Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160614

Some Different Methods via the Solution of Volterra Integral Equation

M. A. Elsayed

Department of Basic Science High Institute for Engineering Elshorouk Academy, Cairo, Egypt

Received: 12 Sep. 2022, Revised: 20 Oct. 2022, Accepted: 27 Oct. 2022
Published online: 1 Nov. 2022

Abstract: In this paper, we establish, in a general case, the Volterra integral equation (**VIE**)from the initial value problems(**IVPs**). Also, some analytical and numerical methods are used to obtain the solution of **VIE** with a continuous kernel. In the numerical applications, the researcher based the Runge-Kutta and Trapezoid rules on the Simpson rule. This reference gives a fast convergence in the solution, a convergent error, and less than the previous traditional methods. Many numerical examples using Maple 18 are considered, and the estimated error, in each case, is computed.

Keywords: Initial (boundary) value problems, Volterra (Fredholm) integral equation, the iteration method, numerical results

1 Introduction

Many problems in mathematical physics [1], theory of elasticity [2,3,4], hydrodynamics [5], quantum mechanics [6,7], and contact problems in the theory of elasticity [8,9], take the form of **IVPs** or **BVPs**.

The theory of integral equations has close contact with many different areas of different sciences. These additional problems have led researchers to establish other methods for solving integral equations of various kinds.

In [10], Diego and Lima used collocation methods for a class of weakly singular integral equations. In [11], Mirzaee and Hoseini used the collocation method for solving Volterra-Fredholm integral equations (**V-FIEs)** with continuous kernels. In [12], Wang and Wang used the Taylor polynomial method for solving mixed**V-FIEs** of the second kind with continuous kernels. In [13], Paripour and Kamyar used new bases function to obtain the solution of nonlinear **V-FIEs** numerically with continuous kernels. In [14,15], Abdou and collage discussed the numerical solution of the quadratic integral equation using Chebyshev polynomials. In [14], and they discussed the behavior of the resolution of a mixed integral equation in two-dimensional problems in [15]. In [16], Ata and Sahin confirmed the **BVP** of the stokes flow with hermit surfaces into an integral equation; then, they used the iteration method to solve the integral equation. In [17], Kuzmina and Marchevsky used the vertex method to solve the investigated integral equation of the airfoil surface line discretization of curvilinear panels. In [18], Lienert and Tumulkastudied **VIE**from relativistic quantum physics and discussed its solution numerically. In [19], Matoog established an integral equation with a generalized potential kernel from an axisymmetric contact problem and discussed its solution using the orthogonal polynomials method. In addition, Matoog [20] addressed the resolution of the integral nuclear equation in quantum physics problems. In [21], Alharbi and Abdou established **the BVP's FIE**of the second kind and discussed its solution numerically. In [22], Nemati et al. used the orthogonal polynomial method in the Legendre form to discuss the numerical solution of a class of two-dimensional nonlinear **VIEs**. In [23], Baksheesh used the Galerkin approximation method for solving **VIEs** of the first kind with a convolution kernel. In [24], Abdou and Alharbi used the spectral relationships methods to discuss the solution of **FIE** with a singular kernel. In [25], Brezinski andZalglia used extrapolation methods to obtain the numerical solution of nonlinear **FIEs** with the continuous kernel. In [26], Hafez and Youssri used spectral relationships in the form of Legendre-Chebyshev to discuss the numerical solution of nonlinear **VIE** with a stable kernel. In [27], Abdou and Awad used an asymptotic method to solve **FIEs** in some domains. In [28], Basseem and Alalyani used the Toeplitz matrix

* Corresponding author e-mail: dr.mohamed.a.elsayed@gmail.com

method to solve a quadratic integral equation. In [29], Abdou et al. discussed the analytic solution of **F-VIEs** with a phase-lag term in time.

The theory of ordinary differential equations is a fruitful source of integral equations. In the quest for the representation formula for the solution of a linear differential equation in such a manner to include the boundary conditions or initial conditions explicitly, one is always led to an integral equation. Once the **BVPs** or the **IVPs** have been formulated in terms of integral equations, it becomes possible to solve this problem quickly.

