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A Computational Numerical Study of Burger Equation with Fractal Fractional Derivative

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Abstract: We can observe the idea of fractal medium in a number of real-world problems. In this work, we demonstrate that the idea of the fractal derivative describes the fluid's movement inside these media in addition to serving as a representation of the fractal sharps. We develop the solution of the viscous Burger equation using various fractal-fractional derivative kernels in this study. We use Newton's Polynomial approach to solve the fractional Burgers equation in order to solve the numerical procedure. We give simulations for various values of the fractal dimensions to demonstrate the applicability of the current approach in fractal media.

Keywords: Viscous Burger equation, Newton's polynomial, Adams Bash-forth method, fractal-fractional integral operator.

1 Introduction

A travelling wave with the front sharpening is provided by the solution to a non-linear partial differential equation (PDE), known as a Viscous Burgers equation. This formula represents the diffusive waves equation in mathematics, and it is mostly used in fluid dynamics. Numerous physical problems exist, such as one-dimensional sound waves in a viscous media, waves in a fluid filled with viscous elastic tubes, shock waves in a viscous medium, and magnetohydrodynamic waves in a medium with limited electric conductivity, turbulence, etc. All of these examples can be used to illustrate this equation [1].
Burgers introduced the viscous Burger partial differential equation for a field $q(x,t)$ [2].

$$\frac{\partial q}{\partial t}+q\frac{\partial q}{\partial x}=\kappa\frac{\partial^2 q}{\partial x^2} \tag{1}$$

where κ represents kinematic viscosity and q represents the vector of preserved quantity, such as momentum, energy, or mass. This is the well-known Naiver Stokes equation, which does not use a stress gradient and applies to compressible waft. Burger's equations are extended through several studies in novel ways to provide a more precise explanation for non-linear occurrences. The utilisation of novel methodologies was acknowledged in the literature for the numerical approximations of the Viscous Burgers equation [3,4,5,6,7]. Additionally, several authors are quite interested in this fractional order Burgers equation [8,9,10,11,12,13].

The numerical method for solving the fractal fractional Burgers equation is presented in this study. In the Caputo meaning, the derivative in the temporal direction is referred to as an ABC derivative, or Atangana–Baleanu fractional derivative. We apply the Admas-Bashforth approach to solve the fractional non-linear PDE.
The numerical technique for the viscous Burger equation with fractal derivative was established in this study. Strict results were now established in the analysis of several statistical properties of the Burger equation. Partial differential equations and ordinary differential equations are frequently used to describe and illustrate problems that arise in

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everyday life. Numerous models in the scientific and engineering domains, including fluid dynamics, diffusion, wave, heat, elasticity, potential, stochastic, and quantum mechanics, are described by the non-linear PDEs. These equations' solutions are handled analytically or numerically through the development of effective techniques and algorithms. The so-called Burger equations are used to model the interplay between reaction mechanisms, diffusion transports, and convection effects. They are also used to represent instability in one-dimensional space and to model nonlinear phenomena in engineering and applied mathematics, such as wall motion in liquid crystals. In many cases and applications, numerical approaches and solutions are better. Piecewise polynomial estimations have emerged as the most important technique for solving difficult problems in research and engineering applications. The Adomian method, B-spline quasi-interpolation method, homotopy analysis method, tanh-coth method, quadrature technique method, homotopy perturbation method, and finite-difference method are some of the methods used to solve the Burger problem. Burger equations can be solved numerically using a variety of techniques that involve solving ill-conditioned linear systems of equations.
This study presents a thorough analysis of the Burger equation, where the model deals withThis study presents a thorough examination of the Burger equation, which deals with non-localities in the model.

Since their introduction recently, non-local differential and integral operators with fractional order and fractal dimension have shown promise as potent mathematical tools for simulating intricate real-world issues that are beyond the scope of classical and non-local differential and integral operators with single order [14].
The paper is organized in the following manner: Section 2 contains some preliminaries from fractional calculus which are required for the current article. Section 3 is the central portion of the paper which presents the proposed numerical scheme and solution of Burger equation with fractal fractional derivative. In 4 the error analysis is presented. Numerical solution of Burger equation with different fractal-fractional derivative is presented in 5. Numerical simulation is presented in 6 and at the laof the paper is described in Section 7.

2 Preliminaries

A few fundamental concepts and characteristics of fractional calculus theory are covered in this section.

