

# On Tikhonov Regularization Method in Calibration of Volatility Term-structure

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**Abstract:** In this paper, we consider inverse problem arising in calibration of time-dependent volatility function from the Black-Scholes model and analyze its ill-posedness phenomena. The forward operator of the inverse problem under some consideration decomposes into an inner linear convolution operator and an outer nonlinear Nemytskii operator given by a Black-Scholes function. Using Chebyshev collocation method, we transfer the inner linear operator to a linear system. Since the resulting matrix equation is badly ill-conditioned, a regularized solution is obtained by employing the Tikhonov regularization method, while the choice of the regularization parameter are based on generalized cross-validation(GCV) and L-curve criterions. Numerical case studies illustrate the efficiency and accuracy of the presented method.

**Keywords:** Option pricing , Implied volatility, Chebyshev collocation method, Inverse problem, Tikhonov regularization

## 1 Introduction

In recent years several models have been created to price financial security products. Financial securities (options, futures and forward contracts) have become essential tools for corporations and investors over the past few decades. Options can be used, for example, to hedge assets and portfolios in order to control the risk due to the movement in stock prices. European options and American options are the two major types of options. An European Call(Put) option is a financial derivative that certifies the holder's right but not obligation to buy (for a Call option) or sell (for a put option) a specific amount of an underlying security, for a fixed price  $K$  (exercise price), at a fixed future time  $T$  (maturity or expiry). Since an option scrutinizes a right it has a certain option value or option price. Classical option pricing theory was suggested by Black and Scholes [1] and extended by Merton [2].

In the Black-Scholes world there is the important quantity of volatility. Volatility is a measure of the amount of fluctuation in the asset price, i.e., a measure of the randomness. It has a major impact on the option value. Knowing the volatility function allows for a better understanding of underlying stochastic process of option price. Most option traders invert the Black-Scholes

formula to determine the volatility (Called the implied volatility) from the market option price. The implied volatility of an option pricing model that depends on its life and defines as function of time to maturity is called volatility of term-structure. Often traders use this volatility (for more details see [3,4,5]).

The mathematical problem that arises here consists in finding (calibrating) a time-dependent volatility function defined on a finite time interval  $I := [0, T]$  from the term structure on  $I$  of observed prices of vanilla Call options with a fixed strike  $K > 0$ . In the fact, we want to convert observed measurements (option prices) into information about volatility function that we are interested in and it isn't observable. It has been observed to be an ill-posed problem in the sense that reconstruction of volatility is unstable with respect to errors in the data. Therefore the calibration of volatility function is an inverse problem. Existence and uniqueness and some properties of the solution to this problem were established in [6]. Researchers in literature have used different methods for approximating volatility function. For example, in [7] authors used maximum entropy regularization (MER) to find an estimation of volatility function and in [8] authors explore the theoretical and numerical application of local regularization methods for identifying volatility function. In this work, we use Tikhonov regularization method with

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general cross validation (GCV) and L-curve criterions to resolve ill-posedness of the problem.

This paper is organized as follows. In Section 2 we first discuss the original Black-Scholes model and expose the mathematical model for pricing European option when the model has time-dependent parameters. Then we state the calibration of time dependent volatility function from the option pricing model as an inverse problem and investigate ill-posedness phenomena. Discretization of the problem obtained from previous section in the form linear system using Chebyshev collocation method will done in Section 3. In Section 4 we describe Tikhonov regularization method for resolving the ill-posed problem of option pricing and introduce GCV and L-curve criterions for determining regularization parameter. We illustrate in Section 5 the accuracy and efficiency of the method with numerical examples. Finally Section 6 concludes.

## 2 Mathematical formulate

The price of the asset or underlying derivative  $s(t)$  follows a Geometric Brownian motion  $w(t)$ , meaning that  $s$  satisfies the following stochastic differential equation (SDE):

$$ds(t) = \mu s(t)dt + \sigma s(t)dw(t), \quad (1)$$

the trend or drift  $\mu$  (measures the average rate of growth of the asset price), the volatility  $\sigma$  (measures the standard deviation of the returns), and no dividends are paid in that time period.

