Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.18576/jsapl/100106

Asymptotic Inference for Periodic Time-Varying Bivariate Poisson $INGARCH\,(1,1)$ Processes

Ahmed Ghezal

Department of Mathematics and Computer Sciences, University Center of Mila, Algeria

Received: 2 Oct. 2022, Revised: 22 Nov. 2022, Accepted: 2 Dec. 2022
Published online: 1 Jan. 2023

Abstract: In this paper, we would like to propose an extension of bivariate Poisson integer valued GARCH (shortly, BINGARCH) processes to periodically time-varying coefficients one. In these models, the parameters are allowed to switch periodically between different seasons. The main motivation of this new model is capable of modeling bivariate time series of counts. So, a necessary and sufficient condition for the periodically stationary in the mean, is established, while providing the closed-form expression for the mean. Furthermore, we show that the conditional maximum likelihood estimator (CMLE) of the parameter of the model is strongly consistent and asymptotically normal.

Keywords: Time series of counts, Periodic $INGARCH$, Bivariate Poisson distribution, Stationarity in the mean, Asymptotic properties.

1 Introduction

Bivariate (multivariate) integer-valued $GARCH$ time series models, proposed by Liu [6], are capable of capturing the serial dependence between two time series of counts, which has many applications, such as, in epidemiology, environmental, biology, accidents analysis and many others. These models are based on time-invariant parameter assumption, in addition a few attempts to model $BINGARCH$ time series of counts, for example, Liu [6] considers $BINGARCH$ models constructed via the trivariate reduction and proves the stationarity and ergodicity under certain conditions. Andreassen [1] verifies the strong consistency of the $CMLE$ of $BINGARCH$ models. Lee et al. [5] considers the problem of testing for a parameter change in $BINGARCH$ models and shows the asymptotic normality of $CMLE$. However, it was widely recognized that many economic, financial and environmental integer-valued time series, exhibit a periodicity feature in their some specific structures which cannot be taken into account and described by time-invariant parameter integer-valued time series models. So, it is possible to consider a $BINGARCH$ model whose coefficients are periodic in time series exhibiting structural changes in season (see Bentarzi and Bentarzi [2], for more qualitative discussion and references therein). Now firstly, we give a necessary and sufficient condition for the periodically stationary in the mean. Secondly, the aim of this paper is to analysis the asymptotic properties of the $CMLE$ of periodic $BINGARCH$ models.
The main contributions of this paper can be summarized as follows. In section 2, we set out the main assumptions underlying introduce the periodic $BINGARCH$ model. In the next section, we give a necessary and sufficient condition for the periodically stationary in the mean, thus, the closed-form expressions for the mean is obtained. The consistency and asymptotic normality of the $CMLE$ is proved, in section 4. Section 5 concludes the paper.

2 The model and main assumptions

Let $\underline{X}_t' = \left(X_t^{(1)}, X_t^{(2)}\right)$ be the bivariate random vector of counts at time t, where $\left(X_t^{(1)}\right)_{t\geq 1}$ and $\left(X_t^{(2)}\right)_{t\geq 1}$ are the two time series of counts with the conditional distribution following a Poisson distribution with conditional mean $\lambda_t^{(1)}$ and

* Corresponding author e-mail: a.ghezal@centre-univ-mila.dz

$\lambda_t^{(2)}$, respectively. So, a periodically correlated integer-valued process $(\underline{X}_t)$ defined on some probability space $(\Omega, \Im, P)$ is called a periodic bivariate Poisson $INGARCH\,(p,q)$ process with period $s>0$ (shortly, $P\mathscr{B}INGARCH_s\,(p,q)$), if it is given by

$$\begin{cases} \underline{X}_t \,|\mathscr{F}_{t-1} \backsim \mathscr{B}\mathscr{P}\left(\lambda_t^{(1)}, \lambda_t^{(2)}, \phi_t\right) \\ \underline{\lambda}_t := \left(\lambda_t^{(1)}, \lambda_t^{(2)}\right)' = \underline{a}_0(t) + \sum_{i=1}^{p} A_i(t)\underline{X}_{t-i} + \sum_{j=1}^{q} B_j(t)\,\lambda_{t-j} \end{cases}. \tag{2.1}$$

In (2.1), $\mathscr{F}_t$ is the $\sigma-$algebra representing knowledge of the full past up to time t, the parameters $\underline{a}_0'(t) = \left(a_0^{(1)}(t), a_0^{(2)}(t)\right)$, $A_i(t)$, $B_j(t)$, $1 \leq i \leq p$, $1 \leq j \leq q$ are 2×2 matrices and $\phi_t = Cov\left(X_t^{(1)}, X_t^{(2)} \middle| \mathscr{F}_{t-1}\right)$ are subject to the following assumptions

Assumption 1*Are non-negative coefficients with* $\phi_{.} \geq 0$.

