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On Study Nonlocal Integro Differential Equation Involving The The Caputo-Fabrizio Fractional Derivative And q-Integral Of The Riemann Liouville Type

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Abstract: In this work, the existence and uniqueness of a solution for the integro-differential equation that contains the Caputo-Fabrizio fractional derivative and the q-integral of the Riemann Liouville type will be investigated. The continuous dependence of the solution is studied. The Schauder fixed-point theorem is used to prove the existence of a solution to the addressed equation. In addition, we obtain a numerical solution for the proposed problem using a merge of finite difference with trapezoidal methods and a merge of cubic-b spline with trapezoidal methods. The definition of Caputo-Fabrizio fractional derivative and Riemann-Liouville q integral will be used. The finite difference and cubic-b spline methods will be applied to the derivative part, and the trapezoidal method will be applied to the integral part. Then, the problem will be converted into a system of algebraic equations that can be solved together to get the solution. Finally, some examples are provided for comparing the numerical solutions obtained by using the proposed methods with the exact solutions of those. It It has been shown that the method is effective and easy to implement..

Keywords: Fractional derivative, q-integro-differential equation, Existence and Uniqueness of solution, Numerical solutions

1 Introduction

In the last two decades, mathematicians and physicists have become much more interested in fractional calculus and quantum calculus (q-calculus), which offer an effective way to describe a variety of real-world dynamical phenomena that arise in engineering and scientific fields like physics, biology, electrochemistry, chemistry, economics, electromagnetic control theory, and viscoelasticity. In view of the wide range of applications of fractional calculus and q-calculus, it is difficult for researchers to obtain direct solutions to most fractional and q-fractional differential and integro-differential equations. As a result, it is necessary to discuss the existence and uniqueness of solutions to various fractional integro-differential equations. Many results have been obtained by researchers regarding the existence and uniqueness of solutions to various fractional integro-differential equations [1,2,3,4,5,6]. Also, many researchers interested in studying the existence of solutions to q-fractional integro-differential equations[7,8,9,10,11]. At the same time, a large number of numerical solutions of various types of integro-differential equations have been obtained [12,13,14,15,16,17,18]. In 2020, the authors investigated the analytical solution for a first-order nonlinear Fredholm integro-differential equation:

$$u'(x) = f(x) + \int_a^b g(x,t,u'(t))dt, \quad u(a) = \alpha,$$

where $u(x)$ is the unknown function and $f(x)$ is the known function. In addition, they study the numerical solution using finite difference and Simpson's methods [19]. Also, in 2022 they investigated the existence and uniqueness of the following Fredholm–Volterra integro-differential equation:

$$u''(x) = F\left(x, u(x), \int_a^b f(x,t,u'(t))dt,\right.$$

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$$\int_a^x g(x,t,u'(t))dt\Big)$$

with the nonlocal and boundary condition:

$$\sum_{k=0}^{m} a_k u(\tau_k) = \alpha_0, \quad u'(a) = \beta_0.$$

In addition, they used the merge of finite difference with trapezoidal methods to solve it numerically[20].

Now, we study the analytical and numerical solutions for the following nonlocal fractional q integro-differential equation:

$$u''(t) = f\Big(t, u(t), {}^{CF}D^{\alpha}u(t), I_q^{\beta} g(t,u'(t))\Big), \quad t \in (0,1], \tag{1}$$

with the q-nonlocal condition:

$$(1-q)\tau\sum_{i=0}^{n} q^i u(q^i\tau) = \alpha_0, \quad u'(0) = \beta_0, \quad \tau \in (0,1] \tag{2}$$

where ${}^{CF}D^{\alpha}u(t)$ is the Caputo-Fabrizio fractional derivative, I_q^{β} is the fractional q-integral of the Riemann Liouville type of order $\beta \geqslant 0$, α_0, β_0 are constants, and $q, \alpha \in (0,1)$. The definitions of the Caputo-Fabrizio fractional derivative and q-integral of the Riemann Liouville type will be applied to prove the existence and uniqueness. Then, the finite difference method or the cubic b-spline method will be applied to the derivative part and the trapeziodal method will be applied to the integral part to convert this equation into algebric equations that can be solved together to obtain the solution to the problem.

The rest of the paper is organised as follows: some basic concepts of fractional calculus and q-calculus, which will be needed in our paper, are introduced in Section 2. Section 3 gives the integral representation of the problem. In Section 4, we use the Schauder fixed point theorem to discuss the existence of the solution. Section 5 is devoted to discussing the solution's uniqueness, while the continuous dependence on the constant α_0 of the problem will be introduced in Section 6. Section 7 includes a summary of the numerical techniques that will be used in our paper. In Section 8, we apply the assumptions of the existence theorem to some examples and solve them numerically by using finite-trapezoidal and cubic-trapezoidal methods to demonstrate their efficiency of them. Finally, the conclusion section will be introduced.

2 Basic concepts

In this section, we introduce some important definitions related to q-calculus and fractional calculus.

Definition 1.[21] *For any number* κ

$$[\kappa]_q = \frac{1-q^{\kappa}}{1-q},$$

where $q \in (0,1)$.