Definition 21*The fractal-fractional derivative of with order $\delta-\beta$ in the Liouville-Caputo sense is defined as follows*

$$ {}^{FF-C}\mathfrak{D}_{0,t}^{\beta,\delta}\{f(t)\} = \frac{1}{\Gamma(m-\beta)}\frac{d}{dt^{\delta}}\int_0^t (t-s)^{m-\beta-1}\left(\frac{d}{ds^{\varepsilon}}f(s)\right)ds, \tag{2}$$

where $m-1<\beta,\delta\leq m\in\mathbb{N}$ and $\frac{df(s)}{ds^{\delta}}=\lim\limits_{t\to s}\frac{f(t)-f(s)}{t^{\delta}-s^{\delta}}$.

Definition 22*The Atangana-Baleanu fractal-fractional derivative of $f(t)$ with order $\delta-\beta$ in the Liouville-Caputo sense is defined as follows*

$$ {}^{FF-ABC}\mathfrak{D}_{0,t}^{\beta,\delta}\{f(t)\} = \frac{AB(\beta)}{(1-\beta)}\int_0^t E_{\beta}\left(-\frac{\beta}{1-\beta}(t-s)^{\beta}\right)\left(\frac{d}{ds^{\delta}}f(s)\right)ds, \tag{3}$$

where $0<\beta,\quad\delta\leq 1$ and $AB(\beta)=1-\beta+\frac{\beta}{\Gamma(\beta)}$.

Definition 23*The Liouville-Caputo fractal-fractional integral of $f(t)$ with order β is defined as follows:*

$$ {}^{FF-C}\mathfrak{I}_{0,t}^{\beta}\{f(t)\} = \frac{\delta}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\delta-1}f(s)ds. \tag{4}$$

Definition 24*The Atangana-Baleanu fractal-fractional integral of $f(t)$ with order β is defined as follows:*

$$ {}^{FF-AB}\mathfrak{I}_{0,t}^{\beta,\delta}\{f(t)\} = \frac{\beta\delta}{AB(\beta)}\int_0^t s^{\beta-1}(t-s)^{\beta-1}f(s)ds + \frac{\delta(1-\beta)t^{\delta-1}}{AB(\beta)}f(t). \tag{5}$$

3 Numerical Solution for Burger Equation

The Viscous Burger differential equation with classical fractal fractional derivative is given as

$$ \frac{dq(z,t)}{dt^{\delta}} + q(z,t)\frac{\partial q(z,t)}{\partial z} = \kappa\frac{\partial^2 q(z,t)}{\partial z^2} \tag{6}$$

where κ is a positive constant.
By taking $\widetilde{Q}(z,t)=\kappa\frac{\partial^2 q(z,t)}{\partial z^2}-q(z,t)\frac{\partial q(z,t)}{\partial z}$. The Eq. (6) becomes

$$\frac{dq(z,t)}{dt^{\delta}}=\widetilde{Q}(z,t). \tag{7}$$

Applying fractal operator on Eq. (7), we have

$$q(z,t)-q(z,0)=\int_0^t \delta\varepsilon^{\delta-1}\widetilde{Q}(z,\varepsilon)d\varepsilon. \tag{8}$$

When $t_{p+1}=(p+1)\Delta t$ and with $Q(z,\varepsilon)=\delta\varepsilon^{\delta-1}\widetilde{Q}(z,\varepsilon)$ it becomes

$$q(z,t_{p+1})-q(z,0)=\int_0^{t_{p+1}} Q(z,\varepsilon)d\varepsilon. \tag{9}$$

and at the point $t_p=p\Delta t$

$$q(z,t_p)-q(z,0)=\int_0^{t_p} Q(z,\varepsilon)d\varepsilon. \tag{10}$$

or we have

$$q(z,t_{p+1})-q(z,t_p)=\int_{t_p}^{t_{p+1}} Q(z,\varepsilon)d\varepsilon. \tag{11}$$

When $Q(z,\varepsilon)$ is approximated using Newton's polynomial, the result is

$$\begin{aligned}R_p(\varepsilon)&=Q(z,\varepsilon_{p-2})+\frac{Q(z,\varepsilon_{p-1})-Q(z,\varepsilon_{p-2})}{\Delta t}(\varepsilon-t_{p-2})\\&+\frac{Q(z,\varepsilon_p)-2Q(z,\varepsilon_{p-1})+Q(z,\varepsilon_{p-2})}{2(\Delta t)^2}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1}).\end{aligned} \tag{12}$$

