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Linear Elliptic Control Problems of Infinite Order with Pointwise State Constraints

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Abstract: This paper deals with optimal control problems of systems governed by elliptic operator of infinite order with constraints on the state. Optimal control problem for linear elliptic equation is investigated. Existence of solution for elliptic control problem of infinite order with pointwise state constraints is obtained. Existence of optimal control of systems governed by elliptic operator of infinite order with pointwise state constraints is verified. Optimality conditions are given and reqularity of the optimal solution is investigated. Lagrange multipliers in the optimality condition of systems governed by elliptic operator of infinite order with pointwise state constraints are measures.

Keywords: Linear control problem, infinite order operator, elliptic equations, pointwise state constraints, optimality conditions, Lagrange mulipliers

1 Introduction

This paper is concerned with distributed control problems of infinite order with constaint on the state y. Our main interest is the derivation of optimality conditions. Let consider the optimal control problem to be minimize:

$$(P)\begin{cases} \min J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}, & (1)\\ \text{subject to the state equationsin}\Omega & (2)\\ y^{|w|}|_\Gamma = 0|w| = 0,1,2,\cdots \\ \text{and to the pointwise state constraints} \\ v\in K \quad and \quad |y(v,x)|\leq 1 \text{ for all } \quad x\in\Omega & (3)\end{cases}$$

where K be a nonempty, convex and closed subset of $L^2(\Omega)$ and y_d is a fixed element of $L^2(\Omega)$ and $\lambda \geq 0$. The operator A denotes elliptic operator of infinite order having the form [1,2].

$$Ay = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y \quad a_\alpha > 0. \tag{4}$$

This operator is bounded self-adjoint mapping $W_0^\infty\{a_\alpha,2\}$ onto $W^{-\infty}\{a_\alpha,2\}$.
where

$$W^\infty\{a_\alpha,2\} = \{\phi\in C^\infty(R^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha\|D^\alpha\phi\|_2^2 < \infty\}$$

be a Sobolev space of infinite order of periodic functions $\phi(x)$ defined on all boundary Γ of R^n, $n\geq 1$ where $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is the canonical norm with space $L^2(R^n)$ all functions are assumed to be the real valued on

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots(\partial x_n^{\alpha_n})},$$

where $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_n)$ is a multi-index for differentiation $|\alpha| = \sum_{i=1}^{\infty}\alpha_i$.

And the space $W_0^\infty\{a_\alpha,2\}$ is the set of all functions of $W^\infty\{a_\alpha,2\}$which vanish on the boundary Γ of R^n.

$$W_0^\infty\{a_\alpha,2\} = \left\{\phi(x)\in C_0^\infty(R^n) : \|\phi\|^2 = \sum_{|\alpha|=0}^{\infty} a_\alpha\|D^\alpha\phi\|_2^2 < \infty, \; D^{|w|}\phi|_\Gamma = 0,\ |w| = 0,1,\cdots\right\}$$

From above, $W^\infty\{a_\alpha,2\}$ is everywhere dense in $L^2(R^n)$ with topological inclusions and $W^{-\infty}\{a_\alpha,2\}$ dense the topological dual space with respect to $L^2(R^n)$, so we have the following chain

$$W^\infty\{a_\alpha,2\} \subseteq L^2(R^n) \subseteq W^{-\infty}\{a_\alpha,2\}$$

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Analogous to the above chain we have

$$W_0^\infty\{a_\alpha,2\} \subseteq L^2(R^n) \subseteq W_0^{-\infty}\{a_\alpha,2\}$$

To set our problem, we introduce a continuous bilinear form on $W_0^\infty\{a_\alpha,2\}$

$$a(u,v) = (Au,v)_{L^2(R^n)}$$

There are several papers dealing with control problems with state constraints [3]. In these problems, there exists a fundamental difference between integral and pointwise state constraints. Lagrange mulipliers in the optimality conditions are integrable functions in the first case and measure in the second case. The choice of the functional spaces is very important for proving the existence of Lagrange multiplier associated with the constraint in the state, Bonnans and Casas [4]. One of the first paper in distributed optimal control problems with state constraints was written by Mossino [5] . In this paper, the first aim is to prove the existance of a Lagrange multiplier which allows one to derive the optimality conditions. For the elliptic case with quadratic objective and linear equation of infinite order, this obtained by El-Zahaby et. al [6] . The papers which a close conection to our work, we refer to [3,5,7,8,9].

