Stochastic Differential Equations with Transformed Anticipating Conditions

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Abstract: Stochastic differential equations can be specified with anticipatory initial value constraints (IVC) which is useful in many applications where future filtration is known. However, such specification leads to different results than those used in the usual Itô's calculus, and choice of transformations, at IVC, affects the orientation of equivalent isometry moments and related quantities. To solve this problem, the paper analyzes linear stochastic differential equations with anticipatory initial value constraints specified by an exponential transformation. The conditions for general solutions when such function is defined are derived from Itô's lemma, Taylor, and Fourier Series. Exact solutions are found using the derived conditions as well as those found by other research works. The article also derives numerical scheme for stochastic differential equations with anticipating initial conditions from first principles, Euler-Muruyama and Monte-Carlo methods. The results show that, the Euler-Muruyama method without a Monte-Carlo extension gives reliable numerical solutions for stochastic differential equations with anticipatory initial conditions. The Monte-Carlo extension introduces a slight smoothing effect on the estimated numerical solution.

Keywords: Stochastic Differential Equations, Exponential Anticipating Initial Condition, Euler-Muruyama Method, Monte-Carlo Extension

1 Introduction

Anticipating stochastic models are useful in modeling random phenomena in which the future filtration of noise driving a solution of diffusion time evolution equations is assumed also to affect initial value constraints. In financial mathematics, this could be useful in modeling inside trading. Consider a problem of the form

$$\begin{cases} dY_t = \gamma(t)Y_t dW_t + \alpha(t)Y_t dt, \ t \in [a,b]. \\ Y_a = f(y;a,b). \end{cases} \tag{1}$$

The problem is called diffusion equation with anticipatory initial conditions (SDew) [1] if $f(y;a,b) = f(W(b) - W(a))$. The function $f(y;a,b)$ is the transformation and argument $W(b) - W(a)$ are total change in Gauss type Brownian noise driving Y_t. If $Y_a = y$, then, using Ito's lemma it can be proved [2] that the unique solution of the problem (1) is given by

$$Y_t = y\exp\left\{ \int_a^t \left(\alpha(s) - \frac{\gamma(s)^2}{2} \right) ds + \int_a^t \gamma(s)ds \right\}. \tag{2}$$

Many authors, for example [3], have examined analytical characteristics of stochastic differential with anticipatory initial value constraints through the use of Itô's and Mallavin calculus with varying levels of success. The most important of the successes is the realization that solution process of SDew is different from the ordinary one. That is the most important question which authors have tried to answer is this: given that an anticipatory is defined, what are the conditions for existence of unique solution, finiteness of second moment of its solutions under equivalent Ito's isometry, well-posedness

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of drift-free measure, conformation of the SDew with the Fokker-Planck equation (FPE); and right form of option pricing formula with the defining initial conditions?

At the moment significant proportion of the question is not yet fully answered, and few authors have focused in this area of research as most results are theoretical and described in topological language which is generally hard to decipher [4]. [5] consolidated the study of SDew in which focus was put on metric specified initial conditions based on all available results, related to SDew's, in literature at that time. The author tried to deduce physical probability spaces for underlying solutions of diffusion equality satisfying the Fokker-Planck equation (FPE). This approach as seen in many recent studies, for instance [6] and [7], works well only under very simplifying assumptions which are capable of misrepresenting reality. Of course another approach would be to maintain the SDew and use numerical Euler method followed by optimal methods that can capture probability density as numerical one as seen in [8].

First attempt to apply this technique but in SDew can be seen in [9] and [10]. Pure forms of the numerical method were used and conditions that account for non-linear convergence to average through application of FPE or complex Fourier series stability transformations were avoided. This study gap was filled by [11], [12] and [13] only in analytical paradigm but not numerical one.

A deeper look into SDew in which coefficients satisfy both Lipchitz continuity and linear growth property, there by implying uniqueness in existential time evolution satisfying the SDew, under a wide range of transformations at initial value constraints was made by [14]. The attributes of linear and logarithmic transformations derived by these authors have made their ways to applications in dynamic programming techniques with random spikes as in [15] and [16].

The danger using such functions in initial conditions, given that the random specified time evolution is an SDew, can be seen in [17]. These authors show that the probability of getting negative variance and second moment under these transformation does not vanish, especially when the author decides to shift the problem to drift-free framework for a pricing application. One way of dealing with this problem is to use exponential transformations [18]. [19] and [20] use these concept to solve multivariate security valuation problems, in which the volatility is it self a diffusion time evolution, but in non-anticipating conditions.