Assumption 2*Are periodic in t, with period s, i.e.,* $\underline{a}_0(t) = \underline{a}_0(t+ms)$, $A_i(t) = A_i(t+ms)$, $B_j(t) = B_j(t+ms)$ *and* $\phi_t = \phi_{t+ms}$ *for* $1 \leq i \leq p$, $1 \leq j \leq q$ *and* $m \in \mathbb{N}$.

Since $P\mathscr{B}INGARCH_s\,(1,1)$ models are typically used in applications, thus, we focus more on the course of the first-order $P\mathscr{B}INGARCH_s$. Now, setting $t = sn+v$, $v = 1,\ldots,s$ and $n \in \mathbb{N}$, for $P\mathscr{B}INGARCH_s\,(1,1)$ with periodic notations, can be replaced with the following equivalent form

$$\begin{cases} \underline{X}_{sn+v} \,|\mathscr{F}_{sn+v-1} \backsim \mathscr{B}\mathscr{P}\left(\lambda_{sn+v}^{(1)}, \lambda_{sn+v}^{(2)}, \phi_v\right) \\ \underline{\lambda}_{sn+v} = \underline{a}_0(v) + A_1(v)\underline{X}_{sn+v-1} + B_1(v)\underline{\lambda}_{sn+v-1} \end{cases}, \tag{2.2}$$

where $A_1(.) = (a_{kl}(.))_{1\leq k,l\leq 2}$ and $B_1(.) = (b_{kl}(.))_{1\leq k,l\leq 2}$. A lot of models can be defined from (2.2) includes, as special cases,

i.Standard $\mathscr{B}INGARCH_1\,(1,1)$: This model, obtained by assuming the functions $\underline{a}_0(.)$, $A_1(.)$, $B_1(.)$ and $\phi_{.}$ constant, or equivalently by assuming that the $s = 1$ (see, e.g., Liu [6]; Cui and Zhu [4]).
ii.Periodic integer-valued $GARCH$ models ($PINGARCH_s\,(1,1)$): This model, obtained in a univariate random vector of counts at time t (see, e.g., Bentarzi and Bentarzi [2]).

*Remark.*First of all, it is worth noting that the symbolic $\underline{X}_t\,|\mathscr{F}_{t-1} \backsim \mathscr{B}\mathscr{P}\left(\lambda_t^{(1)}, \lambda_t^{(2)}, \phi_t\right)$ represents the bivariate Poisson distribution whose probability mass function is given by

$$P\left(X_t^{(1)} = x_1, X_t^{(2)} = x_2 \middle| \mathscr{F}_{t-1}\right) = \exp\left\{-\left(\lambda_t^{(1)} + \lambda_t^{(2)} - \phi_t\right)\right\} \frac{\left(\lambda_t^{(1)} - \phi_t\right)^{x_1}}{x_1!} \frac{\left(\lambda_t^{(2)} - \phi_t\right)^{x_2}}{x_2!}$$
$$\times \sum_{i=0}^{x_1 \wedge x_2} C_{x_1}^i C_{x_2}^i i! \left(\frac{\phi_t}{\left(\lambda_t^{(1)} - \phi_t\right)\left(\lambda_t^{(2)} - \phi_t\right)}\right)^i,$$

where $\phi_t < \lambda_t^{(1)} \wedge \lambda_t^{(2)}$.

*Remark.*Recent explanations of bivariate Poisson distribution allows for modeling dependence between $X_t^{(1)}$, $X_t^{(2)}$ and ϕ_t, so there are really three random variables $Z_t^{(1)}$, $Z_t^{(2)}$ and $Z_t^{(3)}$, which follow independent Poisson distributions with parameters $\lambda_t^{(1)} - \phi_t$, $\lambda_t^{(2)} - \phi_t$, ϕ_t, respectively, such that $X_t^{(1)} = Z_t^{(1)} + Z_t^{(3)}$ and $X_t^{(2)} = Z_t^{(2)} + Z_t^{(3)}$ (see, Cui and Zhu [4] for more qualitative discussion).