The paper is organized as follows: In section one, we formulate linear elliptic control problem of infinite order with constraints on the state. The existence of optimal control is given in section two. In section three, we derive optimality conditions for control problem of infinite order with pointwise state constraints. In section four, we give the existence of the Lagrange multiplier μ.

In this paper, we use the following assumption

Assumption 1:

Let $\Omega \subset R^N$ be a bounded domain in R^N with Lipschtiz continuous boundary and suppose that $\lambda \geq 0$, $y_d \in L^2(\Omega)$.

2 Existance of Optimal Control

The following theorem follows from the Lax-Milgram Lemma and for a proof we refers to [10].

Theorem 1. *With Assumption 1 holding there exist a unique weak solution $y \in W^\infty\{a_\alpha,2\}$ to (2), for every $u in L^2(\Omega)$ i.e.*

$$\sum_{|\alpha|=1}^{\infty} \int_{R^n} (D^\alpha y)(x)(D^\alpha \phi)(x)\,dx + \int_{R^n} q(x)y(x)\phi(x)\,dx$$
$$= \int_\Omega u\phi\,dx \text{ for all } \phi \in W^\infty\{a_\alpha,2\}$$

Furthermore

$$\|t\|_{W^\infty\{a_\alpha,2\}} \leq C\|u\|_{L^2(\Omega)} \tag{5}$$

Proof. We apply the Lax-Milgram Lemma in $V = W^\infty\{a_\alpha,2\}$. To this end we define a continuous bilinear form by

$$a[y,\phi] = \sum_{|\alpha|=1}^{\infty} \left((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), \phi\right)_{L^2(R^n)} + (q(x)y(x),\phi(x))_{L^2(R^n)}$$

where $q(x)$ is a real valued function from $L_2(R^n)$ such that

$$q(x) \geq \nu \qquad 1 \geq \nu > 0.$$

The ellipticity of A is sufficient from the coerciveness of $a(y,v)$ on $W^\infty\{a_\alpha,2\}$, see [8] In fact

$$\begin{aligned}
a(y,y) &= (Ay,y) \\
&= \sum_{|\alpha|=1}^{\infty} (a_\alpha D^\alpha y(x), D^\alpha y(x))_{L^2(R^n)} + q(x)(y(x),y(x))_{L^2(R^n)} \\
&\geq \sum_{|\alpha|=1}^{\infty} (a_\alpha D^\alpha y(x), D^\alpha y(x))_{L^2(R^n)} + \nu(y(x),y(x))_{L^2(R^n)} \\
&= \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha y\|^2_{L^2(R^n)}\right) + \nu\|y(x)\|^2_{L^2(R^n)} \\
&= \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha y\|^2_{L^2(R^n)}\right) + \nu \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha y\|^2_{L^2(R^n)}\right) \\
&\quad - \nu \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha y\|^2_{L^2(R^n)}\right) + \nu\|y(x)\|^2_{L^2(R^n)} \\
&= \nu\|y\|^2_{W^\infty\{a_\alpha,2\}} + (1-\nu)\sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha y\|^2_{L^2(R^n)}\right)
\end{aligned}$$

Then

$$a(y,y) \geq \nu\|y\|^2_{W^\infty\{a_\alpha,2\}}. \tag{6}$$

Hence, the assumptions of Lax-Milgram Lemma are satisfied. The boundedness of the functional F is again a consequence of Caushy Shwartz inequality. Indeed, we have

$$\begin{aligned}
|F(\phi)| = |(u,\phi)_{L^2(\Omega)}| &\leq \|u\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \\
&\leq \|u\|_{L^2(\Omega)}\|\phi\|_{W_0^\infty\{a_\alpha,2\}}
\end{aligned}$$

So that

$$\|F\|_{V^*} \leq \|u\|_{L^2(\Omega)}$$

Lax-Milgram Lemma yields the existence of a unique solution y to (2). Hence we find that

$$\|y\|_{W^\infty\{a_\alpha,2\}} \leq C_a\|F\|_{V^*} \leq C_a\|u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}$$

which proves (5).