The motivation for this paper is as follows: what can general solution of a linear SDew look like given that the transformation in initial value conditions is maintained as exponential under different assumptions of nature of the solution? Another follow up question is as follows: how does this solution compare with other solutions for other types of transformations? Related to this question is as follows: How can one solve these problems numerically using the different forms of Euler method?

To solve the problem, this paper derives the general and unique solution of the stochastic differential equation with $f(y;a,b) = f\left(\exp\left(W(b) - W(a)\right)\right)$. Through the use of Ito's Calculus adapted for anticipatory initial constraints, we are able derive conditions which such limitations satisfied under different forms of general solutions. We also supply numerical implementation for the same when anticipatory initial conditions are transformed by $f(y;a,b) = f\left(\exp\left(W(b) - W(a)\right)\right)$ from first principles.

The rest of the paper is as follows: in section 2, we introduce the basic notations, definitions and theorems. Section 3 gives the motivational example. In section 4, the generalization of the model is given. Numerical simulations are carried out in section 5. Section 6 concludes the paper.

2 Basic Notations Definitions and Theorems

The notation $C^{(k)}$ stands for the set of all functions whose $k-$th derivatives are continuous. For example, $C^{(\infty)}$ stands for the set of functions that are infinitely differentiable. The symbol $\mathscr{A}^{(o)}$ stands for the set of analytic functions. For example, $f \in \mathscr{A}^{(o)} \iff f(x) \in C^{(\infty)}$ and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)x^n}{n!}, \; a \in \mathbb{R}. \tag{3}$$

We define a Fourier Transformation of $f(x)$ and its inverse as

$$f^*(c) = \int_{\forall x \in \mathbb{R}} f(x)e^{2\pi ixc}dc, \tag{4}$$

and

$$f(x) = \int_{\forall x \in \mathbb{R}} f^*(c) e^{2\pi i c x} dx. \tag{5}$$

We assume that the standard Brownian motion $\{W_t\}$ is a local Martingale with respect to sequence $(\mathscr{F})_{t \in \mathbb{R}^+ \cup \{0\}}$ in that for each s, $t \in \mathbb{R}^+ \cup \{0\}$, $s < t \Rightarrow W_t - W_s$ if independent of $\mathscr{F}_s$, where $\mathscr{F}_t$ is a filtration of the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ that has probability measure $\mathbb{P}$. If X_t is adapted with respect to filtration $(\mathscr{F})_{t \in \mathbb{R}^+ \cup \{0\}}$, the statement $X_t \in \mathscr{L}$ implies that X_t is integrable.

[21] give the solution of the linear stochastic differential equation with anticipating condition given that prior solution can be factored across the anticipating initial conditions, time, and space. Their results can be summarized in the following theorem. We supply the transformation in the theorem.

Theorem 1.[1, 3]. *Suppose that*

$\star Z(t, x, y) = Z_1(t) Z_2(x) Z_3(y)$,
$\star Z_1 \in C^{(1)}(\mathbb{R})$, $Z_2 \in C^{(2)}(\mathbb{R})$, and $Z_3 \in C^{(\infty)}(\mathbb{R})$ *and*
$\star dX_t = \gamma_t dW_t + \alpha_t dt$.

Then,

$$dZ\left(t, X_t, e^{W_b - W_a}\right) = \frac{\partial Z_t\left(t, X_t, e^{W_b - W_a}\right)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 Z_t\left(t, X_t, e^{W_b - W_a}\right)}{\partial x^2} (dX_t)^2$$
$$+ \frac{\partial^2 Z\left(t, X_t, e^{W_b - W_a}\right)}{\partial x \partial y} dX_t dW_t + \frac{\partial Z_t\left(t, X_t, e^{W_b - W_a}\right)}{\partial t} dt, \ y = W_b. \tag{6}$$

3 Motivation Example

Using the definition in Equation (1) and the conditions in the Theorem 1, we put

$$Z_t = f(\exp(W(b) - W(a))) \exp\left\{ \int_a^t \left(\alpha(s) - \frac{\gamma(s)^2}{2} \right) ds + \int_a^t \gamma(s) ds \right\}. \tag{7}$$

We consider a case where $f(y; a, b) = y$, $a = 0$, $b = 1$, $\gamma(t) = 1$, and $\alpha = 0$. Then the prior solution information is given by $Z_t = e^{W(1)} \exp\left(W_t - \frac{t}{2}\right)$. Applying Theorem 1 on $Z_t = e^{W(1)} \exp\left(W_t - \frac{t}{2}\right)$, we obtain the following stochastic differential equation

$$dZ_t = Z_t dW_t + Z_t dt. \tag{8}$$

This means that $Z_t = e^{W(1)} \exp\left(W_t - \frac{t}{2}\right)$ cannot be the solution of the differential equation in the case so chosen, which is

$$\begin{cases} dY_t = Y_t dW_t, \ t \in [0, 1]. \\ Y_0 = e^{W(1)}. \end{cases} \tag{9}$$