*Remark.*The model proposed in this paper is capable of capturing dependence between the two time series $\left(X_t^{(1)}\right)$ and $\left(X_t^{(2)}\right)$, provided that one of the following requirements have to be met: $\phi_{.} > 0$, or the coefficient matrices $A_1(.)$ and $B_1(.)$ are not both diagonal.

3 The desired outcome of the periodically stationary in the mean

In this section, we shall focus our attention on giving a necessary and sufficient condition for the $\mathscr{PB}INGARCH_s(1,1)$ process $(\underline{X}_t)$ satisfying (2.2) to be periodical stationary in the $\mathbb{L}_1$ sense. This probabilistic property has also been studied in the symmetric periodic case $PINGARCH_s(1,1)$ (e.g., Bentarzi and Bentarzi [2]) and periodic bilinear case $PINBL_s(1,0,1,1)$ (e.g., Bentarzi and Bentarzi [3]). Therefore, we can obtain the closed-form expression for the mean of $\mathscr{PB}INGARCH_s(1,1)$ process. Next, the periodical stationary in the $\mathbb{L}_1$ of the model in this paper is given in the following theorem

Theorem 1. *The $\mathscr{PB}INGARCH_s(1,1)$ process defined by (2.2) has a periodical stationary in the mean, if and only if,*

$$\rho_1 := \rho\left(\prod_{v=0}^{s-1}(A_1(v)+B_1(v))\right) < 1. \tag{3.1}$$

Moreover, under this condition the mean is given by, for all $v \in \{1,\ldots,s\}$,

$$\begin{aligned} E\{\underline{X}_{sn+v}\} &= E\{E\{\underline{X}_{sn+v}|\mathscr{F}_{sn+v-1}\}\} \\ &= \left(I_{(2)} - \prod_{l=0}^{s-1}(A_1(v-l)+B_1(v-l))\right)^{-1}\sum_{u=0}^{s-1}\left\{\prod_{l=0}^{u-1}(A_1(v-l)+B_1(v-l))\right\}\underline{a}_0(v-u). \end{aligned}$$

Proof. The idea of proof is to use the conditional mean of the process $(\underline{X}_t)$, we find $E\{\underline{X}_t\} = E\{\underline{\lambda}_t\}$ and

$$\begin{aligned} E\{\underline{\lambda}_t\} &= \underline{a}_0(t) + A_1(t)E\{\underline{X}_{t-1}\} + B_1(t)E\{\underline{\lambda}_{t-1}\} \\ &= \underline{a}_0(t) + (A_1(t)+B_1(t))E\{\underline{\lambda}_{t-1}\}, \end{aligned}$$

by iteration, we obtain $E\{\underline{X}_t\} = \left\{\prod_{u=0}^{t-1}(A_1(t-u)+B_1(t-u))\right\}E\{\underline{X}_0\} + \sum_{u=0}^{t-1}\left\{\prod_{l=0}^{u-1}(A_1(t-l)+B_1(t-l))\right\}\underline{a}_0(t-u)$, with the convention $\prod_{v=0}^{-1}A_1(v) = I_{(2)}$, from which, using periodic notation, we have, for all $v \in \{1,\ldots,s\}$,

$$\begin{aligned} E\{\underline{X}_{sn+v}\} &= \left\{\prod_{u=0}^{sn+v-1}(A_1(v-u)+B_1(v-u))\right\}E\{\underline{X}_0\} + \sum_{u=0}^{sn+v-1}\left\{\prod_{l=0}^{u-1}(A_1(v-l)+B_1(v-l))\right\}\underline{a}_0(v-u) \\ &= \left\{\prod_{u=0}^{s-1}(A_1(v-u)+B_1(v-u))\right\}^n\left\{\prod_{u=0}^{v-1}(A_1(v-u)+B_1(v-u))\right\}E\{\underline{X}_0\} \\ &+ \left\{\prod_{u=0}^{s-1}(A_1(v-u)+B_1(v-u))\right\}^n\sum_{u=0}^{v-1}\left\{\prod_{l=0}^{u-1}(A_1(v-l)+B_1(v-l))\right\}\underline{a}_0(v-u) \\ &+ \sum_{k=0}^{n-1}\left\{\prod_{l=0}^{s-1}(A_1(v-l)+B_1(v-l))\right\}^k\sum_{u=0}^{s-1}\left\{\prod_{l=0}^{u-1}(A_1(v-l)+B_1(v-l))\right\}\underline{a}_0(v-u), \end{aligned}$$