Definition 2.[21] *The q-derivative of* $u(t)$ *can be defined as follows:*

$$(D_q u)(t) = \frac{u(t)-u(qt)}{t-qt},$$

$$\lim_{q\to 1} D_q u(t) = \frac{du(t)}{dt}.$$

Definition 3.[22] *A q-analogue of the common Pochhammer symbol which is called a q-shifted factorial is defined by*

$$(\kappa;q)_n = \begin{cases} 1, & n=0, \\ \prod_{i=0}^{n-1}(1-\kappa q^i), & n \in \mathbb{N}, \end{cases}$$

also,

$$(\kappa;q)_{\infty} = \prod_{i=0}^{\infty}(1-\kappa q^i), \quad n \in \mathbb{N}.$$

Definition 4.[22] *The q-gamma function is defined as*

$$\Gamma_q(\kappa) = \frac{(q;q)_{\infty}}{(q^{\kappa};q)_{\infty}}(1-q)^{1-\kappa},$$

and satisfies $\Gamma_q(\kappa+1) = [\kappa]_q\Gamma_q(\kappa), \qquad \Gamma_q(1) = 1.$

Definition 5.[23] *Let* $u(t)$ *be a function defined on* $[0,1]$. *The fractional q-integral of the Riemann-Liouville type of order* $\beta \geqslant 0$ *is given by*

$$(I_q^{\beta}u)(t) = \begin{cases} u(t), & \beta = 0, \\ \frac{1}{\Gamma_q(\beta)}\int_0^t (t-qs)^{\beta-1}u(s)d_qs. \end{cases} \tag{3}$$

Lemma 1.[23] *For* $\beta > 0$, *using q-integration by parts, we have*

$$(I_q^{\beta}1)(t) = \frac{t^{(\beta)}}{\Gamma_q(\beta+1)}. \tag{4}$$

Definition 6.[24] *(Caputo–Fabrizio fractional derivative). Let* $\alpha \in (0,1)$, *the Caputo–Fabrizio fractional derivative of order* α *of a function* $u(t)$ *is defined by*

$${}^{CF}D^{\alpha}u(t) = \frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}u'(s)ds. \tag{5}$$

We can see in the original definition [25] there is a normalization factor $M(\alpha)$ *in the Caputo–Fabrizio fractional derivative which satisfies* $M(0) = M(1) = 1$. *This factor* $M(\alpha)$ *is chosen to be the identity in a later paper [26].*

3 Integral representation

Consider the problem (1)-(2) with the following assumptions:

1. $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ satisfies Caratheodory condition i. e., f is measurable in t for any $u,\phi,\mu\in\mathbb{R}$ and continuous for almost all $t\in[0,1]$. There exist a function $v_1(t)\in L_1[0,1]$ and a positive constant $d_1>0$, such that
$$|f(t,u,\phi,\mu)|\le v_1(t)+d_1|u|+d_1|\phi|+d_1|\mu|.$$
2. $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies Caratheodory condition. There exist a function $v_2(t)\in L_1[0,1]$ and a positive constant $d_2>0$, such that
$$|g(t,v)|\le v_2(t)+d_2|v|.$$
3.
$$\sup_{t\in[0,1]}\int_0^t v_1(\theta)d\theta\le M_1,$$
$$\sup_{t\in[0,1]}\int_0^t I_q^\beta v_2(\theta)d\theta\le M_2.$$
4. $2d_1+d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2}+\frac{d_1d_2}{(\beta+1)\Gamma_q(\beta+1)}<1.$

Lemma 2. *The solution to problem (1)-(2), if it exists, can be represented by the following integral equation:*
$$u(t)=\frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Big[\alpha_0-(1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau}v(\theta)d\theta\Big]+\int_0^t v(\theta)d\theta, \tag{6}$$
where,
$$v(t)=\beta_0+\int_0^t f\Big(\theta,\frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Big[\alpha_0-(1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau}v(\theta)d\theta\Big]+\int_0^\theta v(s)ds,\ \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds,I_q^\beta g(\theta,v(\theta))\Big)d\theta. \tag{7}$$

Proof. Integrating both sides of (1), we get
$$u'(t)=u'(0)+\int_0^t f\Big(\theta,u(\theta),{}^{CF}D^\alpha u(\theta),I_q^\beta g(\theta,u'(\theta))\Big)d\theta,\quad t\in(0,1]. \tag{8}$$

Using (5), we obtain
$$u'(t)=u'(0)+\int_0^t f\Big(\theta,u(\theta),\frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}u'(s)ds,I_q^\beta g(\theta,u'(\theta))\Big)d\theta,\quad t\in(0,1]. \tag{9}$$

Let $u'(t)=v(t)$ in (9), we obtain
$$v(t)=\beta_0+\int_0^t f\Big(\theta,u(\theta),\frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds,I_q^\beta g(\theta,v(\theta))\Big)d\theta,\quad t\in(0,1], \tag{10}$$
where
$$u(t)=u(0)+\int_0^t v(s)ds,\quad t\in(0,1], \tag{11}$$
using the nonlocal condition (2), we get
$$(1-q)\tau\sum_{i=0}^n q^i u(q^i\tau)=u(0)(1-q)\tau\sum_{i=0}^n q^i+(1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau}v(\theta)d\theta, \tag{12}$$
then,
$$u(0)=\frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Big[\alpha_0-(1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau}v(\theta)d\theta\Big], \tag{13}$$
we obtain (6) and (7) from (10), (11) and (13). This complete the proof.