We consider the $Y_t = e^{W(1) - c(t)} \exp\left(W_t - \frac{t}{2}\right)$ to be the solution of the problem in the Equation (9), where $c(t)$ is diffusion free. The term $Z_t dt$ is the main term that makes the stochastic differential equation in Problem (7) different from Equation (7). Using the Ito's Lemma in Theorem 1, we find

$$\frac{\partial Y_t}{\partial t} = -(c'(t) + \frac{1}{2}) \Lambda(t, x, y), \ \text{where} \ \Lambda(t, x, y) = K \exp\left(W_t - \frac{t}{2}\right), \ K = e^{W(1)}. \tag{10}$$

$$\frac{\partial^2 Y_t}{\partial x^2} = \Lambda(t, x, y) = \frac{\partial Y_t}{\partial x} = \frac{\partial^2 Y_t}{\partial x \partial y}, \ x_t = W_t. \tag{11}$$

$$dY_t = \Lambda(t, x, y) dW_t - (c'(t) - 1) \Lambda(t, x, y) dt. \tag{12}$$

For $Y_t = e^{W(1) - c(t)} \exp\left(W_t - \frac{t}{2}\right)$ to be the solution of the Problem (9), we need $c'(t) - 1 = 0 \Rightarrow c(t) = t$ in the Equation (17). Hence, the solution of the problem is

$$Y_t = \exp\left(W(1) + W_t - \frac{3t}{2}\right). \tag{13}$$

Thus, the solution form $Y_t = q(t, e^{W(1)}) \exp\left(W_t - \dfrac{t}{2}\right)$. Let $q(t, e^{W(1)}) = q(t, y)$, $y = W(1)$, then

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial t}\left(e^{y-t}\right) = -e^{y-t} = -\frac{\partial}{\partial y}\left(e^{y-t}\right) = -\frac{\partial q}{\partial y}. \tag{14}$$

$$q(0, y) = e^{y-0} = e^y. \tag{15}$$

This is different from the work of [22] that requires $q(0, y) = y$. It can agree with this work if and only if $y = e^{W(1)}$. Hence, the solution is

$$Y_t = e^{W(1)-t} \exp\left(W_t - \frac{t}{2}\right) = \exp\left(W(1) + W_t - \frac{3t}{2}\right). \tag{16}$$

4 Generalization

4.1 Under the Factorization Theorem

Using the definition in Equation (1) and the conditions in the Theorem 1, we put

$$Z_t = \exp\left\{\int_a^t \left(\alpha(s) - \frac{\gamma(s)^2}{2}\right)ds + \int_a^t \gamma(s)ds\right\}. \tag{17}$$

For the case where Theorem 1 is assumed to hold, we assume that the solution of Equation (1) under the initiatial conditions $f(y; a, b) = f(\exp(W(b) - W(a)))$ is given by

$$Y_t = [f(\exp(W(b) - W(a)))Z_t] - [c(t, \exp(W(b) - W(a))Z_t]. \tag{18}$$

Considering $dY_t = d[f(\exp(W(b) - W(a)))Z_t] - d[c(t, \exp(W(b) - W(a))Z_t]$, we assume that $c(\cdot)$ can be written as sum of product of factors, just like [23], so that

$$d[c(t, \exp(W(b) - W(a))Z_t] = d\left[\sum_{n=0}^{\infty} c_n(t)\exp((W(b) - W(a))^n)Z_t\right]$$

Using Theorem (1), we have

$$d[c(t, \exp(W(b) - W(a))Z_t] = \left[\sum_{n=0}^{\infty} c_n(t)\exp((W(b) - W(a))^n)\right]dZ_t + 0 \cdot (dZ_t)^2$$

$$+ \left[\sum_{n=0}^{\infty} c_n(t)n(W(b) - W(a))^{n-1}\exp((W(b) - W(a))^n)\right]dZdW_t$$

$$+ \left[\sum_{n=0}^{\infty} c_n'(t)\exp((W(b) - W(a))^n)Z_t\right]dt.$$

Therefore,

$$d[c(t, \exp(W(b) - W(a))Z_t] = \left[\sum_{n=0}^{\infty} c_n(t)\exp((W(b) - W(a))^n)\right]dZ_t.$$

$$+ \left[\sum_{n=0}^{\infty} c_n(t)n(W(b) - W(a))^{n-1}\exp((W(b) - W(a))^n)\right]\gamma(t)Zdt$$

$$+ \left[\sum_{n=0}^{\infty} c_n'(t)\exp((W(b) - W(a))^n)Z_t\right]dt. \tag{19}$$