Inserting the polynomial we have

$$\begin{aligned}q^{p+1}&=q^p+Q(z,\varepsilon_{p-2})\Delta t+\int_{t_p}^{t_{p+1}}\frac{Q(z,\varepsilon_{p-1})-Q(z,\varepsilon_{p-2})}{\Delta t}(\varepsilon-t_{p-2})d\varepsilon\\&+\int_{t_p}^{t_{p+1}}\frac{Q(z,\varepsilon_p)-2Q(z,\varepsilon_{p-1})+Q(z,\varepsilon_{p-2})}{2(\Delta t)^2}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})d\varepsilon.\end{aligned} \tag{13}$$

so we get

$$\begin{aligned}q^{p+1}&=q^p+Q(z,\varepsilon_{p-2})\Delta t+\frac{Q(z,\varepsilon_{p-1})-Q(z,\varepsilon_{p-2})}{\Delta t}\int_{t_p}^{t_{p+1}}(\varepsilon-t_{p-2})d\varepsilon\\&+\int_{t_p}^{t_{p+1}}\frac{Q(z,\varepsilon_p)-2Q(z,\varepsilon_{p-1})+Q(z,\varepsilon_{p-2})}{2(\Delta t)^2}\int_{t_p}^{t_{p+1}}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})d\varepsilon.\end{aligned} \tag{14}$$

After simplification,

$$\int_{t_p}^{t_{p+1}}(\varepsilon-t_{p-2})d\varepsilon=\frac{5}{2}(\Delta t)^2,\quad \int_{t_p}^{t_{p+1}}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})d\varepsilon=\frac{23}{6}(\Delta t)^3 \tag{15}$$

and inserted in Eq. (14) to get at the point z_k

$$\begin{aligned}q_k^{p+1}&=q_k^p+Q(z_k,t_{p-2})\Delta t+[Q(z_k,t_{p-1})-Q(z_k,t_{p-2})]\frac{5}{2}(\Delta t)\\&+[Q(z_k,t_p)-2Q(z_k,t_{p-1})+Q(z_k,t_{p-2})]\frac{23}{12}(\Delta t).\end{aligned} \tag{16}$$

and we get,

$$q_k^{p+1} = q_k^p + \frac{5}{12}\left(Q(z_k,t_{p-2})\right)\Delta t - \frac{4}{3}\left(Q(z_k,t_{p-1})\right)\Delta t + \frac{23}{12}\left(Q(z_k,t_p)\right)\Delta t. \tag{17}$$

Thus, we get the following approximation

$$\begin{aligned} q_k^{p+1} = q_k^p &+ \frac{5}{12}\delta t_{p-2}^{\delta-1}\left(\widetilde{Q}(z_k,t_{p-2})\right)\Delta t - \frac{4}{3}\delta t_{p-1}^{\delta-1}\left(\widetilde{Q}(z_k,t_{p-1})\right)\Delta t \\ &+ \frac{23}{12}\delta t_p^{\delta-1}\left(\widetilde{Q}(z_k,t_p)\right)\Delta t. \end{aligned} \tag{18}$$

So solution of the Burger equation is

$$\begin{aligned} q_k^{p+1} = q_k^p &+ \frac{5}{12}\delta t_{p-2}^{\delta-1}\left\{\kappa\frac{q_{k+1}^{p-2} - 2q_k^{p-2} - q_{k-1}^{p-2}}{h^2} + q_k^{p-2}\frac{q_{k+1}^{p-2} - q_{k-1}^{p-2}}{2h}\right\}\Delta t \\ &- \frac{4}{3}\delta t_{p-1}^{\delta-1}\left\{\kappa\frac{q_{k+1}^{p-1} - 2q_k^{p-1} - q_{k-1}^{p-1}}{h^2} + q_k^{p-2}\frac{q_{k+1}^{p-1} - q_{k-1}^{p-1}}{2h}\right\}\Delta t \\ &+ \frac{23}{12}\delta t_p^{\delta-1}\left\{\kappa\frac{q_{k+1}^{p} - 2q_k^{p} - q_{k-1}^{p}}{h^2} + q_k^{p-2}\frac{q_{k+1}^{p} - q_{k-1}^{p}}{2h}\right\}\Delta t. \end{aligned} \tag{19}$$

3.1 Stability Analysis of Fractional Burger Equation

This section provides a detailed presentation of the stability study performed on the numerical approach used to solve the given equation.