Assume that  $C$  is the Call option value,  $k$  exercise price and let  $r$  denote the risk-free interest rate (constant for  $0 \leq t \leq T$ ). If the market is complete (there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, all securities and credits are available at any time and any size, all variables are perfectly divisible and may take any real number, individual trading will not influence the price and there are no arbitrage opportunities), which means that any asset can be replicated with a portfolio of other assets in the market (see [9]), we can find the Call value of the European option. Under the above assumptions and using Ito's lemma, the Call option value obtains as the following boundary value problem of the Black-Scholes equation [1]

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} + rs \frac{\partial C}{\partial s} - rC &= 0, \quad 0 < s < \infty, 0 \leq t < T, \\ C(s, T) &= (s - K)^+ = \max(s - k, 0), \quad 0 \leq s < \infty, \\ C(0, t) &= 0, \quad 0 \leq t \leq T, \\ C(s, t) &= s - Ke^{-r(T-t)}, \quad s \rightarrow \infty. \end{aligned} \quad (2)$$

We consider in this paper a different kind of the Black-Scholes model, which is more realistic and focused on time-dependent functions over the interval  $[0, T]$  using a generalized geometric Brownian motion as stochastic process for the price  $s(t)$  of an asset, on which options are written. With constant drift  $\mu > 0$ , time-dependent volatilities  $\sigma(t)$ , a dividend yield  $sqdt$  in a time step  $dt$  and a standard Wiener process  $w(t)$ , the stochastic differential equation becomes

$$dS(t) = (\mu - q)S(t)dt + \sigma(t)S(t)dw(t), \quad 0 < t < T. \quad (3)$$

When the parameters  $r$  and  $q$  also become deterministic functions of time, it follows from stochastic and analytic considerations on arbitrage-free markets has to be modified as follows:(see [5])

$$-\frac{\partial C}{\partial t} = \frac{\sigma^2(t)}{2} s^2 \frac{\partial^2 C}{\partial s^2} + [r(t) - q(t)]s \frac{\partial C}{\partial s} - r(t)C, \quad (4)$$

$$0 < s < \infty, \quad t > 0.$$

where  $C$  is the price of the derivative security. When we apply the following transformations:

$$y = \ln s, \quad u = C \cdot \exp\left(\int_0^t r(\tau)d\tau\right), \quad (5)$$

then becomes

$$\frac{\partial u}{\partial t} = \frac{\sigma^2(t)}{2} \frac{\partial^2 u}{\partial y^2} + [r(t) - q(t) - \frac{\sigma^2(t)}{2}] \frac{\partial u}{\partial y}, \quad (6)$$

$$-\infty < y < \infty, \quad t > 0.$$

Given the initial condition  $u(y, 0)$ , the solution to (6) can be expressed as

$$u(y, t) = \int_{-\infty}^{\infty} u(\xi, 0) \phi(y - \xi, t) d\xi, \quad (7)$$

where

$$\phi(y, t) = \frac{\exp\left(-\frac{(y + \int_0^t [r(\tau) - q(\tau) - \frac{\sigma^2(\tau)}{2}]d\tau)^2}{2 \int_0^t \sigma^2(\tau)d\tau}\right)}{\sqrt{2\pi \int_0^t \sigma^2(\tau)d\tau}}. \quad (8)$$

For an asset with current asset price  $X := X(0) > 0$  at time  $t = 0$  we consider a family of European vanilla Call options with a fixed strike  $K > 0$ , a time dependent risk-free interest rate  $r(t) \geq 0$ , dividend yield  $q(t) \geq 0$  and maturities  $t$  varying through the whole interval  $I$ . Then it follows from stochastic considerations (for details see [5]) that the associated fair prices  $C(t)$  ( $0 < t \leq T$ ) of these

options satisfy on an arbitrage-free market the equation

$$C(t) = X e^{-\int_0^t q(\tau) d\tau} \varphi\left(\frac{\ln \frac{X}{k} + \int_0^t [r(\tau) - q(\tau) - \frac{\sigma^2(\tau)}{2}] d\tau}{\sqrt{\int_0^t \sigma^2(\tau) d\tau}}\right) - k e^{-\int_0^t r(\tau) d\tau} \varphi\left(\frac{\ln \frac{X}{k} + \int_0^t [r(\tau) - q(\tau) - \frac{\sigma^2(\tau)}{2}] d\tau}{\sqrt{\int_0^t \sigma^2(\tau) d\tau}}\right) - \sqrt{\int_0^t \sigma^2(\tau) d\tau}, \tag{9}$$

where

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{x^2}{2}} dx, \tag{10}$$

moreover, the payoff of a European Call at expiry provides

$$C(0) = \max(X - k, 0). \tag{11}$$

The Black-Scholes-type formula (9) and (11) is originally derived for positive continuous volatility functions, but it also yields well-defined values  $C(t) \geq 0 (t \in I)$  if the functions  $\sigma^2(t), r(t)$  and  $q(t)$  are Lebesgue-integrable and almost everywhere finite and positive. Therefore the direct problem of option pricing model with time-dependent parameters can be expressed as the following:

**Direct Problem.** The European Call price formula for Black-Scholes option pricing model, with parameters  $X > 0, r(\tau) > 0, q(\tau) \geq 0, \tau \geq 0, s > 0$  and time dependent volatility function  $\sigma(\tau) > 0$  is the following:

$$C_{BS} = \begin{cases} X e^{-\int_0^\tau q(u) du} \varphi(d_1) - k e^{-\int_0^\tau r(u) du} \varphi(d_2), & s > 0, \\ \max(X - k e^{-\int_0^\tau r(u) du}, 0), & s = 0, \end{cases} \tag{12}$$

where  $C_{BS} := C_{BS}(X, k, r(\tau), q(\tau), \tau, s)$  and

$$d_1 = \frac{\ln \frac{X}{k} + \int_0^t [r(u) - q(u) - \frac{\sigma^2(u)}{2}] du}{\sqrt{\int_0^t \sigma^2(u) du}},$$

$$d_2 = d_1 - \sqrt{\int_0^t \sigma^2(u) du}. \tag{13}$$

The European put price formula can be deduced in a similar manner. Also we can reformulate above solution in terms of the auxiliary function

$$b(t) := \int_0^t \sigma^2(\tau) d\tau \tag{14}$$

concisely as the following

$$C(t) = C_{BS}(X, K, r(t), q(t), t, b(t)), \quad t \in I. \tag{15}$$

The option price obtained from the Black-Scholes pricing framework is function of parameters: asset price  $s$ , strike price  $k$ , riskless interest rate  $r$ , dividend yield  $q$ , time to expiry  $t$  and volatility  $\sigma$ . Except for the volatility parameter, the other parameters are observable quantities. The difficulties of setting volatility value in the price formulas lie in the fact that the input value should be the forecast volatility value over the remaining life of the option rather than an estimated volatility value (historical volatility) from the past market data of the asset price. Since  $\sigma(t)$  cannot be solved explicitly in terms of  $X, s, r, q, t$  and option price  $C$  from the pricing formulas, the determination of the implied volatility has been devoted a lot of attention of mathematicians in recent years. In what follows, we try to state the inverse problem arising in option pricing model.

We consider  $C(t)$  is the exact value of European Call option and  $C^\delta(t)$  noisy data option pricing of equation (15) such that,

$$\|C^\delta(t) - C(t)\|_{L^2(I)} = \left( \int_I (C^\delta(\tau) - C(\tau))^2 d\tau \right)^{\frac{1}{2}} \leq \delta,$$

we want to find appropriate approximations  $u^\delta(t) := \sigma_\delta^2(t)$  from exact function  $u(t) := \sigma^2(t)$  by the following accuracy

$$\|u^\delta(t) - u(t)\|_{L^1(I)} = \int_I (u^\delta(\tau) - u(\tau)) d\tau \leq \delta.$$

According to the notations used in [7], we write the nonlinear forward operator equation  $F(u(t)) = C(t)$  such that

$$F : D(F) \subset L^1(I) \longrightarrow L^2(I),$$

$$F = N \circ J,$$

where  $D(F) = \{u(t) \in L^1(0, T); u(t) \geq 0 \text{ a.e. in } [0, T]\}$ , with the inner linear convolution operator

$$J : L^1(I) \longrightarrow L^2(I),$$

$$J[v](t) =: \int_0^t v(\tau) d\tau,$$

and the outer nonlinear Nemytskii operator

$$N : D_+ \cap L^2(I) \subset L^2(I) \longrightarrow L^2(I),$$

$$[N(b)](t) := C_{BS}(X, K, r, t, b(t)),$$

where  $D_+$  is the set of nonnegative functions over the interval  $I$ .

For identifying  $u(t)$ , first we can find uniquely  $b^\delta(t)$  based on previous theorem corresponding to  $C^\delta(t)$  by the following nonlinear Nemytskii operator

$$[N(b^\delta)](t) := C_{BS}^\delta(X, K, r, t, b(t)), \tag{16}$$

and then, determine  $u^\delta(t)$  by the following linear Volterra integral operator

$$J[u^\delta](t) =: \int_0^t u(\tau) d\tau = b^\delta(t), \quad (17)$$

such that

$$\|u^\delta(t) - u(t)\|_{L_1(t)} \leq \delta.$$

Now for simplifying, we write above linear Volterra integral equation in the form  $Au = b$ , where the operator  $A$  is defined for  $u$  by

$$A[u(\tau)] = \int_0^t u(\tau) d\tau = b(t), \quad t \in [0, T], \quad (18)$$

where  $u(t) := \sigma^2(t)$ . Then  $A \in L(U)$ , the space of continuous linear operators from  $U \equiv L_2[0, T]$  to  $U$ . We will assume throughout that the data  $b(t)$  is such that there exists a unique solution  $\bar{u} \in U$  of equation (19) and, in particular, we require that  $b(0) = 0$ .