thus $E\{\underline{X}_{sn+v}\}$, $v \in \{1,\ldots,s\}$, converges, as $n \longrightarrow \infty$, if and only if the Condition (3.1) holds.■

Example 1. In this example, the Condition (3.1) for some subclass with particular case are simplified, where we find

Specification	ρ_1	$E\{\underline{X}_t\}$
$\text{Standard}_1(1,1)$	$\rho(A_1(1)+B_1(1)) < 1$	$(I_{(2)} - A_1(1) - B_1(1))^{-1}\underline{a}_0(1)$
$PINGARCH_s(1,1)$	$\prod_{v=0}^{s-1}(a_{11}(v)+b_{11}(v)) < 1$	$(1-\rho_1)^{-1}\sum_{u=0}^{s-1}\{\prod_{l=1}^{u}(a_{11}(l)+b_{11}(l))\}a_0^{(1)}(v-u)$

Table 1: Conditions (3.1) for the existence of $E\{\underline{X}_t\}$ for certain models.

4 Estimation

In the present section, we consider the conditional maximum likelihood estimator for estimating the parameters of $\mathscr{PB}INGARCH_s$ model gathered in vector $\underline{\theta}' := \left(\underline{\theta}_1', \underline{\theta}_2', \underline{\phi}'\right) \in \Theta := \Theta_1 \times \Theta_2 \times \Theta_3 \subset \mathbb{R}^{11s}$ where $\underline{\theta}_1' := (\underline{a}_0^{(1)\prime}, \underline{a}_1^{(1)\prime}, \underline{a}_1^{(2)\prime}, \underline{b}_1^{(1)\prime}, \underline{b}_1^{(2)\prime})$, $\underline{\theta}_2' := (\underline{a}_0^{(2)\prime}, \underline{a}_2^{(1)\prime}, \underline{a}_2^{(2)\prime}, \underline{b}_2^{(1)\prime}, \underline{b}_2^{(2)\prime})$ and $\underline{\phi}' := (\phi_1,\ldots,\phi_s)$ with $\underline{a}_0^{(j)} := \left(a_0^{(j)}(1),\ldots,a_0^{(j)}(s)\right)'$, $\underline{a}_i^{(j)} := (a_{ij}(1),\ldots,a_{ij}(s))'$ and $\underline{b}_i^{(j)} := (b_{ij}(1),\ldots,b_{ij}(s))'$ for all $i,j = 1,2$. The true parameter value denoted by $\underline{\theta}_0 \in \Theta \subset \mathbb{R}^{11s}$ is unknown and should be estimated. The period s are assumed to be known and fixed. For this aim, let $\{\underline{X}_1,\ldots,\underline{X}_n; n = sN\}$ be an observations from model (2.2). Then, constructing the logarithm of the conditional likelihood function based on the observation $\{\underline{X}_1,...,\underline{X}_n\}$, up to a constant free of $\underline{\theta}$, by $L_{sN}(\underline{\theta}) = \sum_{t=1}^{N}\sum_{v=0}^{s-1} l_{st+v}(\underline{\theta})$ with

$$l_t(\underline{\theta}) = X_t^{(1)}\log\left(\lambda_t^{(1)}(\underline{\theta}) - \phi_t\right) + X_t^{(2)}\log\left(\lambda_t^{(2)}(\underline{\theta}) - \phi_t\right) - \left(\lambda_t^{(1)}(\underline{\theta}) + \lambda_t^{(2)}(\underline{\theta}) - \phi_t\right)$$
$$+\log\left(\sum_{i=0}^{X_t^{(1)}\wedge X_t^{(2)}} C^i_{X_t^{(1)}} C^i_{X_t^{(2)}} i!\left(\frac{\phi_t}{\left(\lambda_t^{(1)}(\underline{\theta}) - \phi_t\right)\left(\lambda_t^{(2)}(\underline{\theta}) - \phi_t\right)}\right)^i\right).$$

