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Analysis of Fractional Differential Equations with Antagana-Baleanu Fractional Operator

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Abstract: We provide new approximation solutions to nonlinear fractional order differential equations with Atangana-Baleanu operator using the natural variational iteration method in this paper. To confirm the suggested method's high accuracy, certain specific instances are given, and the resulting solutions are compared to the exact and analytical data. The findings show that, for lower degree of approximations, natural variational iteration method converge quickly to accurate solutions of the given problems.

Keywords: Approximate solution, fractional-order differential equations, natural transform, variational iteration method.

1 Introduction

There are numerous phenomena in physics, biology, chemistry, engineering, finance, and other applied sciences that are represented by PDEs. In recent years, there has been a special interest in fractional PDEs, especially nonlinear ones, because of their influence in many applied sciences, such as diffusion of biological populations, fluid flow, electromagnetic waves, control theory of dynamical systems, and so on [1,2].

Fractional calculus, a fast-developing branch of mathematics, is the study of the integrals and derivatives of functions of any order. It has been gaining popularity among scientists working on a range of issues due to the excellent results gained when different tools from this calculus were utilized to simulate specific real-world situations. What makes this calculus interesting to learn is the diversity of fractional operators. The range of fractional operators makes it easy to choose the one that will produce the best results. Fractional calculus has many applications in the field of electrical, electrochemistry, statistics, and probability [3].

Many sophisticated and efficient approaches have been devised and developed to discover the solutions of fractional PDEs [3,4,5,6,7,8,9,10,11,12,13,14,15]. Our aim is to present the coupling method of NT and VIM, which is called as the NVIM, and to used it to solve the fractional-order PDEs.

2 Basic Concepts

Definition 1.[13] *Let* $f \in H^1(\varepsilon_1, \varepsilon_2), \varepsilon_1 > \varepsilon_2$, *the ABC sense for* $0 < \kappa < 1$ *is*

$$ {}^{ABC}D_t^{\kappa}(f(t)) = \frac{B(\kappa)}{1-\kappa}\int_0^t f^{'}(\vartheta)E_{\kappa}\left(-\kappa\frac{(t-\vartheta)^{\kappa}}{1-\kappa}\right)d\vartheta \tag{1}$$

where $B(0) = B(1) = 1$.

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Definition 2.[14] *The NT is*

$$N(f(t)) = R(u,s) = \int_0^{\infty} e^{-st} f(ut)dt. \tag{2}$$

The LT can be obtained by the NT by [15],

$$R(s,u) = \frac{1}{u}\int_0^{\infty} e^{-st/u} f(t)dt = \frac{1}{u}F(\frac{s}{u}). \tag{3}$$

Definition 3.[15] *The inverse natural transform of a function is definned by*

$$N^{-1}(R(u,s)) = f(t) = \frac{1}{2i\pi}\int_{p-i\infty}^{p+i\infty} e^{st/u} R(u,s)dt, \quad s,u>0, \tag{4}$$

Lemma 1.*Let $N(f(t))$ is the natural transform of $f(t)$,then the natural transform of the fractional derivative with Atangana-Baleanu operator in caputo sense of $f(t)$ for $\kappa \in (0,1)$ is*

$$N({}^{ABC}D_t^{\kappa}(f(t))) = \frac{B(\kappa)}{1-\kappa+\kappa(\frac{u}{s})^{\kappa}}\left(R(u,s)-\frac{1}{s}f(0)\right). \tag{5}$$

*Proof.*From [15], Laplace transform of Atangana-Baleanu-Caputo operator of $f(t)$ is

$$L({}^{ABC}D_t^{\kappa}(f(t))) = \frac{B(\kappa)}{1-\kappa}\frac{s^{\kappa}F(s)-s^{\kappa-1}f(0)}{s^{\kappa}+\frac{\kappa}{1-\kappa}}, \tag{6}$$

after a few simple steps, the following relationship can be obtained

$$L({}^{ABC}D_t^{\kappa}(f(t))) = \frac{B(\kappa)}{1-\kappa+\kappa s^{-\kappa}}\left(F(s)-\frac{1}{s}f(0)\right), \tag{7}$$

from relation (3) , we get

$$N({}^{ABC}D_t^{\kappa}(f(t))) = \frac{B(\kappa)}{1-\kappa+\kappa(\frac{u}{s})^{\kappa}}\left(\frac{1}{u}F(\frac{s}{u})-\frac{1}{s}f(0)\right) = \frac{B(\kappa)}{1-\kappa+\kappa(\frac{u}{s})^{\kappa}}\left(R(u,s)-\frac{1}{s}f(0)\right).$$