In the remainder of this paper,we establish the **VIE**of the second kind from the **IVP.** Section three discusses various methods to analyze the **VIEs** with the continuous kernel. Section four uses numerical methods to solve the **VIE** with the continuous kernel, and the error in different starting algorithms is computed.

2 Volterra Integral Equation and Initial Value Problem

There is a fundamental relationship between the **IVPs** and **VIE**. In general, let us consider the linear differential equation of order n.

$$\frac{d^n y}{ds^n}+A_1(s)\frac{d^{n-1}y}{ds^{n-1}}+\cdots+A_{n-1}(s)\frac{dy}{ds}+A_n(s)y \\ =F(s,y(s)) \tag{1}$$

with the initial conditions.

$$y(a)=q_0,\ y'(a)=q_1,\ \ldots,\ y^{(n-1)}(a)=q_{n-1} \tag{2}$$

where the functions $A_1, A_2, \ldots, A_n$ and F are defined and continuous in $a\le s\le b$

Introduce the unknown function

$$\frac{d^n y}{ds^n}=g(s). \tag{3}$$

Hence, we get

$$\begin{aligned}
\frac{d^{n-1}y}{ds^{n-1}} &= \int_a^s g(t)\,dt+q_{n-1}\\
\frac{d^{n-2}y}{ds^{n-2}} &= \int_a^s (s-t)\,g(t)\,dt+(s-a)\,q_{n-1}+q_{n-2}\\
\vdots \qquad & \vdots \qquad \vdots\\
\frac{dy}{ds} &= \int_a^s \frac{(s-t)^{n-2}}{\Gamma(n-1)}g(t)\,dt+\frac{(s-a)^{n-2}}{\Gamma(n-1)}q_{n-1}\\
&+\frac{(s-a)^{n-3}}{\Gamma(n-2)}q_{n-2}+\cdots+\frac{(s-a)}{\Gamma(2)}q_2+q_1
\end{aligned}$$

$$\begin{aligned}
y=&\int_a^s \frac{(s-t)^{n-1}}{\Gamma(n)}g(t)\,dt+\frac{(s-a)^{n-1}}{\Gamma(n)}q_{n-1}\\
&+\frac{(s-a)^{n-2}}{\Gamma(n-1)}q_{n-2}+\cdots+\frac{(s-a)}{\Gamma(2)}q_1+q_0.
\end{aligned} \tag{4}$$

Now, if we multiply relation (3) and (4) by $1,\ A_1(s),\ A_2(s),\ \ldots,\ A_n(s)$, and using the following connection (Kanwal[30])

$$\int_a^s\int_a^{s_n}\cdots\int_a^{s_3}\int_a^{s_2} L(s_1)\,ds_1 ds_2\cdots ds_{n-1}ds_n = \\ \frac{1}{\Gamma(n)}\int_a^s (s-t)^{n-1}L(t)\,dt$$

$\Gamma(n)$ Is the gamma function, we find that the **IVP** (1) and (2) reduce to the nonlinear **VIE** of thesecond kind.

$$\begin{aligned}
g(s)=f(s)+\int_a^s k(s,t)\,g(t)\,dt\ ,\ k(s,t)\\
=\sum_{i=1}^n A_i(s)\,\frac{(s-t)^{i-1}}{(i-1)!}
\end{aligned} \tag{5}$$

$$\begin{aligned}
f(s)=&F\left(s,\int_a^s \frac{(s-t)^{n-1}}{\Gamma(n)}g(t)\,dt+\frac{(s-a)^{n-1}}{\Gamma(n)}q_{n-1}\right.\\
&\left.+\frac{(s-a)^{n-2}}{\Gamma(n-1)}q_{n-2}+\cdots+\frac{s-a}{\Gamma(2)}q_1+q_0\right)\\
&-q_{n-1}A_1(s)-((s-a)\,q_{n-1}+q_{n-2})\,A_2(s)-\ldots\\
&-\left(\frac{(s-a)^{n-1}}{\Gamma(n)}q_{n-1}+\cdots+\frac{s-a}{\Gamma(2)}q_1+q_0\right)A_n(s)
\end{aligned} \tag{6}$$

The **VIE** can be obtained from the integro-differential equation as the following.