Considering $d\left[f\left(\exp\left(W(b)-W(a)\right)\right)Z_t\right]$, we have

$$
\begin{aligned}
d\left[f\left(\exp\left(W(b)-W(a)\right)\right)Z_t\right] = {} & \left[f\left(\exp\left(W(b)-W(a)\right)\right)\right]dZ_t \\
& + 0.(dZ_t)^2 \\
& + \exp\left(W(b)-W(a)\right)f'\left(\exp\left(W(b)-W(a)\right)\right)\gamma(t)Zdt \\
& + 0 \cdot dt
\end{aligned}
$$

To achieve the solution of Equation (1) under the condition $f(y;a,b) = f\left(\exp\left(W(b)-W(a)\right)\right)$, we should have

$$
\begin{aligned}
0 = {} & \exp\left(W(b)-W(a)\right)f'\left(\exp\left(W(b)-W(a)\right)\right)\gamma(t)Z \\
& + \left[\sum_{n=0}^{\infty}c'_n(t)\exp\left((W(b)-W(a))^n\right)Z_t\right] \\
& + \left[\sum_{n=0}^{\infty}c_n(t)n(W(b)-W(a))^{n-1}\exp\left((W(b)-W(a))^n\right)\right]\gamma(t)Z.
\end{aligned}
\tag{20}
$$

Hence, we have the following conditions that must be satisfied by $c = c(t,\exp\left(W(b)-W(a)\right))$ and $f = f\left(\exp\left(W(b)-W(a)\right)\right)$

$$
\gamma(t)\frac{df}{dy} + \frac{\partial c}{\partial t} + \gamma(t)\frac{\partial c}{\partial y} = 0.
\tag{21}
$$

4.2 Factorization Theorem not Satisfied

If factorization theorem is not satisfied, we assume that

$$
Y_t = f(\exp\left(W(b)-W(a)-c(t,\exp\left(W(b)-W(a)\right))\right))Z_t.
\tag{22}
$$

we also assume that $c(t,\exp\left(W(b)-W(a)\right)) = c(t)$. Using Taylor Series, we have

$$
\begin{aligned}
dY_t = {} & -c'(t)\exp\left(W(b)-W(a)-c(t)\right)f'(\exp\left(W(b)-W(a)-c(t)\right))Z_t dt \\
& + \exp\left(W(b)-W(a)-c(t)\right)f'(\exp\left(W(b)-W(a)-c(t)\right))Z_t dW_t \\
& + f(\exp\left(W(b)-W(a)-c(t)\right))dZ_t \\
& + \exp\left(W(b)-W(a)-c(t)\right)f'(\exp\left(W(b)-W(a)-c(t)\right))dW_t dZ_t \\
& + \frac{1}{2}\left(\exp\left(W(b)-W(a)-c(t)\right)f'(\exp\left(W(b)-W(a)-c(t)\right))\right. \\
& + \left.\exp(2W(b)-2W(a)-2c(t))f''(\exp\left(W(b)-W(a)-c(t)\right))\right)Z_t dt.
\end{aligned}
\tag{23}
$$

For Y_t to be a solution, we must have

$$
c'(t) = \gamma(t) + \frac{1}{2}\left(1 + e^{W(b)-W(a)-c(t)}\frac{f''(e^{W(b)-W(a)-c(t)})}{f'(e^{W(b)-W(a)-c(t)})}\right).
\tag{24}
$$

5 Numerical Simulations

Numerical simulations are performed for Wiener process, Euler-Muruyama and Monte-Carlo methods.