3 Analysis of the Method

Suppose that FPDE with AB-Caputo operator

$${}^{ABC}D_t^{\kappa}v(x,t)+L(v(x,t))+M(v(x,t)) = f(x,t), \tag{8}$$

with initial condition $v(x,0) = v_0(x)$,

Applying the NT to (8):

$$\frac{B(\kappa)}{1-\kappa+\kappa(\frac{u}{s})^{\kappa}}\left(N(v(t))-\frac{1}{s}v(0)\right) = N[f(x,t)-L(v)-M(v)], \tag{9}$$

by substituting initial condition of eq.(8)

$$\bar{v} = \frac{1}{s}v_0(x) - \frac{1-\kappa+\kappa(\frac{u}{s})^{\kappa}}{B(\kappa)}N[L(v)+M(v)-f(x,t)]. \tag{10}$$

Appling VIM:

$$\bar{v}_{n+1} = \bar{v}_n + \lambda \left(\bar{v}_n - \frac{1}{s} v_0(x) + \frac{1-\kappa+\kappa(\frac{u}{s})^\kappa}{B(\kappa)} N[L(v_n) + M(v_n) - f(x,t)] \right), \tag{11}$$

where λ is the Lag. mult., since $0 < \kappa < 1$,then $\lambda = -1$, after applying the inverse of the NT to eq.(11)

$$v_{n+1} = v_0(x) - N^{-1} \left(\frac{1-\kappa+\kappa(\frac{u}{s})^\kappa}{B(\kappa)} N[L(v_n) + M(v_n) - f(x,t)] \right), \tag{12}$$

where is the initial iteration is $v_0(x,t) = v_0(x)$, consequenty, we have

$$v(x,t) = \lim_{k\to\infty} v_k(x,t).$$

4 Convergence Analysis

Now, define the operator $A[v]$ as ,

$$A[v] = -N^{-1} \left(\frac{1-\kappa+\kappa(\frac{u}{s})^\kappa}{B(\kappa)} N[L(v_n) + M(v_n) - f(x,t)] \right), \tag{13}$$

and also components $w_k, k = 0,1,2,...,$

$$\begin{aligned} &w_0 = v_0 \\ &w_1 = A[w_0] \\ &w_2 = A[w_0 + w_1] \\ &\vdots \\ &w_{k+1} = A[w_0 + w_1 + ... + w_k], \end{aligned} \tag{14}$$

as a result, we get

$$v(x,t) = \lim_{k\to\infty} v_k(x,t) = \sum_{k=0}^{\infty} w_k(x,t) \tag{15}$$

Theorem 1. *Let H is a Hilbert space , and A defined in (13) is an operator from H to H.Then the series $v = \lim_{k\to\infty} v_k = \sum_{k=0}^{\infty} w_k$ defined in (15) converges if $\exists\ 0 < \delta < 1$ s.t $\|w_{n+1}\| \le \delta \|w_n\|,\ k = 0,1,2,3,....$*

Proof. Define $\{S_n\}_{n=0}^{\infty}$ as ,

$$\begin{aligned} S_0 &= w_0 \\ S_1 &= w_0 + w_1 \\ S_2 &= w_0 + w_1 + w_2 \\ &\vdots \\ S_n &= w_0 + w_1 + ... + w_n, \end{aligned} \tag{16}$$

now, we prove that $\{S_n\}_{n=0}^{\infty}$ is a Cuchy sequance in the Hilbert space H.