Hence, we obtain the *CMLE* of $\underline{\theta}_0$ by $\widehat{\underline{\theta}}_n = Arg\max_{\underline{\theta}\in\Theta} L_n(\underline{\theta}) = Arg\min_{\underline{\theta}\in\Theta}(-L_n(\underline{\theta}))$. So, we consider an approximate version $\widetilde{L}_{sN}(\underline{\theta})$, using an arbitrarily chosen initial value $\widetilde{\underline{\lambda}}_1$, $\widetilde{L}_{sN}(\underline{\theta}) = \sum_{t=1}^{N}\sum_{v=0}^{s-1}\widetilde{l}_{st+v}(\underline{\theta})$. Next, the partial derivatives of $l_t(\underline{\theta})$ are expressed as: $\nabla_{\underline{\theta}} l_t(\underline{\theta}) = \Delta_t(\underline{\theta})\underline{E}_t(\underline{\theta})$ and $\nabla^2_{\underline{\theta}} l_t(\underline{\theta}) = \Delta_t(\underline{\theta})\nabla_{\underline{\theta}}\underline{E}_t(\underline{\theta}) + \underline{U}_t(\underline{\theta})$, where

$$\Delta_t(\underline{\theta}) := \begin{pmatrix} \Delta_t^{(1)}(\underline{\theta})I_{(4)} & O_{(4)} & \underline{O}_{(4)} \\ O_{(4)} & \Delta_t^{(2)}(\underline{\theta})I_{(4)} & \underline{O}_{(4)} \\ \underline{O}'_{(4)} & \underline{O}'_{(4)} & \Delta_t^{(3)}(\underline{\theta}) \end{pmatrix}, \underline{E}_t(\underline{\theta}) := \begin{pmatrix} \nabla_{\underline{\theta}_1}\lambda_t^{(1)}(\underline{\theta}_1) \\ \nabla_{\underline{\theta}_2}\lambda_t^{(2)}(\underline{\theta}_2) \\ 1 \end{pmatrix},$$
$$U_t(\underline{\theta}) := \left(U_{t,i}^{(j)}(\underline{\theta})\right)_{1\le i,j\le 3} \odot \left(\underline{E}_t(\underline{\theta})\otimes\underline{E}'_t(\underline{\theta})\right),$$

wherein

$$1+\Delta_t^{(k)}(\underline{\theta}) = \frac{X_t^{(k)}}{\lambda_t^{(k)}(\underline{\theta}_k)-\phi_t} - \frac{\xi_1(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)}{\xi_0(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)\left(\lambda_t^{(k)}(\underline{\theta}_k)-\phi_t\right)}, k=1,2,$$
$$1+\sum_{k=1}^{3}\Delta_t^{(k)}(\underline{\theta}) = \frac{\xi_1(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)}{\xi_0(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)\phi_t}, \ \xi_m(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t) = \sum_{i=0}^{X_t^{(1)}\wedge X_t^{(2)}} C^i_{X_t^{(1)}}C^i_{X_t^{(2)}} i! i^m \psi^i(\underline{\lambda}_t(\underline{\theta}),\phi_t),$$
$$\psi(\underline{\lambda}_t(\underline{\theta}),\phi_t) = \phi_t\left(\lambda_t^{(1)}(\underline{\theta}_1)-\phi_t\right)^{-1}\left(\lambda_t^{(2)}(\underline{\theta}_2)-\phi_t\right)^{-1},$$