5.1 Numerical Simulation of Wiener Process

We define the Brownian motion as the stochastic process $(W_t)_{t\in\mathbb{R}\cup\{0\}}$ such that $W(0) = 0$; $W_t \sim N(0,t)$; and $0 < t_1 < t_2 < \cdots \Rightarrow W(t_1), W(t_2) - W(t_1), \cdots$ are independent [24]. Under these axioms it can be proved that $\mathbb{V}ar\left(W_t - W_s\right) = |t-s| \Rightarrow W(t) - W(s) \sim \sqrt{|t-s|}N(0,1)$ [25]. Using this definition authors such as [26] and [20] adopt the following algorithm for simulating Brownian Motion:

$$
W_0 = 0, \quad W_{j+1} = W_j + \Delta W_j, j \geq 0, \quad \Delta W_j \sim \sqrt{\Delta t_j}N(0,1).
\tag{25}
$$

Figure 1 shows the simulated Wiener process.
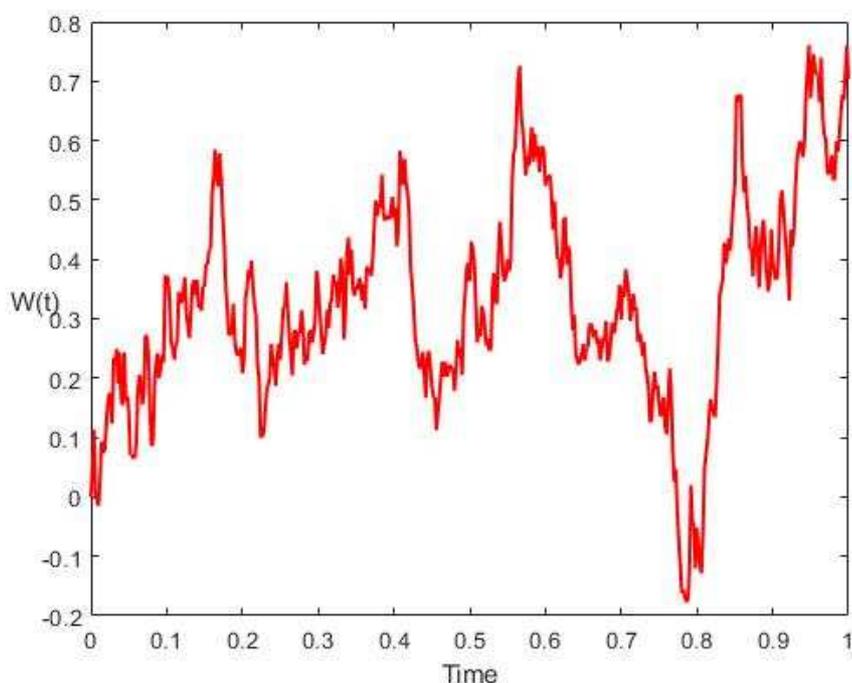
Fig. 1: Simulated Wiener Process

5.2 Euler-Muruyama Numerical Methods

Euler-Muruyama Method (EMM) is usually used to simulate give approximate solution of stochastic differential equation. EMM assumes that X_t is an Ito's Process and hence, follows the following SDE [27]

$$dX_t = f(X_t)dW_t + G(X_t)dt, \ X(0) = x, \ 0 \le t \le T. \tag{26}$$

Clearly, Equation (26) shows that anticipating initial conditions are not satisfied. This paper would like to modify EMM so that they can be used for transformed anticipating conditions. Using this differential equation the algorithm for EMM is usually given by the following difference equation

$$X_0 = x, \ X_{j+1} = X_j + G(X_j)\Delta t + F(X_j)\Delta W_j, \ j \ge 0. \tag{27}$$

The Algorithm (27) is not very accurate in approximating solution to SDE's. [28] improves this algorithm through inspiration from Monte-Carlo simulation and definition of Brownian Integration by choosing $\Delta W_j = \sum\limits_{k=j\tau-\tau+1}^{j\tau} dW_k$. The graph in Figure 2 shows the difference between direct use of EMM and EMM-Monte Carlo (EMMC) for solving the Geometric Brownian Motion used in Black-Scholes Model for option pricing. It can be seen that there is no significant difference between the two methods. The Geometric Brownian motion is given by the SDE $dX_t = rX_tdt + \sigma X_tdW_t, \ X(0) = x, \ 0 \le t \le T$. Using classical Ito's lemma or the Equation (1), the exact solution is given by

$$X_t = x\exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}N(0,1)\right\}. \tag{28}$$