Let us denote the linear map $v \to y(v)$ by T, $T : L^2(\Omega) \to C(\bar{\Omega})$. An obvious consequence of (1) is that T is continuous. Let B be the following subset:

$$B = \{z \in C_0(\Omega) : \|z\|_\infty \leq 1\}$$

and let I_B be the indicator of B

$$I_B(z) = \begin{cases} +\infty, \text{ if } & z \in B; \\ [0], \text{ if } & z \in B; \end{cases} \quad (7)$$

Then the problem (P) can be stated in the following way:

$$Inf\{J(v) + (I_B oT)(v)\} \quad for \quad all v \in K \quad (8)$$

. **Assumption 2:** Let $v_0 \in K$ such that $Tv_0 \in B$.

Theorem 2. *Let K be nonempty, closed , convex and bounded set in $L^2(\Omega)$, $\lambda > 0$. Suppose that Assumption 2 holds. Then the problem (P) has a unique optimal solution.*

The proof is standard in [11,12], it is enough to note that $J(v) + (I_B oT)(v)$ is a lower semicontinuous convex functional which prove the existence of optimal control. Note also that, even if $\lambda = 0$, the functional J is strictly covex and so we deduce the uniqueness of the solution.

3 Optimality Conditions

To derive optimality conditions, we will use Assumption 2:

We will denote by $M(\Omega)$ the space of all real and regular Borel measures on Ω endowed with the norm

$$\|\mu\|_M(\Omega) = |\mu|(\Omega) \quad (9)$$

According to the Riesz representation theorem, $M(\Omega)$ is the dual space of $C_0(\Omega)$ and the following equality holds:

$$|\mu|(\Omega) = \sup_{z\in B} \int_\Omega z d\mu \quad (10)$$

and $\mu \in M(\Omega)$

$$\int_\Omega y d\mu = \sup_{z\in B} \int_\Omega z d\mu, \qquad y \in B, \quad (11)$$

Theorem 3. *Under Assumption 2 . Let $\lambda > 0$ be given, $u \in K$ is a solution of the problem (P) if the variational inequality*

$$\int_\Omega (y(u) - y_0)(y(v) - y(u)dx + \lambda \int_\Omega u(v-u)dx + \int_\Omega T^*\mu(v-u)dx \geq 0 \qquad for \quad all \quad v \in K \quad (12)$$

holds.

$$\mu \in \partial I_B(Tu). \quad (13)$$

Proof. u is a solution of the problem (P) if and only if u minimizers in $L^2(\Omega)$ the functional

$$\phi(u) = J(v) + (I_B oT)(v) + I_K(v) \quad (14)$$

where I_K is the indicator of K.

But $\phi(u) = Inf_{v\in L^2(\Omega)}\phi(v)$ if and only if $0 \in \partial\phi(u)$.

Under Assumption 2 and the standard formulas for subdifferentials of convex functions(Rockafellar [8]) we obtain

$$0 \in J'(u) + T^* o\partial I_B(Tu) + \partial I_K(u) \quad (15)$$

which is equivalent to $u \in K$ and the existence of $\mu \in \partial I_B(Tu)$ such that

$$\int_\Omega (y(u) - y_0)(y(v) - y(u)dx + \lambda \int_\Omega u(v-u)dx + \int_\Omega T^*\mu(v-u)dx \geq 0 \qquad for \quad all \quad v \in K \quad (16)$$

$$\mu \in \partial I_B(Tu). \quad (17)$$

Definition 1. *The weak solution $p \in W^\infty\{a_\alpha, 2\}$ of the adjoint or dual equation*

$$\begin{aligned} Ap &= y - y_d & in \quad \Omega \\ p^{|w|}|_\Gamma &= 0 & on \quad \Gamma \end{aligned} \quad (18)$$

with $y_d \in L^2(\Omega)$.