4 Existence of solution

Theorem 1. *Let the assumptions $1-4$ be satisfied. Then, (7) has at least one solution $v\in C[0,1]$.*

Proof. Define the operator H associated with the integral equation (7) by
$$Hv(t)=\beta_0+\int_0^t f\Big(\theta,\frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Big[\alpha_0-(1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau}v(\theta)d\theta\Big]+\int_0^\theta v(s)ds,\ \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds,I_q^\beta g(\theta,v(\theta))\Big)d\theta.$$

Let $Q_r = \{v \in \mathbb{R} : \|v\|_C \leq r\}$, where $r = \frac{|\beta_0|+M_1+\frac{d_1|\alpha_0|}{(1-q)\tau\sum_{i=0}^n q^i}+d_1M_2}{1-(2d_1+d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2}+\frac{d_1d_2}{(\beta+1)\Gamma_q(\beta+1)})}$.
Then, for $v \in Q_r$, we have

$$
\begin{aligned}
\|Hv(t)\|_C \leq & \Bigg|\beta_0 + \int_0^t f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 \\
& -(1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v(\theta)d\theta\Bigg] + \int_0^\theta v(s)ds, \\
& \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds, I_q^\beta g(\theta, v(\theta))\Bigg)d\theta\Bigg| \\
\leq & |\beta_0| + \int_0^t \Bigg|f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 - \\
& (1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v(\theta)d\theta\Bigg] + \int_0^\theta v(s)ds, \\
& \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds, I_q^\beta g(\theta, v(\theta))\Bigg)\Bigg|d\theta \\
\leq & |\beta_0| + \int_0^t \Bigg[v_1(\theta) + d_1\Bigg|\frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 - \\
& (1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v(\theta)d\theta\Bigg] + \int_0^\theta v(s)ds\Bigg| \\
& + \frac{d_1}{1-\alpha}\int_0^\theta |e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)|ds \\
& + d_1 I_q^\beta |g(\theta, v(\theta))|\Bigg]d\theta \\
\leq & |\beta_0| + M_1 + \int_0^t \Bigg[\frac{d_1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[|\alpha_0| \\
& +(1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} |v(\theta)|d\theta\Bigg] \\
& + d_1\int_0^\theta |v(s)|ds + \frac{d_1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}|v(s)|ds \\
& + d_1 I_q^\beta (v_2(\theta) + d_2|v(\theta)|)\Bigg]d\theta
\end{aligned}
$$

$$
\begin{aligned}
\leq & |\beta_0| + M_1 + \int_0^t \Bigg[\frac{d_1}{(1-q)\tau\sum_{i=0}^n q^i}|\alpha_0| \\
& + d_1\|v\| + d_1\|v\| + \frac{d_1\|v\|(1-e^{\frac{\alpha\theta}{\alpha-1}})}{\alpha} + d_1M_2 \\
& + d_1d_2\|v\|\frac{\theta^\beta}{\Gamma_q(\beta+1)}\Bigg]d\theta \\
\leq & |\beta_0| + M_1 + \frac{d_1|\alpha_0|}{(1-q)\tau\sum_{i=0}^n q^i} + 2d_1r \\
& + d_1r\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2} \\
& + d_1M_2 + \frac{d_1d_2r}{(\beta+1)\Gamma_q(\beta+1)} = r.
\end{aligned}
$$

This proves that $H: Q_r \to Q_r$ and the class of functions $\{Hv(t)\}$ is uniformly bounded in Q_r.
Now, let $t_1, t_2 \in [0,1]$ such that $|t_2 - t_1| < \delta$; then,

$$
\begin{aligned}
|Hv(t_2) - Hv(t_1)| = & \Bigg|\beta_0 \\
& + \int_0^{t_2} f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 - \\
& (1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v(\theta)d\theta\Bigg] + \int_0^\theta v(s)ds, \\
& \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds, I_q^\beta g(\theta, v(\theta))\Bigg)d\theta \\
& -\beta_0 - \int_0^{t_1} f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 - \\
& (1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v(\theta)d\theta\Bigg] + \int_0^\theta v(s)ds, \\
& \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds, I_q^\beta g(\theta, v(\theta))\Bigg)d\theta\Bigg| \\
\leq & \int_{t_1}^{t_2}\Bigg|f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 - \\
& (1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v(\theta)d\theta\Bigg] + \int_0^\theta v(s)ds, \\
& \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds, I_q^\beta g(\theta, v(\theta))\Bigg)\Bigg|d\theta \\
\leq & \int_{t_1}^{t_2} v_1(\theta)d\theta + \frac{d_1|\alpha_0|(t_2-t_1)}{(1-q)\tau\sum_{i=0}^n q^i} \\
& + 2d_1r(t_2-t_1) + \frac{d_1r(1-e^{\frac{\alpha\theta}{\alpha-1}})(t_2-t_1)}{\alpha} \\
& + d_1\int_{t_1}^{t_2} I_q^\beta v_2(\theta)d\theta + d_1d_2r\int_{t_1}^{t_2}\frac{\theta^\beta}{\Gamma_q(\beta+1)}d\theta.
\end{aligned}
$$

This means that the class of functions $\{Hv(t)\}$ is equi-continuous in Q_r.