The equation (18) is ill-posed, which has serious implications in the usual case where we only have available an approximation  $b^\delta \in U$  of  $U$ . The ill-posedness means that the solution  $u^\delta$  of  $Au = b^\delta$  (when such a solution exists) may be arbitrarily far from the solution  $\bar{u}$  of the unperturbed problem. Therefore, some kind of regularization procedure will be needed to solve the problem in the case of perturbed data  $b^\delta$  (for more details see [18]). Then the inverse problem of calibrating the volatility term structure  $\sigma(t)$  from noisy data  $C^\delta(t)$  can be expressed as follows:

**Inverse Problem.** Determining of the time-dependent volatility function  $\sigma(t), (0 < t < T)$ , under the assumptions stated above from noisy observations  $C^\delta(t), (0 < t < T)$  of the maturity-dependent fair price function  $C(t), (0 < t < T)$  in nonlinear forward operator  $F(u(t)) = C(t)$ , where  $u(t) := \sigma^2(t)$ .

### 3 Numerical Approach

In this section we describe in greater details the approximation algorithm adopted in this paper. First using Newton's method [10], we find  $b^\delta(t)$  from  $C^\delta(t)$  in equation (16) and then we try to obtain  $u^\delta(t)$  corresponding to  $C^\delta(t)$  from  $b^\delta(t)$  with noise level  $\delta$  in equation (17). In order to solve the equation (17) by approximation we need to define:

- i. The family of basis functions to approximate the function  $u(t)$ .
- ii. The interpolation nodes,  $t_i$ .

According to the assumptions on the volatility function (bounded over its domain and uniformly Holder continuous on each compact subset of its domain), it's

possible to get a finite dimensional approximation of  $u(t)$  in  $C[0, T]$  by using the least square method [11] as

$$u(t) \simeq P_N(t) = \sum_{i=0}^N c_i \psi_i(t) \quad (19)$$

where  $c_i, i = 0, 1, 2, \dots, N$  are real constants for  $N = 1, 2, \dots$  and  $\psi_i(t)$ s are a set of orthogonal functions[11].

The error made by using a polynomial of order  $N$  to approximate the function given  $u(t)$ , can be easily calculated as:

$$u(t) - P_N(t) = \frac{1}{N+1} u^{N+1}(\varepsilon) \Pi_{i=0}^N(t - t_i).$$

this error of the approximation (19) may be further reduced by adding more functions  $\psi_{i+1}(t), \dots$  to the pervious set [11]. Coefficients  $c_i, i = 0, 1, 2, \dots, N$  in the (19) are unknown and if these coefficients are determined, then we get an estimation for  $u(t)$ . Our approach can be justified by appealing to Rivlin's theorem, stating that Chebyshev node polynomial interpolants are nearly optimal polynomial approximants and has been shown to perform well empirically [12]. Chebyshev nodes over  $[0, T]$  are as the following

$$t_i = \frac{1}{2} T \left( 1 + \cos\left(\frac{(2i-1)\pi}{2n}\right) \right), \quad i = 0, 1, 2, \dots, N,$$

as important as the choice of the nodes interpolants is that of a family of functions from which the approximant  $P$  will be drawn. We suggest using a Chebyshev polynomial. The Chebyshev polynomials of the first kind are as the following

$$\psi_i(\tau) = T_i(\tau) = \cos(i \cdot \arccos(\tau)), \quad i = 0, 1, 2, \dots, N. \quad (20)$$

For determining coefficients  $c_i, i = 0, 1, \dots, N$ , we use Collocation method base on Chebyshev polynomials and Chebyshev nodes, namely Chebyshev collocation method.

The Chebyshev collocation method is one of the most efficient tools for the numerical solution of intertemporal optimizing. The principle of these methods is that the solution is represented by a finite Chebyshev series with unknown coefficients. This expression is substituted into the equation and the coefficients are determined so that the equation is satisfied at certain points within the range under consideration. The number of points is chosen so that, along with the conditions of equation, there are enough equations to find the unknown coefficients. The positions of the points in the range are chosen to make small the residual obtained when the approximate solution is substituted into the equation. This residual is minimized if collocation point were roots of Chebyshev polynomial. Lanczos in [11] calls this choice of points the "selected points" principle or the method of collocation.

