; $x,\theta > 0$.
–Aradhana [20]: $f(x) = \frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2e^{-x\theta}$; $x,\theta > 0$.
–Akash [21]: $f(x) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-x\theta}$; $x,\theta > 0$.

based on the criteria: -2logL, Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Kolmogorov-Smirnov Statistic (KS) and its p-value. The results are summarized in Table 5.

Table 5: Goodness of fit criteria for all fitted distributions

Data	*Distribution*	$-2logL$	*AIC*	*BIC*	*KS*	*p-value*
Data 1	**Ola**	**154.205**	**156.205**	**156.978**	**0.203**	**0.463**
	Ishita	**160.047**	**162.047**	**162.820**	**0.262**	**0.184**
	Exponential	**175.171**	**177.171**	**177.944**	**0.447**	**0.002**
	Lindley	**165.198**	**167.198**	**167.970**	**0.334**	**0.043**
	Shanker	**164.898**	**166.898**	**167.670**	**0.331**	**0.046**
	Aradhana	**160.314**	**162.314**	**163.087**	**0.266**	**0.172**
	Akash	**160.059**	**162.059**	**162.832**	**0.263**	**0.183**
Data 2	**Ola**	**174.735**	**176.735**	**178.102**	**0.183**	**0.286**
	Ishita	**183.401**	**185.401**	**186.768**	**0.203**	**0.185**
	Exponential	**209.765**	**211.765**	**213.132**	**0.374**	**0.001**
	Lindley	**194.573**	**196.573**	**197.940**	**0.276**	**0.024**
	Shanker	**192.060**	**194.060**	**195.427**	**0.259**	**0.041**
	Aradhana	**186.069**	**188.069**	**189.436**	**0.217**	**0.130**
	Akash	**184.092**	**186.092**	**187.459**	**0.206**	**0.172**

Table 6: MLEs of the parameters for all fitted distributions and their related confidence intervals

Data	*Distribution*	*MLE*	*Error*	*Lower bound C.I.*	*Upper bound C.I.*
Data 1	**Ola**	**0.091**	**0.008**	**0.075**	**0.107**
	Ishita	**0.034**	**0.005**	**0.025**	**0.044**
	Exponential	**0.011**	**0.003**	**0.006**	**0.017**
	Lindley	**0.023**	**0.004**	**0.015**	**0.030**
	Shanker	**0.023**	**0.004**	**0.015**	**0.031**
	Aradhana	**0.034**	**0.005**	**0.024**	**0.043**
	Akash	**0.034**	**0.005**	**0.025**	**0.044**
Data 2	**Ola**	**0.584**	**0.038**	**0.509**	**0.660**
	Ishita	**0.218**	**0.023**	**0.173**	**0.264**
	Exponential	**0.073**	**0.014**	**0.046**	**0.100**
	Lindley	**0.137**	**0.018**	**0.102**	**0.173**
	Shanker	**0.144**	**0.019**	**0.107**	**0.181**
	Aradhana	**0.204**	**0.022**	**0.161**	**0.247**
	Akash	**0.216**	**0.023**	**0.171**	**0.261**

Table 5 indicates that Ola distribution provides better fit for both data sets than other considered distributions as it has the smallest values of $-2logL$, BIC, AIC and KS with highest p-value. Table 6 give the MLEs of the parameters for all fitted distributions with their corresponding standard error and confidence intervals.

12 Concluding Remarks

In this article, a novel one parameter continuous distribution based on the idea of mixture distributions. Several mathematical properties such as: moments and their related measures, moment generating function, reliability analysis functions, Bonferroni and Lorenz curves, stochastic ordering, mean deviations, order statistics and Rényi entropy are studied and investigated. The suggested distribution's parameter is estimated using maximum likelihood technique. The usefulness of the proposed distribution is illustrated in modeling Covid-19 and ball bearing data sets.

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