Let $v_k(t) \in Q_r$, $v_k(t) \to v(t) (k \to \infty)$, then from the continuity of the two functions f and g, we obtain $f(t, u_k, \phi_k, \mu_k) \to f(t, u, \phi, \mu)$ and $g(t, v_k) \to g(t, v)$ as $k \to \infty$. Also,

$$
\begin{aligned}
\lim_{k\to\infty} Hv_k(t) = & \lim_{k\to\infty}\Bigg[\beta_0 + \int_0^t f\Bigg(\theta, \\
& \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 \\
& -(1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v_k(\theta)d\theta\Bigg] + \int_0^\theta v_k(s)ds, \\
& \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v_k(s)ds, I_q^\beta g(\theta, v_k(\theta))\Bigg)d\theta\Bigg].
\end{aligned}
$$

Using assumptions 1-2 and Lebesgue dominated convergence Theorem [27], we obtain

$$\lim_{k\to\infty} Hv_k(t) = \beta_0 + \int_0^t \lim_{k\to\infty} f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 - (1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v_k(\theta)d\theta\Bigg] + \int_0^\theta v_k(s)ds, \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)} v_k(s)ds, I_q^\beta g(\theta, v_k(\theta))\Bigg)d\theta = Hv(t).$$

Then $Hv_k(t) \to Hv(t)$ as $k\to\infty$. This means that the operator H is continuous in Q_r. Then by Schauder fixed point Theorem [28], there exist at least one solution $v\in C[0,1]$ of the integral equation(7). Thus, based on the Lemma 2, the problem (1)–(2) possess a solution $u\in C[0,1]$.

5 Uniqueness of the solution

Let f and g satisfy the following assumptions:

(i) $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ is measurable in t for any $u,\phi,\mu\in\mathbb{R}$ and satisfy the Lipschitz condition

$$|f(t,u,\phi,\mu) - f(t,u_1,\phi_1,\mu_1)| \le d_1|u-u_1| + d_1|\phi-\phi_1| + d_1|\mu-\mu_1|,$$

(ii) $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is measurable in t for any $v\in\mathbb{R}$ and satisfy the Lipschitz condition

$$|g(t,v) - g(t,w)| \le d_2|v-w|.$$

Theorem 2. *Let the assumptions $(i)-(ii)$ be satisfied, then (7), has a unique solution.*

Proof. Let $v(t), w(t)$ be two solutions of (7), then

$$\begin{aligned}
|v(t)-w(t)| &\le \int_0^t \Bigg| f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 - (1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau} v(\theta)d\theta\Bigg] + \int_0^\theta v(s)ds, \\
&\quad \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}v(s)ds, I_q^\beta g(\theta,v(\theta))\Bigg)d\theta \\
&\quad - f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha_0 - (1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau} w(\theta)d\theta\Bigg] + \int_0^\theta w(s)ds, \\
&\quad \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}w(s)ds, I_q^\beta g(\theta,w(\theta))\Bigg)d\theta\Bigg| d\theta \\
&\le \int_0^t \Bigg[d_1\Bigg|\frac{1}{(1-q)\tau\sum_{i=0}^n q^i} (1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau}(w(\theta)-v(\theta))d\theta \\
&\quad + \int_0^\theta (v(s)-w(s))ds\Bigg| \\
&\quad + d_1\frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}|v(s)-w(s)|ds \\
&\quad + d_1 I_q^\beta|g(\theta,v(\theta)) - g(\theta,w(\theta))|\Bigg]d\theta \\
&\le d_1\int_0^t\Bigg[|w(\theta)-v(\theta)| + |v(s)-w(s)| \\
&\quad + \frac{d_1(1-e^{\frac{\alpha\theta}{\alpha-1}})}{\alpha}|v(s)-w(s)|ds \\
&\quad + \frac{d_2\theta^\beta}{\Gamma_q(\beta+1)}|v(\theta)-w(\theta)|\Bigg]d\theta \\
&\le 2d_1\|w-v\| + d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2}\|v-w\| \\
&\quad + \frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)}\|v-w\| \\
&\le \Bigg(2d_1 + d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2} \\
&\quad + \frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)}\Bigg)\|w-v\|.
\end{aligned}$$

Hence,

$$\Bigg[1 - (2d_1 + d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2} + \frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)})\Bigg]\|w-v\| \le 0.$$

Since $2d_1 + d_1 \frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2} + \frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)} < 1$, then $w(t) = v(t)$ and the solution of the integral equation (7) is unique. Thus, based on the Lemma 3.1, the proplem (1)-(2) possess a unique solution $u(t) \in C[0,1]$.

6 Continuous dependence

Now, the continuous dependence of a solution on a constant α_0 is presented.