*Example 1.*Consider the nonlinear integro-differential equation;

$$\emptyset'(t)-\lambda\int_0^t k(t,s,\emptyset(s))\,ds=f(t)\ ,\ \emptyset(0)=h_0 \tag{7}$$

We adapt (7) to take the form

$$Z(t)-\lambda\int_0^t k\left(t,s,\left(h_0+\int_0^s Z(u)\,du\right)\right)ds \\ =f(t)\ ,\ (\phi'(t)=Z(t))\,. \tag{8}$$

Therefore, (8) is equivalent to a system of nonlinear integral equation

If in (7), we have the exact solution$F(t)=e^t$, $F(0)=1$. Hence, the free term becomes

$$f(t)=e^t-\frac{1}{4}t-\frac{1}{2}t^2e^{2t}+\frac{1}{4}te^{2t}.$$

For solving, numerically, the integral equation.

$$\emptyset'(t)-\lambda\int_0^t ts\phi^2(s)\,ds=e^t-\frac{1}{4}t-\frac{1}{2}t^2e^{2t}+\frac{1}{4}te^{2t}\ ,\ F(0)=1.$$

We have the following results.

(1.1) If $t=0.2$, $N=2$

Table 1: The relation between the exact and numerical solution at $N=2$

T	ϕ	ϕNum.	Error
0	1	1	—
0.1	1.110725724	1.105170918	$1\times(10)^{-3}$
0.2	1.232873127	1.221402758	$5\times(10)^{-2}$

(1.2) If $t=0.2,\ \ N=4$

Table 2: The relation between the exact and numerical solution at $N=4$

T	ϕ	ϕ Num.	Error
0	1	1	—
0.05	1.052626228	1.051271096	0.001355132
0.1	1.107988204	1.105170918	0.002817286
0.15	1.166174648	1.161834243	0.004340405
0.2	1.227246579	1.221402758	0.005843821

*Example 2.*Also, for the integro-differential equation of the second order;

$$\phi''(t)+b(t)\phi(t)+\lambda\int_0^t k(s,t)\phi(s)\,ds=g(t)$$

$$F(0)=a\ ,\ F'(0)=b \tag{9}$$

we can obtain a system of integral equations in the form;

$$\phi(t)+\lambda\int_0^t\int_s^t\int_s^\tau k(u,s)\phi(s)\,du\,d\tau\,ds=H(t)$$
$$H(t)=\alpha+\int_0^t\left(\beta+\int_0^\tau(g(s)+b(s)\phi(s))\,ds\right)d\tau. \tag{10}$$

Numerical results, if in example 2, we consider the exact solution $\phi(t)=e^t$, $b(t)=t$, and $k(t,s)=ts$. Hence, we have $\phi(0)=\phi'(0)=1$ and $g(t)=t+e^t+t^2e^t$.

(2.1): at $t=0$, 0.1 , 0.2 , $N=2$

Table 3: The relation between the exact and numerical solution at $N=2$

T	ϕ	ϕ Num.	Error
0	1	1	—
0.1	1.105170918	1.105170918	$1\times(10)^{-3}$
0.2	1.222553547	1.221402758	$1\times(10)^{-3}$

(2.2): at $t=0$, 0.1 , 0.2 , $N=4$

Table 4: The relation between the exact and numerical solution at $N=4$

T	ϕ	ϕ Num.	Error
0	1	1	—
0.1	1.105300703	1.105170918	$1\times(10)^{-4}$
0.2	1.222214053	1.221402758	$8\times(10)^{-4}$

(2.3): at$t=0$, 0.1 , 0.2 , $N=8$

Table 5: The relation between the exact and numerical solution at $N=8$

T	ϕ	ϕ Num.	Error
0	1	1	—
0.1	1.10526470184721	1.105170918	$9\times(10)^{-5}$
0.2	1.22185497317221	1.221402758	$4\times(10)^{-4}$