and

$$U_{t,k}^{(k)}(\underline{\theta}) = \frac{\Delta_t^{(k)}(\underline{\theta})+1}{\lambda_t^{(k)}(\underline{\theta}_k)-\phi_t} - \frac{\varphi(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)}{\left(\lambda_t^{(k)}(\underline{\theta}_k)-\phi_t\right)^2}, k=1,2, \ \ \phi_t U_{t,i}^{(j)}(\underline{\theta}) = \varphi(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)\psi(\underline{\lambda}_t(\underline{\theta}),\phi_t), 1\le i\ne j\le 2,$$
$$\varphi(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t) = \frac{\xi_2(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)}{\xi_0(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)} - \left(1+\sum_{k=1}^{3}\Delta_t^{(k)}(\underline{\theta})\right)^2,$$
$$U_{t,i}^{(j)}(\underline{\theta}) + U_{t,i}^{(i)}(\underline{\theta}) + U_{t,j-2}^{(j-1)}(\underline{\theta}) = U_{t,j}^{(i)}(\underline{\theta}) + U_{t,i}^{(i)}(\underline{\theta}) + U_{t,j-2}^{(j-1)}(\underline{\theta}) = -\frac{\varphi(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t)}{\phi_t\left(\lambda_t^{(i)}(\underline{\theta}_i)-\phi_t\right)}, i=1,2, j=3,$$
$$\phi_t\sum_{j=1}^{3}U_{t,i}^{(j)}(\underline{\theta}) = U_{t,1}^{(2)}(\underline{\theta})\left(\lambda_t^{(1)}(\underline{\theta}_1)+\lambda_t^{(2)}(\underline{\theta}_2)-2\phi_t\right) + \phi_t^{-1}\varphi(\underline{X}_t,\underline{\lambda}_t(\underline{\theta}),\phi_t) - \left(1+\sum_{k=1}^{3}\Delta_t^{(k)}(\underline{\theta})\right),$$

with the standard symbolics, $I_{(n)}$ denotes the identity matrix and $O_{(k,l)}$ denotes the matrix of order $k\times l$ whose entries are zeros, for simplicity we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$. $\otimes$ (resp. $\odot$) is the usual Kronecker (resp. Hadamard) product of matrices. $\nabla_{\underline{\theta}}$ (resp. $\nabla^2_{\underline{\theta}}$) be the vector (resp. matrix) of the first (resp. second)-order partial derivatives. A similar assumptions in Andreassen [1] and may easily be adapted to use with the periodic case, so the *CMLE* is strongly consistent and asymptotically normal under the following regularity in adapted or new assumptions

Assumption 3 *$\underline{\theta}_0\in\Theta$ and Θ is a compact subset of $\mathbb{R}^{11s}$.*

Assumption 4 *$\underline{a}_{0,\underline{\theta}}(.)$, $A_{1,\underline{\theta}}(.)$ and $B_{1,\underline{\theta}}(.)$ have non-negative entries and $B_{1,\underline{\theta}}(.)$ is full rank for all $\underline{\theta}$.*

Assumption 5 *$\phi_v(\underline{\theta}) < u_1(v)\wedge u_2(v)$ where $(u_1(v),u_2(v))' = \left(I_{(2)} - A_{1,\underline{\theta}}(v)\right)^{-1}\underline{a}_{0,\underline{\theta}}(v)$ for all $v\in\{1,...,s\}$, $\underline{\theta}\in\Theta$.*

Assumption 6 *There exists a $m\in[1;+\infty]$ such that $\left\|A_{1,\underline{\theta}}(v)\right\|_m + 2^{1-m^{-1}}\left\|B_{1,\underline{\theta}}(v)\right\|_m < 1$ for all $v\in\{1,...,s\}$, $\underline{\theta}\in\Theta$.*

where, for a matrix $M \in \mathbb{R}^{2\times 2}$, $\|M\|_m$ denotes the $m-$induced norm of matrix M for $m \in [1;+\infty]$, i.e., $\|M\|_m = \max_{\underline{u}\neq \underline{O}} \left\{ \|M\underline{u}\|_m / \|\underline{u}\|_m, \underline{u} \in \mathbb{R}^2 \right\}$ and $\|\underline{u}\|_m$ is the $m-$norm of the vector $\underline{u}$. When $\|M\|_1$ is the maximum absolute column sum of M, $\|M\|_\infty$ is the maximum absolute row sum. To achieve desired goals in this section, we will need the following intermediate results gathered in the next three lemmas

Lemma 1. *Under Assumptions 3–6, we have*

1. $\sum_{v=1}^{s} E\left\{ \sup_{\underline{\theta}\in\Theta} \left\|\underline{\lambda}_{st+v}\right\| \right\} < \infty$ *and* $\sum_{v=1}^{s} E\left\{ \sup_{\underline{\theta}\in\Theta} \left\|\underline{\widetilde{\lambda}}_{st+v}\right\| \right\} < \infty$.