Theorem 1.*We have an equivalent technique given the Viscous time fractional Burger Equation with Caputo Fractional Derivative.*

$$q^{n+1} = k\frac{\partial^2 q^{n+1}}{\partial x^2} + (1-\phi)q^n + \sum_{j=1}^{n-1}\left(\phi_{j+1} - \phi_j\right)q^{n-j} + \phi_n q^0 + f_q^n \tag{20}$$

where $k = \Gamma(1-\phi)\Delta t^{\phi}$. The formulation of (20) subject to the boundary condition

$$q^{n+1}(0) = q^{n+1}(L) = 0, \quad n \geq 0,$$

is defined by the semi-discretized equation

$$\left(q^{n+1}, g\right) + k\left(\frac{\partial q^{n+1}}{\partial x}, \frac{\partial g}{\partial x}\right) = (1-\phi)(q^n, g) + \sum_{j=1}^{n-1}\left(\phi_{j+1} - \phi_j\right)\left(q^{n-j}, g\right) + \phi_n(q^0, g) + (f^n, g), \tag{21}$$

for all $g \in H^1(S)$. Then problem (21) is unconditionally stable for all $\Delta t > 0$, and satisfies

$$\left\|q^{n+1}\right\|_1 \leq \left\|q^0\right\|_0, \; n = 0, 1, 2 \ldots, N-1.$$

*Proof.*Here we shall adopt induction result. With $n = 0$, we obtain

$$\left(q^1, g\right) + k\left(\frac{\partial q^1}{\partial x}, \frac{\partial g}{\partial x}\right) = \left(q^0, g\right). \tag{22}$$

Using the inequality $\|g\|_0 \leq \|g\|_1$ and supposing $g = q^1$, we have

$$\left\|q^1\right\|_1 \leq \left\|q^0\right\|_0.$$

It should be noted that

$$(u, g) = \int_S qg dx$$

and

$$(q,g)_1 = (q,g) + \left(\frac{\partial q}{\partial x}, \frac{\partial g}{\partial x}\right),$$

and

$$\|g\|_0 = \sqrt{(g,g)} \quad \|g\|_1 = \sqrt{\|g\|^2 + k\left\|\frac{du}{dx}\right\|_0^2}.$$

By mathematical induction, we assume that

$$\left\|q^j\right\|_1 \leq \left\|q^0\right\|_0, \quad j = 1,2,\ldots N, \tag{23}$$

is satisfied. Next, we need to show that $\left\|q^{n+1}\right\|_1 \leq \left\|q^0\right\|_0$. Bear in mind that $g = q^{n+1}$ in (21) yields

$$\begin{aligned}\left(q^{n+1},q^{n+1}\right) + k\left(\frac{\partial q^{n+1}}{\partial x}, \frac{\partial q^{n+1}}{\partial x}\right) &= (1-\phi)(q^n,q^{n+1}) + \sum_{j=1}^{n-1}\left(\phi_{j+1}-\phi_j\right)\left(q^{n-j},q^{n+1}\right) \\ &\quad + \phi_n(q^0,q^{n+1}) + (f^n,q^{n+1})\end{aligned} \tag{24}$$

By using above (23), we obtain

$$\begin{aligned}\left\|q^{n+1}\right\|_1^2 &\leq (1-\phi_1)\left\|q^n\right\|_0\left\|q^{n+1}\right\|_0 + \sum_{j=1}^{n-1}\left(\phi_{j+1}-\phi_j\right)\left\|q^{n-j}\right\|_0\left\|q^{n+1}\right\|_0 \\ &\quad + \phi_n\left\|q^0\right\|_0\left\|q^{n+1}\right\|_0 + \|f^n|\left\|q^{n+1}\right\| \\ &\leq \left\{(1-\phi_1) + \sum_{j=1}^{n-1}(\phi_{j+1}-\phi_j) + \phi_n\right\}\left\|q^0\right\|_0\left\|q^{n+1}\right\|_1, + \|f^n|\left\|q^{n+1}\right\|\end{aligned} \tag{25}$$