By substituting (19) in the (17), for  $b^\delta(t)$ , we have the following equation

$$b^\delta(t) = \int_0^t \left( \sum_{i=0}^N c_i T_i(\tau) \right) d\tau. \quad (21)$$

Based on Chebyshev collocation method, with considering Chebyshev nodes over  $[0, T]$  as selected points, we derive the following linear system

$$b^\delta(t_i) = \int_0^{t_i} \left( \sum_{i=0}^N c_i T_i(\tau) \right) d\tau, \quad i = 0, 1, 2, \dots, N. \quad (22)$$

The above mentioned system of  $N + 1$  equations with  $N + 1$  unknown coefficients  $c_i, i = 0, 1, 2, \dots, N$  will be in the form

$$Au = b. \quad (23)$$

where the matrix  $A$  and the vectors  $u$  and  $b$  are as the following

$$A = \begin{pmatrix} \int_0^{t_0} T_0(\tau) d\tau & \int_0^{t_0} T_1(\tau) d\tau & \dots & \int_0^{t_0} T_N(\tau) d\tau \\ \int_0^{t_1} T_0(\tau) d\tau & \int_0^{t_1} T_1(\tau) d\tau & \dots & \int_0^{t_1} T_N(\tau) d\tau \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^{t_N} T_0(\tau) d\tau & \int_0^{t_N} T_1(\tau) d\tau & \dots & \int_0^{t_N} T_N(\tau) d\tau \end{pmatrix},$$

$$u = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix}, b = \begin{pmatrix} b^\delta(t_0) \\ b^\delta(t_1) \\ \vdots \\ b^\delta(t_N) \end{pmatrix}$$

Since the original the first kind Volterra is ill-posed, the ill-conditioning of the matrix  $A$  in equation (23) still persists. In other words, the condition number of matrix  $A$  increases dramatically with respect to the total number of collocation points and therefore most standard numerical methods cannot achieve good accuracy in solving the matrix equation (23) due to the bad condition number of the matrix  $A$ . For this purpose, the Tikhonov regularization method is applied.

### 4 Tikhonov Regularization Method

In Section 2, we showed calibration of the volatility function in option pricing model is an ill-posed problem. Therefore the condition number of matrix  $A$  in equation (23) is large compared with the number of collocation points. Several regularization methods have been developed for solving an ill-conditional problem [13, 14, 15, 16, 17]. In this work we adapt the Tikhonov regularization method [18] to solve the resulting matrix equation (23).

The Tikhonov regularized solution for equation (23) is

defined as the solution of the following least squares problem:

$$\min_{u \in U} \{ \|Au - b^\delta\|^2 + \alpha^2 \|u\|^2 \}, \quad (24)$$

where  $\|\cdot\|$  denotes the Euclidean norm and  $\alpha$  is called the regularization parameter, which controls the trade-off between fidelity to the data and smoothness of the solution. Equivalently, the solution is defined as the solution of the normal equations on  $U$ ,

$$(A^*A + \alpha I)u = b^\delta, \quad (25)$$

where  $A^* \in L(U)$  is the (Hilbert) adjoint operator associated with  $A$ . Standard Tikhonov regularization theory (which is applicable to first-kind Volterra problems) gives well-known conditions on the selection of  $\alpha = \alpha(\delta)$  so that  $u_{\alpha(\delta)}^\delta \rightarrow \bar{u}$  in  $U$  as  $\delta \rightarrow 0$ .

The determination of a suitable value for the regularization parameter  $\alpha$  is crucial and is still under intensive researches. We apply L-curve and GCV criterions to choose the regularization parameter  $\alpha$  and compare them.

**I. L-curve method.** The L-curve is a plot of the squared estimate norm of the regularized solution  $\|u\|$  against the squared norm of the regularized residual  $\|Au - b\|$  for a range of values of regularization parameters. Hansen [14, 15, 16, 19] proposes to choose the value of the regularization parameter that corresponds to the point at the corner of the curve. The corner is defined to be the point on the L-curve with curvature of largest magnitude. The name "L-curve" implies that the shape of the curve should resemble L letter closely.

**II. GCV method.** Generalized cross-validation (GCV) criterion is to choose the regularization parameter  $\alpha$ . The GCV criterion is a very popular and successful method for choosing the regularization parameter [19]. The GCV method determines the optimal regularization parameter by minimizing the following function:

$$G(\alpha) = \frac{\|Au_\alpha^\delta - b^\delta\|^2}{(\text{trace}(I - AA^T))^2} \quad (26)$$

where  $A^I = (A^T A + \alpha^2 I)^{-1} A^T$  is a matrix which produces the regularized solution  $u_\alpha^\delta$  when multiplied with the right hand side  $b^\delta$ , i.e.,  $u_\alpha^\delta = A^I b^\delta$ .