2. $\lambda_t^{(i)}(\underline{\theta}_i)$ *is twice continuously differentiable with respect to* $\underline{\theta}_i$ $(i = 1,2)$ *and satisfies*

$$\sum_{v=1}^{s} E\left\{ \left(\sup_{\underline{\theta}_i\in\Theta_i} \left\| \nabla_{\underline{\theta}_i} \lambda_{st+v}^{(i)}(\underline{\theta}_i) \right\|_m \right)^4 \right\} < \infty, \ \sum_{v=1}^{s} E\left\{ \left(\sup_{\underline{\theta}_i\in\Theta_i} \left\| \nabla^2_{\underline{\theta}_i} \lambda_{st+v}^{(i)}(\underline{\theta}_i) \right\|_m \right)^2 \right\} < \infty,$$

$$\sup_{\underline{\theta}_i\in\Theta_i} \left\| \sum_{t=1}^{N}\sum_{v=1}^{s} \nabla_{\underline{\theta}_i} \left(\widetilde{\lambda}_{st+v}^{(i)}(\underline{\theta}_i) - \lambda_{st+v}^{(i)}(\underline{\theta}_i) \right) \right\|_m < L\kappa^t \text{ a.s.}, \ \sup_{\underline{\theta}_i\in\Theta_i} \left\| \sum_{t=1}^{N}\sum_{v=1}^{s} \nabla^2_{\underline{\theta}_i} \left(\widetilde{\lambda}_{st+v}^{(i)}(\underline{\theta}_i) - \lambda_{st+v}^{(i)}(\underline{\theta}_i) \right) \right\|_m < L\kappa^t \text{ a.s.},$$

where L *stand for a generic positive integrable random variable and* $\kappa \in (0;1)$ *be a generic constant.*

3. $\sup_{\underline{\theta}\in\Theta} \left\| \sum_{t=1}^{N}\sum_{v=1}^{s} \left(\underline{\widetilde{\lambda}}_{st+v}(\underline{\theta}) - \underline{\lambda}_{st+v}(\underline{\theta}) \right) \right\|_m < L\kappa^t$ *a.s.*

4. $\underline{\omega}' \nabla_{\underline{\theta}_i} \lambda_t^{(i)}(\underline{\theta}_{i,0}) = 0 \Rightarrow \underline{\omega} = \underline{O}$.

5. There is $t \in \mathbb{Z}$ *such that* $\underline{\lambda}_t(\underline{\theta}) = \underline{\lambda}_t(\underline{\theta}_0)$ *a.s.* $\Rightarrow \underline{\theta} = \underline{\theta}_0$.

Lemma 2. *Under Assumptions 3–6, we have*

1. $\max\left(\sup_{\underline{\theta}\in\Theta} \left|\Delta_t^{(i)}(\underline{\theta})\right|, \sup_{\underline{\theta}\in\Theta} \left|\widetilde{\Delta}_t^{(i)}(\underline{\theta})\right| \right) \leq K\|\underline{X}_t\| + 1$, $\max\left(\sup_{\underline{\theta}\in\Theta} \left|U_{t,i}^{(i)}(\underline{\theta})\right|, \sup_{\underline{\theta}\in\Theta} \left|\widetilde{U}_{t,i}^{(i)}(\underline{\theta})\right| \right) \leq K\|\underline{X}_t\|^2$ *for some positive constant* K.

2. $\sup_{\underline{\theta}\in\Theta} \left|\Delta_t^{(i)}(\underline{\theta}) - \widetilde{\Delta}_t^{(i)}(\underline{\theta})\right| \longrightarrow 0$ *a.s. and* $\sup_{\underline{\theta}\in\Theta} \left|U_{t,i}^{(i)}(\underline{\theta}) - \widetilde{U}_{t,i}^{(i)}(\underline{\theta})\right| \longrightarrow 0$ *a.s.*

Lemma 3. *Under Assumptions 3–6, we have*

1. $\left\{ \nabla_{\underline{\theta}} l_t(\underline{\theta}_0), \mathscr{F}_t \right\}$ *forms a periodically stationary martingale difference sequence.*

2. $\sum_{v=1}^{s} E\left\{ \sup_{\underline{\theta}\in\Theta} \left\| \nabla_{\underline{\theta}} l_{st+v}(\underline{\theta}_0) \left(\nabla_{\underline{\theta}} l_{st+v}(\underline{\theta}_0)\right)' \right\|_m \right\} < \infty$ *and* $\sum_{v=1}^{s} E\left\{ \sup_{\underline{\theta}\in\Theta} \left\| \nabla^2_{\underline{\theta}} l_{st+v}(\underline{\theta}_0) \right\|_m \right\} < \infty$.