We pick an example, where $x = 0.8$, $r = 20\%$, $\sigma = 35\%$, and $T = 1$. We compare EMM and EMMC approximation with the exact solution derived from Equation (28). In the process of deriving the major result in this paper can inform the numerical methods for estimating solutions of the SDEw's. To apply the Euler-Muruyama numerical method to the
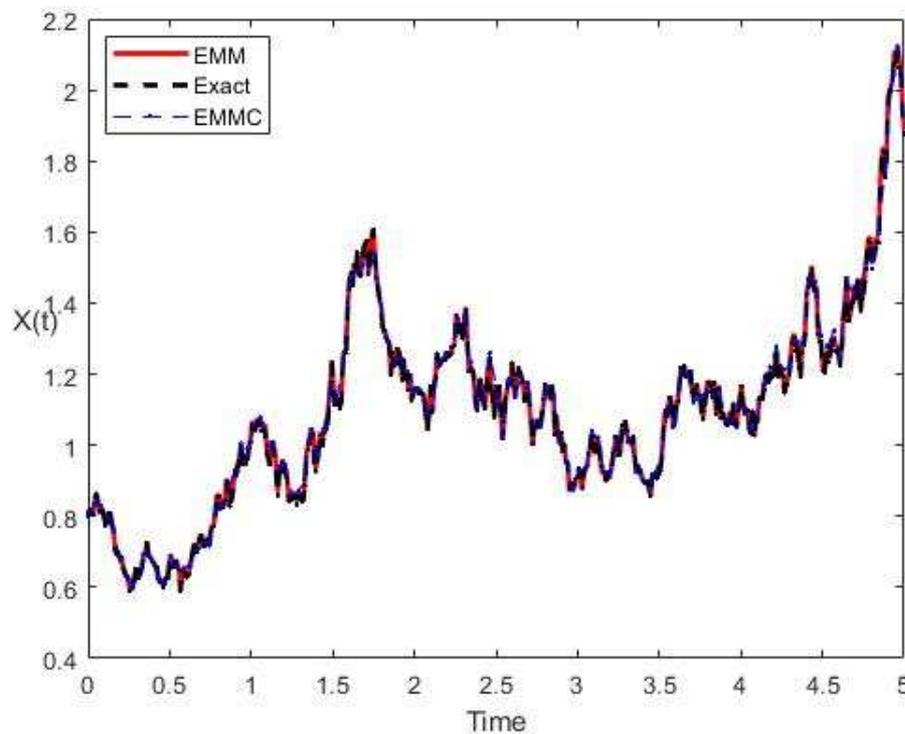
Fig. 2: Numerical Approximation for Solution of Geometric BM

stochastic differential equations with anticipating initial conditions we construct the following scheme.

$$Z_0 = 1. \tag{29}$$

$$Z_{j+1} = Z_j + G(Z_j)\Delta_j + F(Z_j)\Delta_j + F(Z_j)\Delta W_j. \tag{30}$$

$$F(Z_j) = \alpha(t_j)X_j. \tag{31}$$

$$G(Z_j) = \gamma(t_j)X_j. \tag{32}$$

$$K(y_j, t_j) = \sum_{i=1}^{N} \alpha(t_j)f'(y, t_j)\Delta t_j. \tag{33}$$

$$H_j = f(W_N - W_0) - K(W_N - W_0). \tag{34}$$

$$X_{i+1} = Z_{j+1}H_j, \ X_0 = f(W_N - W_0), \ j = 0, 1, \cdots, N-1. \tag{35}$$

We use this method to solve three equations both with exponential initial conditions and those with linear conditions. We also compare the EMM and EMMC.

5.2.1 SDEw without a Drift Coefficient

The analysis in previous sections has showed that the solution of the system

$$\begin{cases} dY_t = Y_t dW_t, \ t \in [0,1]. \\ Y_0 = e^{W(1)}. \end{cases} \tag{36}$$

is given by $Y_t = e^{W(1)-t} \exp\left(W_t - \frac{t}{2}\right)$. We solve this numerically a MATLAB2021a code. A comparison of results in Figures 2 and 3 show that EMM and EMMC do not produce different results when estimating a stochastic differential
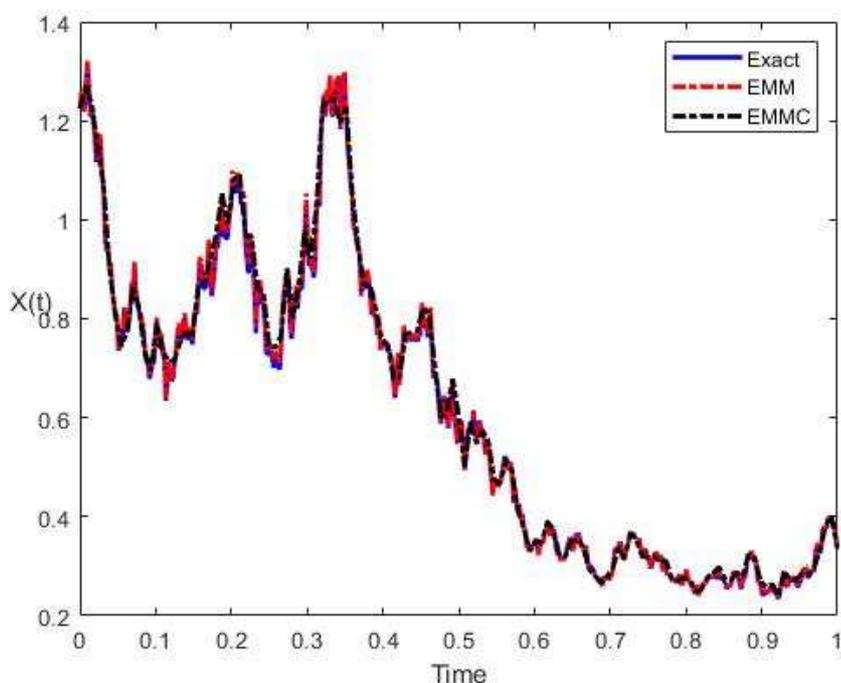
Fig. 3: SDEw without Drift Coefficient