6.1 Continuous dependence on α_0

Definition 7. *The solution $u(t) \in C[0,1]$ of (1)-(2) depends continuously on α_0, if*

$$\forall \varepsilon > 0, \exists \quad \delta_0(\varepsilon) \quad s.t\ |\alpha_0 - \alpha_0^*| < \delta_0 \Rightarrow \|u - u^*\| < \varepsilon,$$

where u^* is the solution of

$$u^{*\prime\prime}(t) = f\left(t, u^*(t), {}^{CF}D^\alpha u^*(t), I_q^\beta g(t, u^{*\prime}(t))\right), \quad t \in (0,1], \tag{14}$$

with the q-nonlocal condition

$$(1-q)\tau \sum_{i=0}^{n} q^i u^*(q^i \tau) = \alpha_0^*, \quad u^{*\prime}(0) = \beta_0. \tag{15}$$

Theorem 3. *If the assumptions 1-4 of theorem (2) are satisfied, then the solution of (1)-(2) is continuously dependent on α_0.*

Proof. Let $u(t)$, $u^*(t)$ be two solutions of (1)-(2) and (14)-(15) respectively. Then,

$$\begin{aligned}
|v(t) - v^*(t)| = \Bigg| \int_0^t \Bigg[f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Big[\alpha_0 - \\
(1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v(\theta)d\theta\Big] + \int_0^\theta v(s)ds, \\
\frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)} v(s)ds, I_q^\beta g(\theta, v(\theta))\Bigg) d\theta \\
-f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Big[\alpha_0^* - \\
(1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v^*(\theta)d\theta\Big] + \int_0^\theta v^*(s)ds, \\
\frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)} v^*(s)ds, I_q^\beta g(\theta, v^*(\theta))\Bigg) d\theta\Bigg|
\end{aligned}$$

$$\begin{aligned}
\le \int_0^t \Bigg| f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Big[\alpha_0 - \\
(1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v(\theta)d\theta\Big] + \int_0^\theta v(s)ds, \\
\frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)} v(s)ds, I_q^\beta g(\theta, v(\theta))\Bigg) d\theta \\
-f\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Big[\alpha_0^* - \\
(1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} v^*(\theta)d\theta\Big] \\
+\int_0^\theta v^*(s)ds, \frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)} v^*(s)ds, \\
I_q^\beta g(\theta, v^*(\theta))\Bigg)\Bigg| d\theta \\
\le \int_0^t \Bigg[\frac{d_1}{(1-q)\tau\sum_{i=0}^n q^i}|\alpha_0 - \alpha_0^*| \\
+d_1|v^*(\theta) - v(\theta)| + d_1|v(s) - v^*(s)| \\
+d_1\frac{1}{1-\alpha}\int_0^\theta e^{-\frac{\alpha}{1-\alpha}(\theta-s)}|v(s) - v^*(s)|ds \\
+d_1 I_q^\beta |g(\theta, v(\theta)) - g(\theta, v^*(\theta))|\Bigg] d\theta \\
\le \int_0^t \Bigg[\frac{d_1}{(1-q)\tau\sum_{i=0}^n q^i}|\alpha_0 - \\
\alpha_0^*| + 2d_1\|v - v^*\| + \frac{d_1(1 - e^{\frac{\alpha\theta}{\alpha-1}})}{\alpha}|v(s) - v^*(s)|ds \\
+d_1 d_2 \frac{\theta^\beta}{\Gamma_q(\beta+1)}\|v - v^*\|\Bigg] d\theta \\
\le \frac{d_1}{(1-q)\tau\sum_{i=0}^n q^i}|\alpha_0 - \alpha_0^*| \\
+2d_1\|v - v^*\| + d_1\frac{\alpha - (\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}} - 1\right)}{\alpha^2}\|v - v^*\| \\
+\frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)}\|v - v^*\| \\
\le \frac{d_1\delta_0}{(1-q)\tau\sum_{i=0}^n q^i} + \Bigg(2d_1 \\
+d_1\frac{\alpha - (\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}} - 1\right)}{\alpha^2} \\
+\frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)}\Bigg)\|v - v^*\|.
\end{aligned}$$

Hence,

$$\|v - v^*\| \le \frac{\frac{d_1\delta_0}{(1-q)\tau\sum_{i=0}^n q^i}}{1 - (2d_1 + d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2} + \frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)})}.$$

Therefore,

$$|u(t)-u^*(t)| = \left| \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i} \left[\alpha_0 - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} v(\theta)d\theta\right] + \int_0^t v(\theta)d\theta - \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[\alpha_0^* - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} v^*(\theta)d\theta\right] + \int_0^t v^*(\theta)d\theta \right|$$
$$\leq \frac{1}{(1-q)\tau\sum_{i=0}^{m} q^i}|\alpha_0-\alpha_0^*| + 2\|v-v^*\|.$$

Hence,

$$\|u-u^*\| \leq \frac{\delta_0}{(1-q)\tau\sum_{i=0}^{n} q^i} + \frac{\frac{2d_1\delta_0}{(1-q)\tau\sum_{i=0}^{n} q^i}}{1-(2d_1+d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2}+\frac{d_1d_2}{(\beta+1)\Gamma_q(\beta+1)})} = \varepsilon.$$

From the above results, the solution of (1)-(2) is continually dependent on α_0.