In our computation, we used the MATLAB code developed by Hansen [13] for solving the discrete ill-conditioned system of equation (23)

### 5 Experimental Results

In this section we report numerical results to demonstrate the accuracy of presented algorithm for calibrating time dependent volatility function from Call option pricing model. Since in the real market, we observe only noisy

option prices  $C^\delta(t)$  for  $t \in I = [0, T]$  instead of fair option prices  $C_{exact}(t)$ , we find an approximation function  $u^\delta(t)$  of function  $u_{exact}(t)$  corresponding to the noisy data  $C^\delta(t)$ . In all of studied case a randomly distributed perturbation  $\delta \times randn(N)$  is added to the  $C(t_i), i = 0, 1, 2, \dots, N$  to generating the noisy data in the form

$$C^\delta(t_i) = C(t_i) + \delta \times randn(N); i = 0, 1, 2, \dots, N \quad (27)$$

where  $\delta$  dictates the level of noise and  $randn(\cdot)$  is a normal distribution function with zero mean and unit standard deviation and it is realized using the MATLAB function `rand`. Our algorithm is implemented using MATLAB for testing purpose.

To test the accuracy of the approximate solution, we use the root mean square error (RMSE) using a weighted  $l^2 - norm$  as the following:

$$\|\cdot\| := \frac{\|u^\delta(t) - u(t)\|_2}{N+1} = \left( \frac{\sum_{i=0}^N (|u^\delta(t_i) - u(t_i)|)^2}{N+1} \right)^{\frac{1}{2}}$$

where  $N + 1$  is the total number of test points distributed in the domain  $[0, T]$  and  $0 \leq t_i \leq T$ .

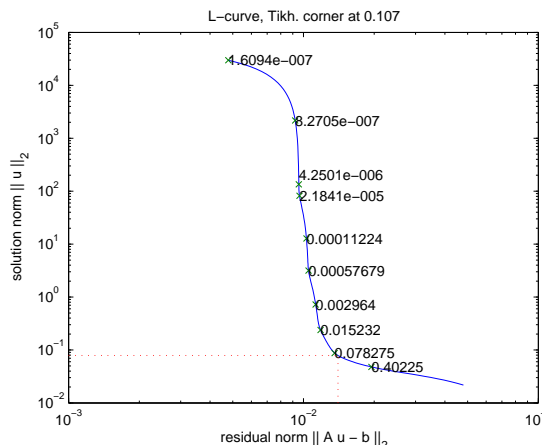
**Example 1.** Consider an 1 year European Call option with the parameters, risk-free rate  $r = 0.05$  per annum, exercise price  $k = 0.5$ , initial stock price  $X = 0.6$ , the level of noise  $\delta = 0.001$  and the following volatility function

$$u(t) = ((t - 0.5)^2 + 0.1)^2.$$

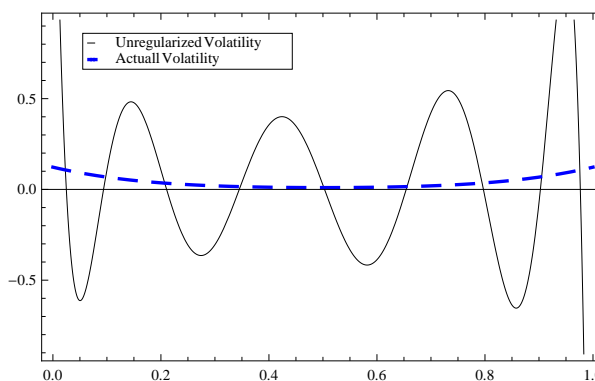
The L-curve plot is shown in Figure1. Figure2 is the approximation volatility function (without regularization) computed using noisy data option pricing compared to the actually volatility function with  $N = 10$ . Figure3 displays regularized volatility function using Tikhonov regularization and L-curve criterion compared to actually volatility function. The RMSE values of volatility function and condition number of resulting matrix in different numbers of collocation points can be found in the Table1.

$N$	$Cond(A)$	$RMSE - Tikh$	$RMSE - Unreg$
5	$1.4053 \times 10^3$	0.0413345	0.358242
10	$1.1835 \times 10^7$	0.0060692	1.70217
15	$4.1355 \times 10^{11}$	0.004197	2.374658
30	$1.8506 \times 10^{18}$	0.00506602	4.41602

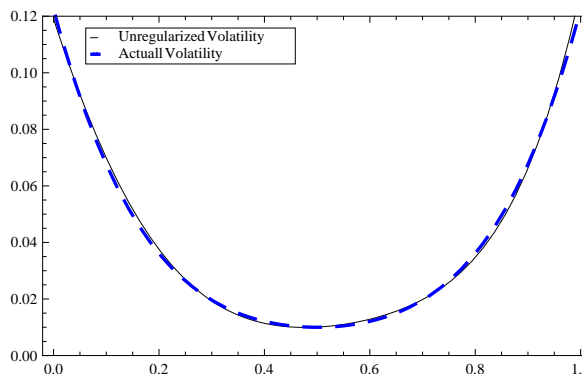
**Table 1:** Accuracy of solutions in Example1.  $N$  indicates number of collocation points,  $Cond(A)$  condition number of resulting matrix,  $RMSE - Tikh$  the root mean square error of regularized solution and  $RMSE - Unreg$  the root mean square error without regularization.