(2.4) at $t=0$, 0.1 , 0.2 , $N=16$

Table 6: The relation between the exact and numerical solution at $N=16$

T	ϕ	ϕ Num.	Error
0	1	1	—
0.1	1.105223461	1.105170918	$5\times(10)^{-5}$
0.2	1.221631916	1.221402758	$2\times(10)^{-4}$

(2.5): at $t = 0$, 0.1 , 0.2 , $N = 32$

Table 7: The relation between the exact and numerical solution at $N = 32$

T	ϕ	ϕ Num.	Error
0	1	1	—
0.1	1.105197254	1.105170918	$2\times(10)^{-5}$
0.2	1.2215099014	1.221402758	$1\times(10)^{-4}$

3 Some Analyticalmethodsfor Solving Volterra Equations

In this section, we discuss some analytic methods to solve the second kind's VIE.

3.1 Theresolving kernel method:

To use the method of resolving kernel, we assume the **VIE**

$$\phi(x) = f(x) + \lambda \int_0^x k(x,t)\,\phi(t)\,dt \tag{11}$$

where $k(x,t)$ is a continuous function for $0 \le x \le a$, $0 \le t \le x$ and $f(x)$ is continuous for $0 \le x \le a$. We shall seek the solution of (11) in the form of infinite power in seriesλ;

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \cdots + \lambda^n\phi_n(x) + \ldots. \tag{12}$$

Then, comparing coefficients of like powers of λ, and by induction, we have

$$\phi_0(x) = f(x)$$

$$\phi_n(x) = \int_0^x k_n(x,t)\,f(t)\,dt \text{ , } k_{n+1}(x,t) = \int_t^x k(x,z)\,k_n(z,t)\,dz. \tag{13}$$

The function $k_n(x,t)$ is called the iterated kernel. We can write the exact solution of the formula (13) to take the form.

$$\phi(x) = f(x) + \sum_{\upsilon=1}^{\infty}\left(\lambda^{\nu}\int_0^x k_{\nu}(x,t)\,f(t)\,dt\right), \quad \left(\phi(x) = \lim_{n\to\infty}\phi_n(x)\right). \tag{14}$$

Define the resolving kernel $R(t,s,\lambda)$ such that

$$R(x,t,\lambda) = \sum_{n=0}^{\infty}\left(\lambda^n k_{n+1}(x,t)\right) \tag{15}$$

Hence, we adapt (14) to take the form

$$\phi(x) = f(x) + \lambda\int_0^x R(x,t,\lambda)\,f(t)\,dt. \tag{16}$$

The formula (16) represents a solution of **VIE** of thesecond kind using a resolving kernel.

*Example 3.*Consider the **VIE**,

$$\phi(x) = f(x) + \lambda\int_0^x (x-t)\,\phi(t)\,dt.$$

Therefore, we have the following.

$$\begin{aligned} k_2(x,t) &= \frac{(x-t)^2}{2}, \\ k_2(x,t) &= \frac{(x-t)^4}{2.4} \\ \vdots \quad &\vdots \quad \vdots \\ k_n(x,t) &= \frac{(x-t)^{2n-2}}{2^{n-1}.(n-1)!} \end{aligned}$$

Thus by the definition of the resolving kernel, It is clear that from the first information about the mathematical shape of the kernel, it is possible to obtain its final form of it. Thus, the analytical structure of this kernel can be obtained, which is called the analytical structure of the solution.

$$\begin{aligned} R(x,t,\lambda) &= \sum_{n=0}^{\infty}\left(\lambda^n k_{n+1}(x,t)\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{\left(\lambda(x-t)^2\right)^n}{2^n n!}\right) \\ &= e^{\frac{\lambda(x-t)^2}{2}}. \end{aligned}$$

Hence, the solution of the integral equation (16) becomes;

$$\phi(t) = f(t) + \lambda\int_0^x e^{\frac{\lambda(x-t)^2}{2}} f(\tau)\,d\tau. \tag{17}$$

For any values of a continuous given function $f(t)$, the formula (17) canbe calculated.