7 Methodology of Numerical Technique.

Now, we want to get the numerical solution of (1)-(2) using the combination of finite difference with trapezoidal and cubic b-spline with trapezoidal methods. To begin with, we can write the problem (1)-(2) as follows:

$$\begin{aligned} &u''(t)-d_1\varphi_1(u(t)) = \\ &v_1(t)+d_1{}^{CF}D^{\alpha}u(t)+d_1I_q^{\beta}g(t,u'(t)), \end{aligned} \tag{16}$$

$$(1-q)\tau\sum_{i=0}^{n} q^i u(q^i\tau) = \alpha_0, \quad u'(0) = \beta_0,$$

where $g(t,u'(t)) = (v_2(t)+d_2\varphi_2(u'(t))$. Then, by using (3) and (5), we can write (16) as follows:

$$\begin{aligned} &u''(t)-d_1\varphi_1(u(t)) = v_1(t) \\ &+d_1\frac{1}{1-\alpha}\int_0^t e^{\frac{-\alpha(t-s)}{1-\alpha}}u'(s)ds \\ &+d_1\frac{1}{\Gamma_q(\beta)}\int_0^t (t-qs)^{\beta-1}(v_2(s)+d_2\varphi_2(u'(s)))d_qs, \end{aligned} \tag{17}$$

where $\varphi_1(u(t)), \varphi_2(u'(t)))$ are nonlinear terms for the unknown function. Now, the interval of integration $[0,t]$ of equation (17) is divided into m equal subintervals of width $h = (t_m-0)/m, m \geqslant 1$, where t_m is the end point we choose for t [29]. By taking $u''_i = u''(t_i), u'_j = u'(s_j)$, $\varphi_1(u_i) = \varphi_1(u(t_i))$, $\varphi_2(u'_j) = \varphi_2(u'(s_j)), v_1(t_i) = v_{1i}, v_2(s_j) = v_{2j}$, let $k_{ij} = (t_i-qs_j)^{\beta-1}$, $K_{ij} = e^{\frac{-\alpha(t_i-s_j)}{1-\alpha}}$. Then, (17) can be written as follows:

$$\begin{aligned} &u''_i - d_1\varphi_1(u_i) = \gamma_i + \frac{d_1}{1-\alpha}\int_0^{t_i} K_{ij}u'_j ds \\ &+\frac{d_1d_2}{\Gamma_q(\beta)}\int_0^{t_i} k_{ij}\varphi_2(u'_j)d_qs, \end{aligned} \tag{18}$$

where $\gamma_i = \gamma(t_i) = v_{1i} + \frac{d_1}{\Gamma_q(\beta)}\int_0^{t_i} k_{ij}(v_{2j})d_qs$. Clearly $k_{ij} = K_{ij} = 0$ for $j > i$.

7.1 A summary of the finite difference-trapezoidal method.

Now, we use the central finite difference method to approximate the differential part of (18), and we approximate the integral part using the trapezoidal method[20] as follows:

1. The derivative part of (18) can be approximated using the central difference as follows

$$\begin{aligned} u''_i &\approx \frac{u_{i+1}-2u_i+u_{i-1}}{h^2}, \\ u'_i &\approx \frac{u_{i+1}-u_{i-1}}{2h}. \end{aligned} \tag{19}$$

2. The integral part of (18) can be approximated using the trapezoidal rule as

$$\begin{aligned} &\int_0^{t_i} K_{ij}u'_jd_qs \approx \frac{h}{2}\left[K_{i0}u'_0 + 2\sum_{j=1}^{n-1} K_{ij}u'_j + K_{im}u'_m\right], \\ &\int_0^{t_i} k_{ij}\varphi_2(u'_j)d_qs \approx \frac{h}{2}\left[k_{i0}\varphi_2(u'_0) + 2\sum_{j=1}^{n-1} k_{ij}\varphi_2(u'_j) + k_{im}\varphi_2(u'_m)\right], \\ &i = 0,1,2,3,\ldots m. \end{aligned}$$

Clearly $K_{ij} = k_{ij} = 0$ for $j > i$.

3.Then, (18) can be written as follows:

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}-d_1\varphi_1(u_i)=\gamma_i + \frac{d_1}{1-\alpha}\frac{h}{2}\Big[K_{i0}(\frac{u_1-u_{-1}}{2h}) + 2\sum_{j=1}^{n-1}K_{ij}(\frac{u_{j+1}-u_{j-1}}{2h}) + K_{im}(\frac{u_{m+1}-u_{m-1}}{2h})\Big] + \frac{d_1d_2}{\Gamma_q(\beta)}\frac{h}{2}\Big[k_{i0}\varphi_2(\frac{u_1-u_{-1}}{2h}) + 2\sum_{j=1}^{n-1}k_{ij}\varphi_2(\frac{u_{j+1}-u_{j-1}}{2h}) + k_{im}\varphi_2(\frac{u_{m+1}-u_{m-1}}{2h})\Big], \quad i=0,1,\dots m. \tag{20}$$

7.2 *A summary of the cubic b-spline-trapezoidal method.*

The interpolation function of the continuous function $u(t)$ on a set of points $0=t_0\le t_1\le t_2\cdots\le t_m=1$ based on cubic b-spline basis functions is defined as follows:

$$u(t_i)=u_i=\sum_{i=-1}^{m+1}C_iB_i^3(t), \quad t\in[0,1], \tag{21}$$

where C_i 's are constants which to be determined and $B_i^3(t)$ forms a basis that was defined in[18]. Now, we get the numerical solution of (18) by using a combination of cubic b-spline with the trapezoidal method as follows:

1.We use the cubic b-spline method to approximate the solution u(t) and its derivative as follows:

$$u_i\approx C_{i-1}+4C_i+C_{i+1},$$

$$u_i'\approx\frac{3}{h}(C_{i+1}-C_{i-1}),$$

$$u_i''\approx\frac{6}{h^2}(C_{i-1}-2C_i+C_{i+1}).$$

2.We approximate the integral part of (18) using the trapezoidal method.