**Fig. 1:** The L-curve plot of Example1 for data with noise level  $\delta = 10^{-3}$ .



**Fig. 2:** The representation of volatility function in Example1 without regularization.



**Fig. 3:** The representation of Tikhonov regularized volatility function with L-curve criterion in Example1.

**Example 2.** Consider an 1 year European Call option with the parameters,  $r = 0.05$ ,  $k = 0.5$ ,  $X = 0.6$ ,  $\delta = 0.001$  and the following volatility function

$$u(t) = 0.1 + \frac{0.9}{1 + 100(2t - 1)^2}.$$

The GCV plot is shown in Figure4. Figure5 is the volatility function computed using noisy data option pricing without regularization compared to the exact volatility function with  $N = 10$ . Figure6 displays regularized volatility function using Tikhonov regularization and GCV criterion compared to actually volatility function. The RMSE values of volatility function and condition number of resulting matrix in different numbers of collocation points can be found in the Table2.

$N$	$Cond(A)$	$RMSE - Tikh$	$RMSE - Unreg$
5	$1.4053 \times 10^3$	0.033377	0.768957
10	$1.1835 \times 10^7$	0.00725633	1.741534
15	$4.1355 \times 10^{11}$	0.0092312	45.6294
30	$1.8506 \times 10^{18}$	0.0817829	16868.6

**Table 2:** Accuracy of solutions in Example2.  $N$  indicates number of collocation points,  $Cond(A)$  condition number of resulting matrix,  $RMSE - Tikh$  the root mean square error of regularized solution and  $RMSE - Unreg$  the root mean square error without regularization.

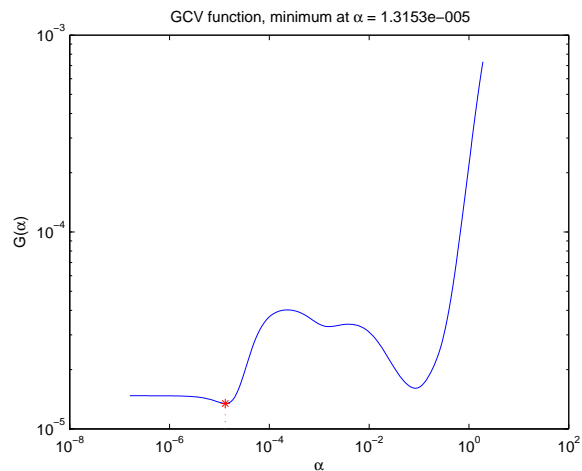
The comparison regularization parameters using L-curve and GCV method for Example1 with the values of RMSE can be found in Table3. The results shows that all the two methods (L-curve and GCV) can achieve good solutions with noise level  $\delta$  and RMSE for both examples are very close.

Method	$Reg - parameter$	$RMSE$
Tikh-L-curve	$\alpha = 1.10613 \times 10^{-14}$	0.004562
Tikh-GCV	$\alpha = 1.19110 \times 10^{-14}$	0.009193

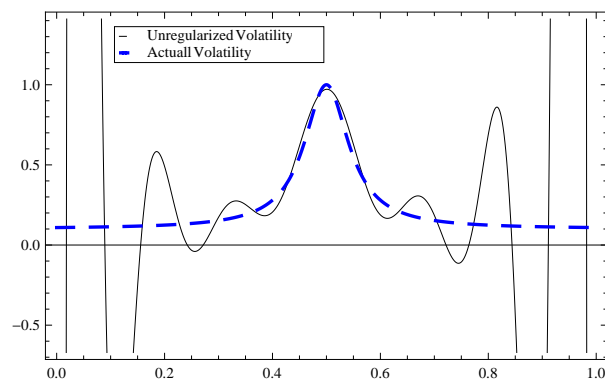
**Table 3:** The comparison of regularization parameters. Tikh-L-curve indicates Tikhonov regularization with parameter selection L-curve and Tikh-GCV indicates Tikhonov regularization with parameter selection GCV.