(1) In (17), if $\lambda = 0.03$, $f(x) = x$, we have

$$\phi(x) = x + 0.03e^{0.015x} - 0.03e^{0.015x-0.5x^2}$$

(2) In (17), if $\lambda = 0.03$, $f(x) = ln(x+1)$, $x \in [0, 0.3]$, we have

Table 8: Describe the numerical solution of equation (17)

x	$\Phi(x)$
0	0
0.05	0.04887633831
0.1	0.09564940844
0.15	0.1405125405
0.2	0.1836328096
0.25	0.2251552867
0.3	0.2652064744

3.2 The successive approximation method:

In (11), we assume that $f(x)$, $k(x,t)$ are continuous in $[0,a]$, $0 \le x \le a$. Then, taking some function $\emptyset_0(x)$ continuous $[0,a]$, then putting the function $\emptyset_0(x)$ in the rightside of (11) $\emptyset(x)$to get a new $\emptyset_1(x)$ which represents the solution ofequation (11). Therefore, we can obtain a sequence of functions

$$\emptyset_n(x) = \{\emptyset_0(x)\ ,\ \emptyset_1(x), \emptyset_2(x), \ldots, \emptyset_n(x), \ldots\}$$

where;

$$\emptyset_n(x) = f(x) + \lambda \int_0^x k(x,t)\,\phi_{n-1}(t)\,dt. \tag{18}$$

The sequence $\phi_n(x)$ converges as $n \to \infty$ to the solution $\emptyset(x)$ of (11). A suitable choice of the "zero" approximation $\emptyset_0(x)$ can lead to a rapid convergence of the sequence $\emptyset_n(x)$ to the resolution of the equation (11).

*Example 4.*For the **VIE**

$$\emptyset(x) = x^2 - \int_0^x (x-t)\,\emptyset(t)\,dt\ ,\ \emptyset_0(x) = 0.$$

We follow $\emptyset_1(x) = x^2$. Then; the approximate solution takes the form

$$\emptyset_n(x) = \sum_{m=1}^{n} (-1)^{m-1} \frac{x^{2m}}{m(2m-1)!} \tag{19}$$

4 Some Numerical Methods

When closed-form solutions to many problems are generally not available, much attention has been focused on numerical methods such as the Galerkin method [31], Runge – Kutta method [32] block, block method [33], Nystrom method [34] and Toeplitz matrixes method [35]. The references [26,27,28,29,30,31,32,33,34,35,36,37,38,39,40] contain extensive literature surveys on purely numerical techniques. More information can be found in Atkinson [41], Baker[42], Delves and Mohamed [43], and Golberg [44] for numerical methods.

First, we consider the **VIE** (11);

$$\emptyset(x) = f(x) + \lambda \int_a^x k(x,t)\,\phi(t)\,dt,\ a \le x < b.$$

It has a unique solution over a finite interval $[a,b]$ where $f(x)$ is a continuous function and $k(x,y)$ satisfies the condition $|k(x,y)| < M$.

4.1 Quadrature methods

We choose a regular mesh using the quadrature rule to solve **VIE** (11) in x and y. For this, we set $x = x_i = a + ih$, $h = \frac{b-a}{N}$. Hence, (11) yields,

$$\emptyset_i = f_i + \lambda h \sum_{j=0}^{i} \omega_{ij} k_{ij} \phi_j + R\ ,\ \{R = R_{i.y}(k(x_i,y)\,\phi(y))\}\,. \tag{20}$$

Here ω_{ij} is the weight function, $R_{i.y}(k(x_i,y)\,\phi(y))$ which represents the error term in the quadrature rule. If we neglect $R_{i.y}$ and assume $\|1 - h\omega_{ij}k_{ij}\| \neq 0$ for any i, we can solve the set of (20) for $\emptyset_i$.