which finally implies that

$$\left\|q^{n+1}\right\|_1 \leq \left\|q^0\right\|_0, \; n = 0,1,2\ldots,N-1.$$

4 Error Analysis

Theorem 2.*Let $y^\delta = f(y,t)$ be a fractal partial differential equation for $\delta > 0$ such that f is bounded, then the numerical solution of y^δ fractal partial differential equation is stable if the following inequality is satisfied.*

$$\begin{aligned}w(x_i,t_{n+1}) &= w(x_i,t_n) + \delta\sum_{j=0}^{n}\left[W_n h^\delta\left\{\frac{\left[(j+1)^{\delta+1}-j^{\delta+1}\right]}{\delta+1} - \frac{(j-1)\left[(j+1)^\delta - j^\delta\right]}{\delta}\right\}\right. \\ &\quad \left. -W_{n-1}h^\delta\left\{\frac{\left[(j+1)^{\delta+1}-j^{\delta+1}\right]}{\delta+1} - \frac{j\left[(j+1)^\delta - j^\delta\right]}{\delta}\right\}\right] + R_n^\delta\end{aligned} \tag{26}$$

where

$$R_n^\delta = \frac{h^2}{8}\left\|W^{(2)}(u,\varsigma)\right\|\sum_{j=0}^{n}\frac{t_{j+1}^\delta - t_j^\delta}{\delta}, \qquad \delta > 0$$

*Proof.*Following the derivation presented earlier

$$w(x_i,t_{n+1}) = w(x_i,t_n) + \delta\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\varepsilon^{\delta-1}W(u,\varepsilon)d\varepsilon$$

where

$$f(u,\varepsilon) = \left\{\frac{(t-t_{n-1})}{(t_n-t_{n-1})}\right\}W_n + \left\{\frac{(t-t_n)}{(t_{n-1}-t_n)}\right\}W_{n-1} + \frac{W^{(2)}(u,\varsigma)}{2!}\prod_{i=0}^{1}(t-t_i) \tag{27}$$

$$w(x_i,t_{n+1}) = w(x_i,t_n) + \delta \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} t^{\delta-1} \left[\left\{ \frac{(t-t_{n-1})}{(t_n-t_{n-1})} \right\} W_n + \left\{ \frac{(t-t_n)}{(t_{n-1}-t_n)} \right\} W_{n-1} \right] dt$$
$$+ \delta \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{W^{(2)}(u,\varsigma)}{2!} \prod_{i=0}^{1} (t-t_i) t^{\delta-1} dt \tag{28}$$

This is equal to

$$w(x_i,t_{n+1}) = w(x_i,t_n) + \delta \sum_{j=0}^{n} \left[W_n h^{\delta} \left\{ \frac{\left[(j+1)^{\delta+1} - j^{\delta+1}\right]}{\delta+1} - \frac{(j-1)\left[(j+1)^{\delta} - j^{\delta}\right]}{\delta} \right\} \right.$$
$$\left. - W_{n-1} h^{\delta} \left\{ \frac{\left[(j+1)^{\delta+1} - j^{\delta+1}\right]}{\delta+1} - \frac{j\left[(j+1)^{\delta} - j^{\delta}\right]}{\delta} \right\} \right] + R_n^{\delta}$$

Let $\|f(t)\|_{\infty} = \sup_{t\in[a,b]} |f(t)|$, we assume that

$$\left\| W^{(2)}(w,\varsigma) \right\|_{\infty} < W < \infty$$

Then one can easily deduce that, the error which is a function of n and the fractal dimension δ

$$R_n^{\delta} = \delta \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{W^{(2)}(w,\varsigma)}{2!} \prod_{i=0}^{1} (t-t_i) t^{\delta-1} dt \tag{29}$$

$$\left\| R_n^{\delta} \right\| \le \delta \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left\| \frac{W^{(2)}(w,\varsigma)}{2!} \prod_{i=0}^{1} (t-t_i) t^{\delta-1} dt \right\|$$
$$\le \delta \left\| \frac{W^{(2)}(w,\varsigma)}{2!} \right\| \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left\| \prod_{i=0}^{1} (t-t_i) \right\| t^{\delta-1} dt \tag{30}$$
$$\le \delta \left\| \frac{W^{(2)}(w,\varsigma)}{2!} \right\| \frac{h^2}{4} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} t^{\delta-1} dt$$