equation without anticipating conditions. Similarly, Figure 3 shows that EMMC and EMM give results that have comparable accuracy. In fact, EMM performs better in this particular SDEw.

Theorem 2. *Consider the differential equation with anticipating initial conditions of the form below*

$$dX_t = A(t)X_t dW_t + B(t)X_t dt, \ X_a = f(W_b - W_a), \ a \leq t \leq b. \tag{37}$$

The solution of this differential equation is given by

$$X_t = [f(W_b - W_a) - K(t, W_b - W_a)]Z_t, \tag{38}$$

where $K(t,y) = \int_a^t A(s)f'(y,s)ds$, $f'(y,s) = f'\left(y - \int_s^t A(u)du\right)$.

5.2.2 SDEw with Linear Initial Condition

Consider a situation where $A(t) = 1 = B(t)$, $0 \leq t \leq 1$, and $f(x) = x = e^{\ln x}$. Then the problem in Theorem 2 can be expressed as $dX_t = X_t dW_t + X_t dt$, $X_0 = W(1)$, $0 \leq t \leq 1$. Similarly $f'(y,s) = 1$, $f(W(1) - W(0)) = W(1)$, $K(t,y) = \int_0^t 1 \cdot 1 ds = t$. Using the definition of Z_t, we have

$$Z_t = \exp\left\{\int_0^t A(s)dW_s + \int_0^t \left(B(s) - \frac{1}{2}A^2(s)\right)ds\right\} = \exp\left\{\int_0^t 1 dW_s + \int_0^t \left(1 - \frac{1}{2} \cdot 1^2\right)ds\right\}$$

$$= \exp\left(W_t + \frac{1}{2}t\right). \tag{39}$$

Hence, the solution can be expressed as follows:

$$X_t = (W(1) - t)Z_t = (W(1) - t)\exp\left(W_t + \frac{1}{2}t\right). \tag{40}$$
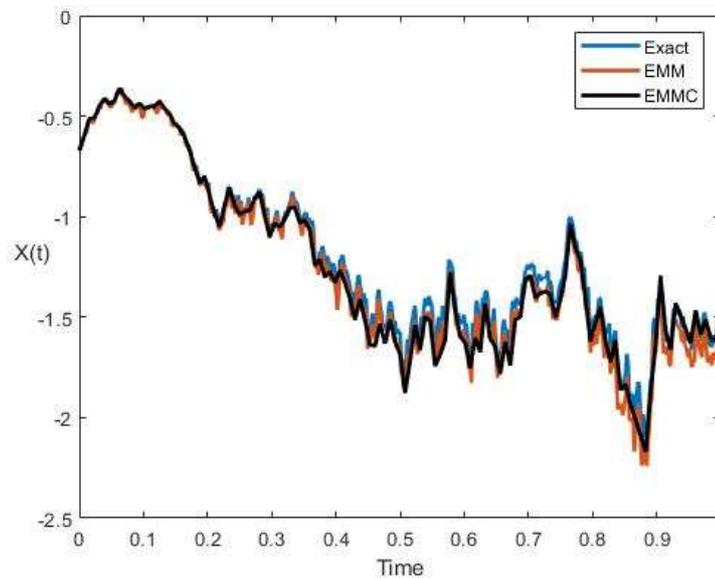
Fig. 4: SDEw with Linear Initial Condition

Clearly, the results shown in Figure 4 also shows that both methods accurately estimate an SDEw with Linear Initial Condition. All the estimated solutions are very close to the exact solution of the SDEw. The results from EMM oscillates in much similar manner like the exact solution. This may suggest an overestimation phenomenon. The results from EMMC oscillate but not as much as those of EMM. They also seem to map between some peaks. This can suggest that the ordinary numerical methods of solving differential equations are sufficient for SDEw's. It can also suggest that more versatile numerical methods like the Stochastic Runge-Kutta, and Milstein Methods can as well give more promising results. The Runge-Kutta method can lead to loss of information of the estimated solutions if Monte-Carlo extension is imputed. However, the first order Stochastic Runge-Kutta Method for linear SDEw's simply reduce to Milstein Methods. Hence, the techniques in this article are sufficient for most SDEw's.