This procedureis numerically very straightforward. However, there remains the problem of choosing a suitable weightω_{ij}. Wenote that, for each$i$, the set $\{\omega_{ij},\ j = 0,1,\ldots,i\}$ represents the weight for $(i+1)$ the point's quadrature rule of Neuton-Cates type, equally spaced points, for the interval $[0,ih]$. For large i, there are many possible choices of rule; for small $i = 1,2,k$, the choice is somewhat limited, yet there seems little point in choosing an accurate let us start by considering the most straightforward possible rule, the repeated trapezoidal rule. The power of degree 1 for each i, then the weight ω_{ij} is given by rule for large i, if we cannot choose an equallyaccurate rule for small i, $\omega_{i0} = w_{ii} = \frac{1}{2}$, $\omega_{ij} = 1$, $j = 1,2,\ldots,\ i-1$. So, (20) reduces to

$$\phi_0 = f_0$$

$$\phi_i = f_i + \lambda h \sum_{j=0}^{i} \omega_{ij} k_{ij} \phi_j\ ,\ (\phi_0 = f_0; i = 1,2,\ldots,N)\,. \tag{21}$$

Equation (21) can be solved successively for $i = 1,2,\ldots,N$ the cost ofthe solution is then $O(h^2)$ so that the Volterra equations are more accessible.

4.2 Multistep method

Consider the integral formula;

$$\phi(x_i) = f(x_i) + \lambda \int_0^{x_i} k(x_i,y)\,\phi(y)\,dy,$$
$$i = 0,1,\ldots,N; (x,y) \in [a,b].$$

Therefore, we have;

$$\emptyset_i = f_i + \lambda \sum_{j=0}^{i} h\omega_{ij} k(x_i, y_j)\phi_j + R, \quad \left(h = \frac{b-a}{n}\right), \quad (22)$$

which represents an equal interval quadrature formula with error R. If the quadrature formula is closed $\omega_{kk} \neq 0$ and $\emptyset_0$, $\emptyset_1$, ..., $\emptyset_{n-1}$ is assumed to be known, then we have a linear equation to obtain the value $\emptyset_n$. Thiscan be solved iteratively by straight forward substitution process when

$$h\left|\omega_{nn} k(nh, nh)\phi_n\right| < 1. \quad (23)$$

The inequality (23) will be satisfied for a small value of h. When (22) is the Trapezoidal rulewith remainder as above or for more accurate computation, we naturally wish to use a higher order quadrature formula. In this case ϕ_1 , ϕ_2 , ..., ϕ_{n-1}, will be needed and aparticular starting procedure is required. If the kernel is sufficiently regular, it is possible to find a power series expansion for ϕ in the neighborhood of the origin from which the necessary starting values can be found by using starting method.

4.3 Starting method

The particular starting procedure method is required for use with quadrature method applied to the solution of (22) by the multistep process. Consider;

$$\phi_i = f_i + \lambda h \sum_{j=0}^{i} \omega_{ij} k_{ij} \phi_j + R, \; i = k, k+1, \ldots, N, \quad (24)$$

and assume the way is of order p, i.e.;

$$R_{iy}(k(x_i, y)\phi(y)) \approx Ah^{p+1}, \quad A \text{ is constant} \quad (25)$$

Assuming that we try to achieve an overall accuracy of $O(h^p)$, thenthe method in (24) should be the local accuracy $O\left(h^{p+1}\right)$.

If the kernel is sufficiently regular, it may be possible to find a Taylor series expansion for x in the neighborhood $x = a$ from which the necessary starting values may be located.

Let us carry out one stage of such a process for a linear Volterra equation using the Trapezoidal rule using a step length h, setting the lower limit $a = 0$, we have;

$$\emptyset(0) = f(0)$$

$$\emptyset(h) = f(h) + \frac{\lambda}{2} h(k(h,0)\phi(0) + k(h,h)\phi(h)) + O\left(h^2\right). \quad (26)$$

Alternatively, Rung-Kutta type rules can be used for a fixed number of initial steps of the quadrature rule, for example define;

$$Rk_0 = hk\left(x_{\frac{1}{2}}, x_0, f_0\right)$$

$$Rk_1 = hk(x_1, x_1, f_1 + Rk_0)$$

$$Rk_2 = hk\left(x_1, x_{\frac{1}{3}}, f_{\frac{1}{3}} + \frac{1}{9}Rk_0 + 2Rk_1\right)$$

Then $\phi_1 = f_1 + \frac{1}{4}(Rk_1 + 3Rk_2)$, we have

$$\phi h = \phi_i + O\left(h^4\right). \quad (27)$$

This could be to provide a start for repeated Simpson's rule.