3.As a result, we can write (18) as follows:

$$\frac{6}{h^2}(C_{i-1}-2C_i+C_{i+1}) - d_1\varphi_1(C_{i-1}+4C_i+C_{i+1}) = \gamma_i+\frac{d_1}{1-\alpha}\frac{h}{2}\Big[K_{i0}\frac{C_1-C_{-1}}{3h} + 2\sum_{j=1}^{n-1}K_{ij}\frac{C_{j+1}-C_{j-1}}{3h}+K_{im}\frac{C_{m+1}-C_{m-1}}{3h}\Big] + \frac{d_1d_2}{\Gamma_q(\beta)}\frac{h}{2}\Big[k_{i0}\varphi_2(\frac{C_1-C_{-1}}{3h}) + 2\sum_{j=1}^{n-1}k_{ij}\varphi_2(\frac{C_{j+1}-C_{j-1}}{3h}) + k_{im}\varphi_2(\frac{C_{m+1}-C_{m-1}}{3h})\Big], \quad i=0,1,\dots,m.$$

8 Test problems

Now, we introduce some numerical examples by using the following two methods:

1.Finite difference-trapezoidal method,
2.cubic b-spline-trapezoidal method.

Test problem 1: In (17), we take $v_1(t) = -0.0595238t^2 + 0.142857e^{-1.t} - 1.14286\sin(t) - 0.142857\cos(t) + 0.0813492\cos(2t) - 0.0396825t\sin(t)\cos(t) - 0.0813492, v_2(t) = \cos^2(t), d_1=\frac{1}{7}, d_2=\frac{1}{9}, \alpha=0.5, \beta=2, q=0.5, \tau=0.2, n=2, \varphi_1(u(t))=u^2(t), \varphi_2(u'(t))=u'^2(t), \alpha_0=0.026108, \beta_0=1.$ Clearly $2d_1+d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2}+\frac{d_1d_2}{(\beta+1)\Gamma_q(\beta+1)}<1$. The exact solution of this problem is $u(t)=\sin(t)$.

As a result, the assumptions of the theorem (1) are clearly satisfied, and therefore the given problem has a continuous solution. Now, we take m = 20 to find the numerical solution of this problem using finite difference-trapezoidal and cubic-trapezoidal approaches.

Table 1: The exact and numerical solutions to test problem 1.

t_i	Exact solutions	Finite-trap.	Abs. error (Finite-trap)	cubic-trap.	Abs. error (cubic-trap)
0.1	0.0998	0.0997	3.850E-5	0.0998	1.803E-5
0.2	0.1986	0.1987	3.854E-5	0.1986	1.799E-5
0.4	0.3894	0.3896	1.909E-4	0.3895	9.207E-5
0.5	0.4794	0.4796	2.660E-4	0.4795	1.307E-4
0.6	0.5646	0.5649	3.402E-4	0.5648	1.708E-4
0.8	0.7173	0.7178	4.866E-4	0.7176	2.566E-4
0.9	0.7833	0.7838	5.590E-4	0.7836	3.030E-4

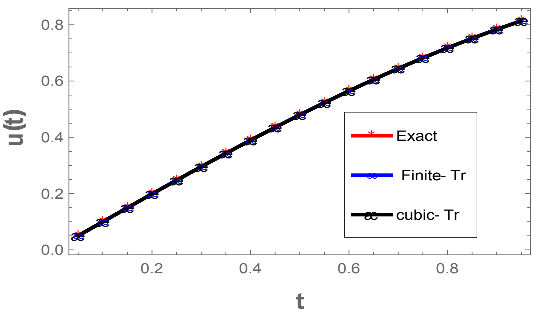

Fig. 1: Comparison between the numerical and exact solutions of test problem 1

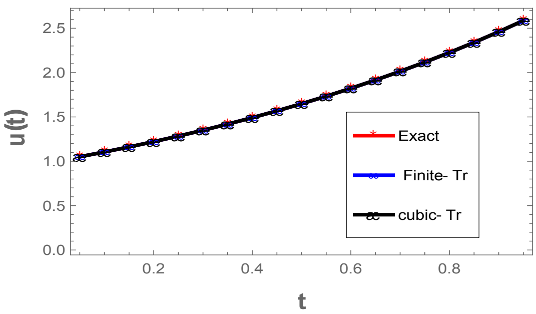

Fig. 2: Comparison between the numerical and exact solutions of test problem 2

Table 1 shows the comparison between the numerical solutions using finite-trapezoidal and cubic-trapezoidal methods with the exact solutions. The results show that both numerical methods are valid and effective. In addition, Figure 1 shows the comparison between the exact solution of test problem 1 and the numerical solution using the cubic b-spline-trapezoidal method and the finite-trapezoidal method. Through our observation of Figure 1, the exact and numerical solutions are very close, indicating that the numerical solutions are good. Furthermore, we study the continuous dependence on α_0 using the cubic b-spline-trapezoidal method. If we take $|\alpha_0 - \alpha_0^*| = 10^{-5} \Rightarrow |u(0.6) - u^*(0.6)| = 5.77108 \times 10^{-5}$. Therfore, u(t) is continuous dependence on α_0.