5.2.3 Exponential Anticipating Non-Martingale Example

Consider a situation where $A(t) = 1 = B(t)$, $0 \leq t \leq 1$, and $f(x) = e^x = f'(x)$. Then the problem in the theorem can be expressed as $dX_t = X_t dW_t + X_t dt, X_0 = e^{W(1)}$, $0 \leq t \leq 1$. It is clear that $y - \int_s^t 1 du = y - t + s$. From this, we can find that $f'(y - t + s) = e^{y-t+s}$. Hence, we realize that $K(y,t) = \int_0^t 1 e^{y-t+s} ds = e^y - e^{y-t}$ and $H(W_1 - W_0) = e^{W_1-W_0} - (e^{W_1-W_0} - e^{W_1-W_0-t}) = e^{W_1-t}$. Hence, the exact solution is given by:

$$X_t = e^{W_1-t} e^{W_1+\frac{t}{2}}. \tag{41}$$

If we use EMM and EMMC, then the difference equations derived above can be used. This can be implemented by a MATLAB2021a code. Figure 5 clearly shows that both EMM and EMMC give a good estimation of solution of SDEw's. The EMMC is still more versatile despite the fact that it exhibits a more smoothing effect.

6 Conclusion

The solution of a stochastic differential equation with anticipating conditions can either be a product of functions with time and initial conditions arguments and it may also fail to be factored. In both cases there exists an ordinary or a partial non-stochastic differential equation that the initial conditions satisfy. In addition, the methods used to derive the conditions are a viable way of constructing a numerical scheme that uses Euler-Muruyama methods. Inclusion of
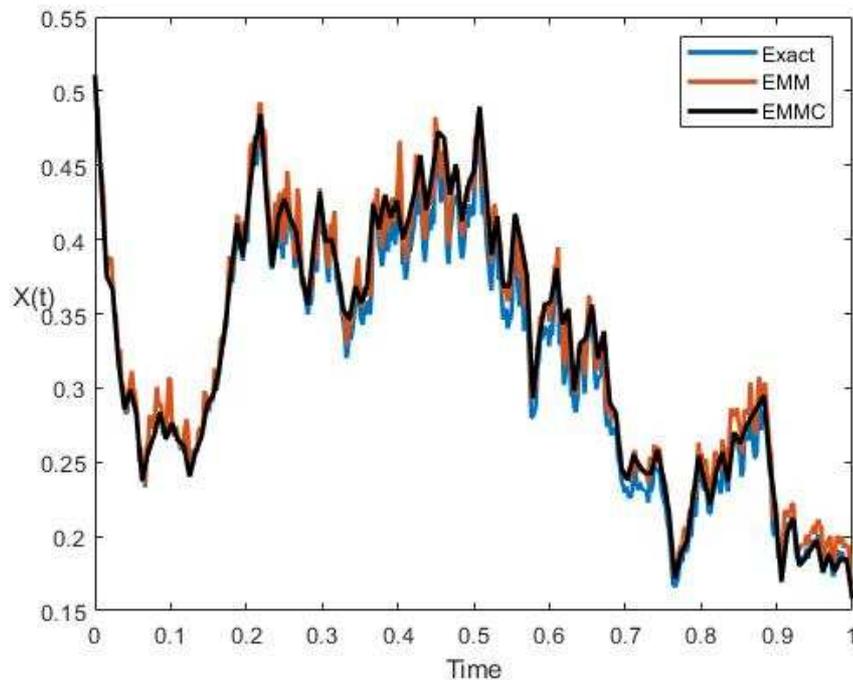
Fig. 5: SDEw with Exponential IC

Monte-Carlo extension in the Euler-Muruyama method does not lower versatility of the Monte-Carlo techniques but has a slight smoothing effect on numerical solution. This makes Euler-Muruyama method produce reliable approximate numerical solutions of stochastic differential equation with anticipating conditions.

Conflict of Interest
Authors declare no conflict of interest as regards to publication of this paper.

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