$$\|S_{n+1} - S_n\| = \|\sum_{i=0}^{n+1} w_i - \sum_{i=0}^{n} w_i\| = \|w_{n+1}\| \le \delta \|w_n\| \le \delta^2 \|w_{n-1}\| \le \cdots \le \|\delta^{n+1}\|w_0\|. \tag{17}$$

For all $n,m \in N, n \geq m$,we have

$$\begin{aligned}\|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+1} - S_m)\| \\ &\leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \ldots + \|S_{m+1} - S_m)\| \\ &\leq \delta^n\|w_0\| + \delta^{n-1}\|w_0\| + \ldots + \delta^{m+1}\|w_0\| \\ &= \delta^{m+1}\|w_0\|(\delta^{n-m-1} + \delta^{n-m-2} + \ldots + 1) \\ &= \frac{1-\delta^{n-m}}{1-\delta}\delta^{m+1}\|w_0\|, \end{aligned} \tag{18}$$

since $(\delta^{n-m-1} + \delta^{n-m-2} + \ldots + 1)$, is a geometric series and $0 < \delta < 1$, then $\lim_{n,m\to\infty} \|S_n - S_m\| = 0$.

Therefore ,$\{S_n\}_{n=0}^{\infty}$ is a Cuchy sequance in the Hilbert space H and therefore produces that the series solution $v(x,t) = \sum_{k=0}^{\infty} w_k(x,t)$, defined in (15) converges.

Theorem 2.*Suppose that the series sotution $\sum_{k=0}^{\infty} w_k(x,t)$ mentioned in (15) is convergent to the solution $v(x,t)$. If $\sum_{k=0}^{\infty} w_k(x,t)$ is used as an approximation to the solution $v(x,t)$ of problem (8) then the maximum error , $E_m(x,t)$ is estimated as $E_m(x,t) \leq \frac{1}{1-\delta}\delta^{m+1}\|w_0\|$*

*Proof.*From theorem 1 , inequality (18)

$$\|S_n - S_m\| \leq \frac{1-\delta^{n-m}}{1-\delta}\delta^{m+1}\|w_0\|, \tag{19}$$

for $n \geq m$, now , as $n \to \infty$ then $S_n \to v(x,t)$ so ,

$$\|v(x,t) - \sum_{k=0}^{m} w_k\| \leq \frac{1-\delta^{n-m}}{1-\delta}\delta^{m+1}\|w_0\|. \tag{20}$$

Also, sine $0 \leq \delta \leq 1$ we have $(1-\delta^{n-m}) < 1$.
Therefor the above inquality becomes

$$\|v(x,t) - \sum_{k=0}^{m} w_k\| \leq \frac{1}{1-\delta}\delta^{m+1}\|w_0\|. \tag{21}$$

5 Application

We will solve two linear and non-linear equations and show tables of solution values and graphs to solve the two equations, we will suppose that $B(\kappa) = 1$.

*Example 1.*Suppose that the linear time-fractional Newell-Whitehead-Segel equation [7] with Atangana-Baleanu-Caputo operator

$$^{ABC}D_t^{\kappa} v(x,t) = v_{xx}(x,t) - 2v(x,t), \ \ t > 0, \ \ 0 < \kappa \leq 1, \tag{22}$$

with initial condition $v(x,0) = e^x$. Applying the fractional natural transform variational iteration method (FNVIM) to (22),we get

$$v_{n+1} = e^x - N^{-1}\left(\left[1 - \kappa + \kappa(\frac{u}{s})^{\kappa}\right] N[2v_n - v_{nxx}]\right) \tag{23}$$

Now, we find the approximate solutions as,

$$\begin{aligned} v_0 &= e^x, \\ v_1 &= e^x\left(\kappa - \kappa\frac{t^{\kappa}}{\Gamma(\kappa+1)}\right), \\ v_2 &= e^x - N^{-1}\left(\left[1 - \kappa + \kappa(\frac{u}{s})^{\kappa}\right] N\left[\kappa e^x\left(1 - \frac{t^{\kappa}}{\Gamma(\kappa+1)}\right)\right]\right) \\ &= e^x\left((1-\kappa+\kappa^2) + (\kappa - 2\kappa^2)\frac{t^{\kappa}}{\Gamma(\kappa+1)}\kappa^2\frac{t^{2\kappa}}{\Gamma(2\kappa+1)}\right), \end{aligned} \tag{24}$$