Test problem 2: In (17), we take $v_1(t) = e^{-0.428571t}(0.777778e^{1.42857t} + e^{5.42857t}(0.0818919\Gamma(1.3,5.t) \quad -0.0818919\Gamma(1.3,4.t)) +0.111111), v_2(t) = \exp(t), d_1 = \frac{1}{9}, d_2 = \frac{1}{7}, \alpha = 0.3, \beta = 1.3, q = 0.2, \tau = 0.5, n = 1, \varphi_1(u(t)) = u(t), \varphi_2(u'(t)) = u'(t), \alpha_0 = 0.747902, \beta_0 = 1.$ Clearly $2d_1 + d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2} + \frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)} < 1$. The exact solution of this problem is $u(t) = \exp(t)$.

As a result, the assumptions of the theorem (1) are clearly satisfied, and therefore the given problem has a continuous solution. Now, we take m=20 to find the numerical solution of this problem using finite difference-trapezoidal and cubic-trapezoidal approaches.

Table 2: The exact and numerical solutions to test problem 2.

t_i	Exact solutions	Finite-trap.	Abs. error (Finite-trap)	cubic-trap.	Abs. error (cubic-trap)
0.2	1.221	1.221	4.342E-5	1.221	9.554E-5
0.4	1.491	1.491	4.765E-6	1.491	2.257E-5
0.6	1.822	1.822	3.004E-5	1.822	7.903E-5
0.8	2.225	2.225	6.500E-5	2.225	2.114E-4

Table 2 shows the comparison between the numerical solutions using finite-trapezoidal and cubic-trapezoidal methods with the exact solutions. The results show that both numerical methods are valid and effective. In addition, Figure 2 shows the comparison between the exact solution of test problem 2 and the numerical solution using the cubic b-spline-trapezoidal method and the finite-trapezoidal method. Through our observation of Figure 2, the exact and numerical solutions are very close, indicating that the numerical solutions are good.

Test problem 3: In (17), we take $v_1(t) = -0.0374139t^{3.5} + 0.0655534t^{1.5} + 0.15625e^{-4.t} - \frac{e^t}{8} - 0.15625, v_2(t) = t^2 - 1, d_1 = \frac{1}{8}, d_2 = \frac{1}{6}, \alpha = 0.8, \beta = \frac{3}{2}, q = 0.4, \tau = 0.5, n = 1, \varphi_1(u(t)) = e^{u(t)}, \varphi_2(u'(t)) = e^{u'(t)}, \alpha_0 = 0.174, \beta_0 = 1.$ Clearly $2d_1 + d_1\frac{\alpha-(\alpha-1)\left(e^{\frac{\alpha}{\alpha-1}}-1\right)}{\alpha^2} + \frac{d_1 d_2}{(\beta+1)\Gamma_q(\beta+1)} < 1$. The exact solution of this problem is $u(t) = t$.

As a result, the assumptions of the theorem (1) are clearly satisfied, then the given problem has a continuous solution. Now, we take m=20 to find the numerical solution of this problem using the finite difference-trapezoidal and cubic-trapezoidal approaches.

Table 3: The exact and numerical solutions to test problem 3.

t_i	Exact solut-ions	Finite-Trap.	Abs. error (Finite-Trap)	cubic-Trap.	Abs. error (cubic-Trap)
0.2	0.2	0.199	7.376E-4	0.199	7.422E-4
0.4	0.4	0.399	2.172E-4	0.399	2.174E-4
0.6	0.6	0.601	1.041E-3	0.601	1.045E-3
0.8	0.8	0.803	3.406E-3	0.803	3.412E-3

Table 3 shows the comparison between the numerical solutions using finite-trapezoidal and cubic-trapezoidal methods with the exact solutions. The results show that both numerical methods are valid and effective. In addition, Figure 3 shows the comparison between the exact solution of test problem 3 and the numerical solution using the cubic b-spline-trapezoidal method and the finite-trapezoidal method. Through our observation of

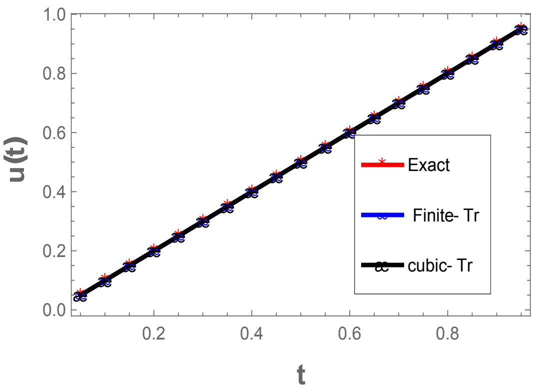

Fig. 3: Comparison between the numerical and exact solutions of test problem 3

Figure 3, the exact and numerical solutions are very close, indicating that the numerical solutions are good.

9 Conclusion

In this paper, by using the Schauder fixed-point theorem, we have established the existence and uniqueness of solution for a nonlocal fractional q integro differential equation. The continuous dependence of the solution on α_0 has been studied. The finite difference-trapezoidal and cubic b-spline-trapezoidal methods has introduced to get the numerical solution to the proposed problem. We applied the assumptions of the existence theorem on three exampes and solved them numerically to demonstrate the accuracy of the two methods used. In the future study, we plan to discuss more general equation to the q fractional integro differential equation with the mixed cindition.

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