because

$$\prod_{i=1}^{n} (t-t_i) < \frac{n!}{4} h^{n+1}$$

so now

$$\left\| R_n^{\delta} \right\| \le \delta \left\| \frac{W^{(2)}(u,\varsigma)}{2!} \right\| \frac{h^2}{4} \sum_{j=0}^{n} \frac{t_{j+1}^{\delta} - t_j^{\delta}}{\delta}$$
$$\le \frac{h^2}{8} \left\| W^{(2)}(w,\varsigma) \right\| \sum_{j=0}^{n} \frac{t_{j+1}^{\delta} - t_j^{\delta}}{\delta}, \qquad \delta > 0 \tag{31}$$

5 Numerical Solution of Burger Equation With Different Fractal-Fractional Derivative

In this section, Numerical scheme for Viscous Berger Equation is introduced with fractal fractional Derivative

5.1 Using Caputo fractal Derivative

Considering the Viscous Burger equation including the Caputo fractal derivative is given as follow:

$${}_{0}^{FFP}D_t^{\delta,\alpha} q(z,t) = \kappa \frac{\partial^2 q(z,t)}{\partial z^2} - q(z,t) \frac{\partial q(z,t)}{\partial z} \tag{32}$$

so we can write above equation as

$$ {}_{0}^{FFP}D_{t}^{\delta,\alpha}q(z,t)=\widetilde{Q}(z,t). \tag{33}$$

Applying Caputo fractal operator on Eq. (33), we have

$$q(z,t)-q(z,0)=\frac{\delta}{\Gamma(\alpha)}\int_{0}^{t}\varepsilon^{\delta-1}\widetilde{Q}(z,\varepsilon)(t-\varepsilon)^{\alpha-1}d\varepsilon. \tag{34}$$

When $t_{p+1}=(p+1)\Delta t$ and with $Q(z,\varepsilon)=\delta\varepsilon^{\delta-1}\widetilde{Q}(z,\varepsilon)$ it becomes

$$q(z,t_{p+1})-q(z,0)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t_{p+1}}Q(z,\varepsilon)(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon. \tag{35}$$

and we can write

$$q(z,t_{p+1})-q(z,0)=\frac{1}{\Gamma(\alpha)}\sum_{n=2}^{m}\int_{t_n}^{t_{n+1}}Q(z,\varepsilon)(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon. \tag{36}$$

Inserting the polynomial we have

$$q^{p+1}=q^{p}+\frac{1}{\Gamma(\alpha)}\sum_{n=2}^{p}\int_{t_n}^{t_{n+1}}\left\{\begin{array}{l}Q(z,\varepsilon_{p-2})+\frac{Q(z,\varepsilon_{p-1})-Q(z,\varepsilon_{p-2})}{\Delta t}(\varepsilon-t_{p-2})\\+\frac{Q(z,\varepsilon_{p})-2Q(z,\varepsilon_{p-1})+Q(z,\varepsilon_{p-2})}{2(\Delta t)^2}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})\end{array}\right\} \tag{37}$$
$$\times(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon.$$

Thus we have

$$q^{p+1}=q^{p}+\frac{1}{\Gamma(\alpha)}\sum_{n=2}^{p}\left\{\begin{array}{l}\int_{t_n}^{t_{n+1}}Q(z,\varepsilon_{p-2})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\\+\int_{t_n}^{t_{n+1}}\frac{Q(z,\varepsilon_{p-1})-Q(z,\varepsilon_{p-2})}{\Delta t}(\varepsilon-t_{p-2})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\\+\int_{t_n}^{t_{n+1}}\frac{Q(z,\varepsilon_{p})-2Q(z,\varepsilon_{p-1})+Q(z,\varepsilon_{p-2})}{2(\Delta t)^2}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\end{array}\right\}. \tag{38}$$

Now, at the point z_k, we set

$$\begin{aligned}q_k^{p+1}&=q_k^{p}+\frac{1}{\Gamma(\alpha)}\sum_{n=2}^{p}Q(z,\varepsilon_{p-2})\int_{t_n}^{t_{n+1}}(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\\&+\frac{1}{\Gamma(\alpha)}\sum_{n=2}^{p}\frac{Q(z_k,\varepsilon_{p-1})-Q(z_k,\varepsilon_{p-2})}{\Delta t}\int_{t_n}^{t_{n+1}}(\varepsilon-t_{p-2})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\\&+\frac{1}{\Gamma(\alpha)}\sum_{n=2}^{p}\frac{Q(z_k,\varepsilon_{p})-2Q(z_k,\varepsilon_{p-1})+Q(z_k,\varepsilon_{p-2})}{2(\Delta t)^2}\\&\times\int_{t_n}^{t_{n+1}}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon.\end{aligned} \tag{39}$$