## 6 Conclusion

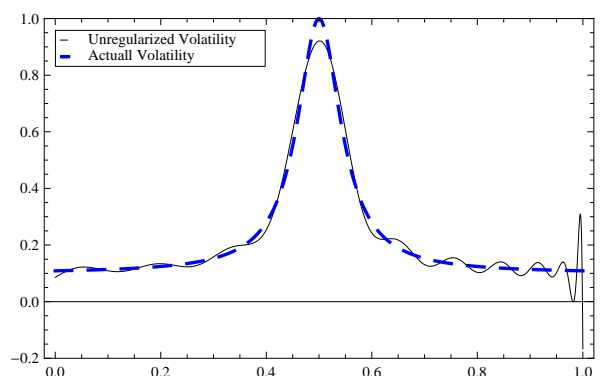
In this study, we considered the inverse problem of determining the unknown time dependent function in option pricing model. The forward operator of the inverse problem under consideration was decomposed into an



**Fig. 4:** The GCV plot of Example2 for data with noise level  $\delta = 10^{-3}$ .



**Fig. 5:** The representation of volatility function in Example2 without regularization.



**Fig. 6:** The representation of Tikhonov regularized Volatility function with GCV criterion in Example2.

inner linear convolution operator and an outer nonlinear Nemytskii operator given by a Black-Scholes function. The inversion of the outer operator led to an ill-posed problem. Using by Chebyshev collocation method, we discrete outer operator in the form an ill-posed linear system. The Tikhonov regularization method used for resolving ill-posedness of system. We checked the ability of two different methods, GCV and L-curve for determining the regularization parameter to estimate a stable solution. Meanwhile, the numerical results showed that the algorithm designed in this paper is stable and the coefficient "volatility function" was recovered very well.

## References

- [1] F. Black and M. Scholes, The pricing of options and corporate liabilities, *The Journal of Political Economy* **81**, 637-654, (1973)
- [2] Robert C. Merton, Theory of rational option pricing, *Bell Journal of Economics and Management Science* **4**, 141-183,(1973)
- [3] J. Hull, *Options, Futures and other Derivatives*, Prentice-Hall, Englewood Cliffs, NJ, (1997).
- [4] P. Wilmott, S. Howison, J. Dewenney, *The Mathematics of Financial Derivatives*, Cambridge University Press, Cambridge, UK, (1995).
- [5] K. Kwok, *Mathematical Models of Financial Derivatives* (Singapore: Springer), (1998).
- [6] T. Hein, B. Hofmann, On the nature of ill-posedness of an inverse problem arising in option pricing, *Inverse Problems* **19**, 1319-1338, (2003).
- [7] B. Hofmann, R. Krmer, On maximum entropy regularization for a specific inverse problem of option pricing, *Journal of Inverse Ill-Posed Problems* **13**, 41-63, (2005).
- [8] Cynthia Lester, Xiaoyue Luo, Ruya Huang, On local regularization for an inverse problem of option pricing, *Applied Mathematics Letters* **24**, 1481-1485, (2011).
- [9] D. Tavella, C. Randall, *Pricing Financial Instruments - The Finite Difference Method*, John Wiley and Sons, Inc., NewYork, (2000).
- [10] C. T. Kelley, *Solving Nonlinear Equations with Newton's Method*, no 1 in *Fundamentals of Algorithms*, SIAM, (2003).
- [11] C. Lanczos, *Applied analysis*, Dover publications (Reprint of 1956 Prentice-Hall ed.),New York, (1988).
- [12] Rivlin, J. Theodore, *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*, 2nd edition, New York: John Wiley and Sons, (1990).
- [13] Hansen PC. Regularization tools: A Matlab package for analysis and solution of discrete ill-posed problems. *Numer Algorithms* **6**, 1-35, (1994).
- [14] Hansen PC. Truncated SVD solutions to discrete ill-posed problems with ill-determined numerical rank. *SIAM J Sci Stat Comput* **11**, 503-18, (1990).
- [15] Hansen PC. Analysis of the discrete ill-posed problems by means of the L-curve. *SIAM Rev* **34**, 561-80, (1992).
- [16] Hansen PC, OLeary DP. The use of the L-curve in the regularization of discrete ill-posed problems. *SIAM J Sci Comput* **14**, 1487-503, (1993).
- [17] Hanke M, Hansen PC. Regularization methods for large-scale problems. *Surv Math Ind* **3**, 253-315, (1993).
- [18] Engl HW, Hanke M, Neubauer A. Regularization of inverse problems. *Mathematics and its applications*, **357**. Dordrecht: Kluwer Academic Publishers, (1996).
- [19] Hansen PC. Rank-deficient and discrete ill-posed problems. Philadelphia: SIAM, (1998).