3. $p\lim \left\| \frac{1}{\sqrt{sN}} \sum_{t=1}^{N}\sum_{v=1}^{s} \nabla_{\underline{\theta}} \left(\widetilde{l}_{st+v}(\underline{\theta}_0) - l_{st+v}(\underline{\theta}_0) \right) \right\|_m = 0$ *and* $p\lim \sup_{\underline{\theta}\in\Theta} \left\| \frac{1}{sN} \sum_{t=1}^{N}\sum_{v=1}^{s} \nabla^2_{\underline{\theta}} \left(\widetilde{l}_{st+v}(\underline{\theta}_0) - l_{st+v}(\underline{\theta}_0) \right) \right\|_m = 0$.

4. $\lim_{n\to\infty} \left(-\frac{1}{sN} \sum_{t=1}^{N}\sum_{v=1}^{s} \nabla^2_{\underline{\theta}} l_{st+v}\left(\underline{\widetilde{\theta}}\right) \right) \overset{a.s}{=} I(\underline{\theta}_0)$, *where* $\underline{\widetilde{\theta}}$ *is any intermediate point between* $\underline{\widehat{\theta}}_n$ *and* $\underline{\theta}_0$, *and the matrix* $I(\underline{\theta}_0)$ *given by* $I(\underline{\theta}_0) := \sum_{v=1}^{s} E_{\underline{\theta}_0} \left\{ \nabla_{\underline{\theta}} l_{st+v}(\underline{\theta}_0) \left(\nabla_{\underline{\theta}} l_{st+v}(\underline{\theta}_0)\right)' \right\} = -\sum_{v=1}^{s} E_{\underline{\theta}_0} \left\{ \nabla^2_{\underline{\theta}} l_{st+v}(\underline{\theta}_0) \right\}$.

The main results of this section is the following theorem

Theorem 2. *Suppose that* $(\underline{X}_t, t \in \mathbb{Z})$ *is generated by (2.2), then under Assumptions 3–6, we have* $\underline{\widehat{\theta}}_n$ *is strongly consistent and* $\sqrt{n}\left(\underline{\widehat{\theta}}_n - \underline{\theta}_0 \right) \rightsquigarrow \mathscr{N}\left(\underline{O}, I^{-1}(\underline{\theta}_0)\right)$ *as* $n \to \infty$.

Proof. The proof of Theorem 2 is based on the previous three lemmas.■

5 Conclusion

In this paper, we have introduced a new model for count data, called bivariate Poisson integer valued *GARCH* model with periodically time-varying coefficients, which is a natural extension of the standard model with time-invariant coefficients, constructed by using trivariate reduction method of independent Poisson variables. Thus, we investigated the features of this new model, we found the necessary and sufficient condition for the periodically stationary in the mean, and the *CMLE* for the parameters are considered and asymptotic properties of the estimators are established.

Acknowledgement

The author would like to thank the editor and anonymous referees for their helpful comments and suggestions.

References

[1] C. M. Andreassen, Models and inference for correlated count data. Ph.D. thesis, Aarhus University 2013.
[2] M. Bentarzi., W. Bentarzi, Periodic integer-valued $GARCH(1,1)$ model. Communications in Statistics - Simulation and Computation., **46** (2), $1167-1188$ (2017).
[3] M. Bentarzi., W. Bentarzi, Periodic integer-valued bilinear time series model. Communications in Statistics - Theory and Methods., **46** (3), $1184-1201$ (2017).
[4] Y. Cui., F. Zhu, A new bivariate integer-valued $GARCH$ model allowing for negative cross-correlation. Test., **27** (2), $428-452$ (2018).
[5] Y. Lee., S. Lee and D. Tjϕstheim, Asymptotic normality and parameter change test for bivariate Poisson $INGARCH$ models. Test., **27** (1), $52-69$ (2018).
[6] H. Liu, Some models for time series of counts. Ph.D. thesis, Columbia University 2012.