Theorem 4. *Under Assumption 2, $\lambda > 0$ be given. Suppose $u \in K$ is a solution of the problem (P) and $p \in L^2(\Omega)$ satisfies the variational inequality*

$$\int_\Omega (p + \lambda u)(v-u)dx \geq 0 \qquad for \quad all \quad v \in K. \quad (19)$$

Proof. Take $y = y(u)$, $p_1 = T^*\mu$ and let $p_2 \in W_0^\infty\{a_\alpha, 2\}$ be the solution of the Dirichlet problem

$$\begin{aligned} Ap_2 &= y - y_d \quad \text{in } \Omega \\ p_2^{|w|} &= 0 \quad \text{on } \Gamma, \quad |w| = 0,1,2,\cdots \text{on } \Gamma \end{aligned} \quad (20)$$

Finally, taking $p = p_1 + p_2$, we have

$$\begin{aligned} &\int_\Omega (y - y_d)(y(v) - y(u)dx + \int_\Omega T^*\mu(v-u)dx \\ &= \int_\Omega A^* p_2(y(v) - y(u)dx + \int_\Omega p_1(v-u)dx \\ &= \int_\Omega p_2(v-u)dx + \int_\Omega p_1(v-u)dx = \int_\Omega p(v-u)dx. \end{aligned} \quad (21)$$

substitute in (16) we obtain (19).

Theorem 5. *Under Assumption 2 . Let $z \in W^\infty\{a_\alpha, 2\}$, then the adjoint state p satifies*

$$\int_\Omega pAzdx = \int_\Omega z(y - y_d)dx + \int_\Omega zd\mu \, for all \, z \in W^\infty\{a_\alpha, 2\}, \quad (22)$$

*Proof.*Now we prove (22). Let $z \in W^{\infty}\{a_\alpha, 2\}$, then

$$\begin{aligned}\int_\Omega pAz \quad dx &= \int_\Omega p_2Az \quad dx + \int_\Omega p_1Az \quad dx \\ &= \int_\Omega A^* p_2 z \quad dx + \int_\Omega T^* \mu Az \quad dx \\ &= \int_\Omega (y-y_d)z \quad dx + \int_\Omega (ToA)zd\mu \\ &= \int_\Omega (y-y_d)z \quad dx + \int_\Omega z \quad d\mu.\end{aligned} \tag{23}$$

4 Study of the Multiplier

Next we prove some properties of the Lagrange multiplier μ. First recall that if the solution u of (P) satisfies $\|y(u)\|_\infty < 1$, then we deduce from (11) that $\mu = 0$. Now we examine the case $\|y(u)\|_\infty = 1$.

Theorem 6.*Let u be the solution of (P) and let $\mu \in M(\Omega)$ satisfying (11). Take $y = Tu$ and consider the sets*

$$\Omega_+ = \{x \in \Omega : y(x) = +1\}, \ \Omega_- = \{x \in \Omega : y(x) = -1\}$$

and

$$\Omega_0 = \{x \in \Omega : |y(x)| < 1\}$$

Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ and $|\mu| = \mu^+ + \mu^-$ the total variation measure of μ. Then μ^+ and μ^- are concentrated in Ω_+ and Ω_-, respectively, that is to say

$$\mu^+(\Omega_- \cup \Omega_0) = \mu^-(\Omega_+ \cup \Omega_0) = |\mu|(\Omega_0) = 0 \tag{24}$$

Moreover

$$\int_\Omega (|y(x)| - 1)d|\mu|(x) = 0. \tag{25}$$

*Proof.*From (10) , $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$ we see that (11) is equivalent to

$$\begin{aligned}\int_\Omega y \quad d\mu^+ - \int_\Omega y \quad d\mu^- &= |\mu|(\Omega) \\ &= \int_\Omega 1 \quad d\mu^+ + \int_\Omega 1 \quad d\mu^-.\end{aligned} \tag{26}$$

This can be written as

$$\int_\Omega (y-1)d\mu^+ - \int_\Omega (y+1)d\mu^- = 0. \tag{27}$$

From $-1 \le y \le +1$ we see that each of the integrals must vanish. The assertion (24) follows now from the definition of Ω_+ , Ω_- and Ω_0. Equation (25) is an obvios consequence of (24)

Summarizing, a control u is optimal for (2) if and only if u satisfies together with optimal state y and adjoint state p the following first order necessary optimal system

$$\begin{aligned} &Ay = u && Ap = y - y_d + \mu \\ &y^{|w|}|_\Gamma = 0, \quad |w| = 0,1,2 && p^{|w|}|_\Gamma = 0 \\ &(p + \lambda \bar{u}, v - \bar{u})_{L^2(\Omega)} \ge 0, && \text{for all} \quad v \in K \end{aligned} \tag{28}$$

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