We start with third approximations of $\phi(a+h)$, $\phi(a+2h)$, $\phi(a+3h)$;

$$\phi_{11} = f_1 + hk(h, 0, f_0)$$

$$\phi_{12} = f_1 + \frac{h}{2}(k(h,0,f_0) + k(h,h,\phi_{11}))$$

$$\phi_{13} = f_{\frac{1}{2}} + \frac{h}{4}\left(k\left(\frac{h}{2}, 0, f_0\right) + k\left(\frac{h}{2}, \frac{h}{2}, \frac{f_0}{2}, \frac{\phi_{12}}{2}\right)\right)$$

Then;

$$\phi_1 = f_1 + \frac{h}{6}\left(k(h,0,f_0) + 4k\left(h, \frac{h}{2}, \phi_{13}\right) + k(h_0, h, \phi_{12})\right). \quad (28)$$

Next, let;

$$\phi_{21} = f_2 + 2hk(2h, h, \phi_1) \quad (29)$$

Then;

$$\phi_2 = f_2 + \frac{h}{3}(k(2h,0,f_0) + 4k(2h,h,\phi_1) + k(2h,2h,\phi_{21})) \quad (30)$$

Finally with;

$$\phi_{31} = f_3 + \frac{3}{2}h(k(3h,h,\phi_2) + k(3h,2h,\phi_2)) \quad (31)$$

we obtain;

$$\phi_3 = f_3 + \frac{3h}{8}(k(3h,0,f_0) + 3k(3h,h,\phi_1) + 3k(3h,2h,\phi_2) + k(3h,3h,\phi_3)) \quad (32)$$

4.4 Repeated Simpson's rule

A convenient and straightforward continuation of Day's starting procedure. Runge-Kutta and Trapezoid rule can be based on Simpson's power in the following manner; for this only ϕ_0 and ϕ_1 is required when r is even, wecan use repeated Simpson's rule immediately to give;

$$\phi_r = f_r + \lambda \frac{h}{3} \sum_{j=0}^{r} \omega_{ij} k(rh, jh) \phi_j \ , \ r = 2 \ , \ 4 \ , \ 6 \ , \ \ldots$$

$$\phi_r = f_r + \lambda \sum_{j=0}^{r-1} \left(\frac{\omega_j k(x_r, y_i) \phi_j}{1 - \lambda \omega_j k(x_r, y_r)} \right)$$

$$\omega_0 = \omega_r = \frac{h}{3} \ , \ \omega_j = \left(3 - (-1)^j\right) \frac{h}{3} \ , \ 1 \leq j \leq r-1. \quad (33)$$

However, when r is odd, a different strategy is required and to maintain the local truncation error of $O\left(h^5\right)$ Simpson's three-eighth rule is used at the upper end to give;

$$\emptyset_r = f_r + \frac{\lambda}{3} h \sum_{j=0}^{r-3} \omega_{r-3} k(rh, jh) \phi_j + \frac{3\lambda}{8} h (k(rh, rh) \phi_r + A),$$
$$r = 3, 5, 7, \ldots$$
$$A = 3k(rh, (r-1)h) \phi_{r-1} + 3k(rh, (r-1)h) \phi_{r-2} + k(rh, (r-3)h) \phi_{r-3}. \quad (34)$$

Therefore, we have;

$$\phi_r = B \left(f_r + \frac{\lambda h}{3} \sum_{j=0}^{r-3} \omega_{r-3} k(rh, jh) \phi_j + \frac{3}{8} \lambda h A \right), \quad (35)$$

where; $\omega_{p0} = \omega_{pp} = 1 \ , \ \omega_{pj} = 3 - (-1)^j \ , \ 1 \leq j \leq p-1$ and

$$B^{-1} = \left(1 - \frac{3\lambda h}{8} k(rh, rh, \phi_r) \right)$$

This means that ϕ_1 it can be calculated by one of three starting algorithms, if $r \geq 2; r$ it is even, we use Simpson's one-third formula, and if $r \geq 2; r$ it is odd, we use Simpson's three eight formula. This method has the advantage that, given a suitable starting value $\emptyset$, all approximatesolution values may be calculated with the same accuracy order.