Thus, the approximate solution of (23) can be written,

$$v(x,t) = e^x \left((1-\kappa+\kappa^2) + (\kappa - 2\kappa^2)\frac{t^\kappa}{\Gamma(\kappa+1)} + \kappa^2 \frac{t^{2\kappa}}{\Gamma(2\kappa+1)} + \cdots \right), \tag{25}$$

when choosing $\kappa = 1$ in eq.(25), it becames

$$v(x,t) = e^x(1 - t + \frac{t^2}{2!} + \cdots), \tag{26}$$

ultimately, the exact solution of equation (22) , $v(x,t) = e^{x-t}$.

*Example 2.*Assuming that the non-linear time-fractional Burger equation [6] with Atangana-Baleanu operator in Caputo sense

$${}^{ABC}D_t^\kappa v(x,t) = v_{xx}(x,t) - v(x,t)v_x(x,t), \ \ t > 0, \ \ 0 < \kappa \leq 1, \tag{27}$$

with initial condition $v(x,0) = x$. Applying the fractional natural transform variational iteration method (FNVIM) to (27),we can obtain

$$v_{n+1} = e^x - N^{-1}\left(\left[1-\kappa+\kappa(\frac{u}{s})^\kappa \right] N[v_n v_{nx} - v_{nxx}] \right) \tag{28}$$

Now, we find the approximate solutions as,

$$\begin{aligned}
v_0 &= x, \\
v_1 &= x - N^{-1}\left(\left[1-\kappa+\kappa(\frac{u}{s})^\kappa \right] N[x] \right) \\
&= x\left(\kappa - \kappa \frac{t^\kappa}{\Gamma(\kappa+1)} \right), \\
v_2 &= x - N^{-1}\left(\left[1-\kappa+\kappa(\frac{u}{s})^\kappa \right] N\left[\kappa e^x \left(1 - \frac{t^\kappa}{\Gamma(\kappa+1)} \right) \right] \right) \\
&= x\left((1-\kappa^2+\kappa^3) + (2\kappa^2 - 3\kappa^3)\frac{t^\kappa}{(\Gamma(\kappa+1))^2} + 2\kappa^3 \frac{t^{2\kappa}}{\Gamma(2\kappa+1)} - \kappa^3 \frac{\Gamma(2\kappa+1)}{(\Gamma(\kappa+1))^2}\frac{t^{3\kappa}}{\Gamma(3\kappa+1)} \right),
\end{aligned} \tag{29}$$

Thus, the approximate solution of (27) can be written,

$$v(x,t) = x\left((1-\kappa^2+\kappa^3) + (2\kappa^2 - 3\kappa^3)\frac{t^\kappa}{(\Gamma(\kappa+1))^2} + 2\kappa^3 \frac{t^{2\kappa}}{\Gamma(2\kappa+1)} - \kappa^3 \frac{\Gamma(2\kappa+1)}{(\Gamma(\kappa+1))^2}\frac{t^{3\kappa}}{\Gamma(3\kappa+1)} + \cdots \right), \tag{30}$$

when choosing $\kappa = 1$ in eq.(30), it becames

$$v(x,t) = x(1 - t + t^2 - \cdots), \tag{31}$$

ultimately, the exact solution of equation (27) , $v(x,t) = \frac{x}{1-t}$.

6 Conclusion

We used FNVIM with ABFO to evaluate the fractional-order DEs in this work. The present technique is used to demonstrate the solutions to cases. The FNVIM result closely resembles the precise solution to the provided issues. The convergence of the fractional-order answers to integer-order solutions was confirmed by a graphical examination of the results. Furthermore, the proposed method is clear, simple, and low-cost to implement; it may be extended to solve additional fractional-order PDEs.

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