4.5 The error in starting algorithm

Here, some examples will be solved using Simpson's method $N = 32$ to study the effect of starting algorithm on the solution.

Example 5.

$$\emptyset(x) = 2x + 3 - \int_0^x (3 + 2(x-y)\emptyset(y))\, dy \ , \ \left(\emptyset(x) = e^{-x}\left(4e^{-x} - 1\right)\right)$$

Example 6.

$$\emptyset(x) = x - 1 + e^{-2x}\left(1 + x^2\right) \int_0^x \left(x^2 e^{-xy} \emptyset(y)\right) dy \ , \ (\emptyset(x) = x)/$$

Example 7.

$$\emptyset(x) = 1 + x - \int_0^x (\emptyset(y))\, dy \ , \ (\emptyset(x) = 1)$$

Example 8.

$$\emptyset(x) = x + 1 - \cos x - \int_0^x (\cos(x-y)\, \emptyset(y))\, dy \ , \ (\emptyset(x) = x)$$

Example 9.

$$\emptyset(x) = \sin x + \int_0^x (\sin(x-y)\, \emptyset(y))\, dy \ , \ \left(\emptyset(x) = \frac{1}{2}\sin x + \sinh x\right)$$

Example 10.

$$\emptyset(x) = x + \int_0^x (\sin(x-y)\, \emptyset(y))\, dy \ , \ \left(\emptyset(x) = x + \frac{x^3}{3}\right)$$

Example 11.

$$\emptyset(x) = x + \int_0^x ((x-y)\emptyset(y))\, dy \ , \ (\emptyset(x) = \sin x\ )$$

Table 9: Max errors in different starting Algorithm

Runge - Kutta	Day's Algorithm	Trapezoid
8.75×10^{-2}	8.75×10^{-2}	8.75×10^{-2}
5.13×10^{-4}	5.13×10^{-4}	5.13×10^{-4}
9.02×10^{-2}	9.02×10^{-2}	9.02×10^{-2}
9.97×10^{-4}	9.97×10^{-4}	9.97×10^{-4}
3.60×10^{-9}	3.61×10^{-9}	5.09×10^{-6}
8.11×10^{-10}	8.11×10^{-10}	5.09×10^{-6}
1.62×10^{-8}	1.62×10^{-8}	5.09×10^{-6}

We note that the first four examples have the same error value with different starting algorithms. While for the last three examples, a difference is found between the starting algorithms. Rung-Kutta's starting algorithm and Day's algorithm have the same error, which is smaller than the error in Trapezoid's algorithm. The difference depends on the shape of the approximated kernel function.

5 Conclusions

From the above work, we can deduce the following:

– The initial value problem in ordinary differential equations leads to the second kind Volterra integral equation

- –The well-known analytic methods for solving **VIE** are resolving kernel, successive approximation method, and Laplace transformation method.
- –The resolving kernel method is based on obtaining the nth approximation of the shape of the kernel. Then write the integral equation with the general structure of the kernel (structure resolving kernel). And then the solution can be found.
- –The successive approximation method assumes that the solution function is a sequence of consecutive solutions. The weak point of this method is that the solution is chosen when the zero approximation is zero, so an approximate solution can be obtained.
- –When the researcher fails to find an analytical solution, he resorts to finding the answer by approximate methods. Among the most famous of these methods for the continuous kernel areQuadrature Method,Multistep method,Starting method,Simpson's rule, Collection method,Galerkin Method, Runge – Kutta method,and block by the block method.

If the integral equation has a discontinuous kernel, we use the following numerical methods: the Nystrom method and the Toeplitz matrixes method.

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Mohamed Elsayed received the PhD degree in mathematics from faculty of science, Alexandria University (2009). The research interest field is integral equation and contact problem. He has published researches in some international journals of mathematics.