The integrals are simplified in order to get a compact solution:

$$\begin{aligned}&\int_{t_n}^{t_{n+1}}(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon=\frac{(\Delta t)^{\alpha}}{\alpha}[(p-n+1)^{\alpha}-(p-n)^{\alpha}]\\&\int_{t_n}^{t_{n+1}}(t_{p+1}-\varepsilon)^{\alpha-1}(\varepsilon-t_{n-2})d\varepsilon=\frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)}\left[\begin{array}{c}(p-n+1)^{\alpha}(p-n+3+2\alpha)\\-(p-n)^{\alpha}(p-n+3+3\alpha)\end{array}\right]\\&\int_{t_n}^{t_{n+1}}(\varepsilon-t_{n-2})(\varepsilon-t_{n-1})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon=\frac{(\Delta t)^{\alpha+1}}{\alpha(\alpha+1)}\\&\left[\begin{array}{c}(p-n+1)^{\alpha}\left[\begin{array}{c}2(p-n)^2+(3\alpha+10)(p-n)\\2\alpha^2+9\alpha+12\end{array}\right]\\+(p-n)^{\alpha}\left[\begin{array}{c}2(p-n)^2+(5\alpha+10)(p-n)\\6\alpha^2+18\alpha+12\end{array}\right]\end{array}\right].\end{aligned} \tag{40}$$

rearranging the above equation, we obtain the following scheme:

$$\begin{aligned} q_k^{p+1} &= q_k^0 + \frac{(\Delta t)^\alpha}{\Gamma(\alpha+1)} \sum_{n=2}^{p} Q(z_k,\varepsilon_{p-2}) \frac{(\Delta t)^\alpha}{\alpha}[(p-n+1)^\alpha-(p-n)^\alpha] \\ &+ \frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{n=2}^{p} Q(z_k,\varepsilon_{p-1}) - Q(z_k,\varepsilon_{p-2}) \begin{bmatrix} (p-n+1)^\alpha(p-n+3+2\alpha) \\ -(p-n)^\alpha(p-n+3+3\alpha) \end{bmatrix} \\ &+ \frac{(\Delta t)^\alpha}{2\Gamma(\alpha+3)} \sum_{n=2}^{p} Q(z_k,\varepsilon_p) - 2Q(z_k,\varepsilon_{p-1}) + Q(z_k,\varepsilon_{p-2}) \\ &\times \begin{bmatrix} (p-n+1)^\alpha \begin{bmatrix} 2(p-n)^2+(3\alpha+10)(p-n) \\ 2\alpha^2+9\alpha+12 \end{bmatrix} \\ +(p-n)^\alpha \begin{bmatrix} 2(p-n)^2+(5\alpha+10)(p-n) \\ 6\alpha^2+18\alpha+12 \end{bmatrix} \end{bmatrix}. \end{aligned} \tag{41}$$

Replacing this $Q(z_k,\varepsilon)$ in our equation, we obtain

$$\begin{aligned} q_k^{p+1} &= q_k^0 + \frac{(\Delta t)^{\alpha+1}}{\Gamma(\alpha+1)} \sum_{n=2}^{p} \delta t_{p-2}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-2} - 2q_k^{p-2} - q_{k-1}^{p-2}}{h^2} + q_k^{p-2} \frac{q_{k+1}^{p-2} - q_{k-1}^{p-2}}{2h} \right\} \\ &\times [(p-n+1)^\alpha - (p-n)^\alpha] \\ &+ \frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{n=2}^{p} \begin{bmatrix} \delta t_{p-1}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-1}-2q_k^{p-1}-q_{k-1}^{p-1}}{h^2} + q_k^{p-1} \frac{q_{k+1}^{p-1}-q_{k-1}^{p-1}}{2h} \right\} \\ -\delta t_{p-2}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-2}-2q_k^{p-2}-q_{k-1}^{p-2}}{h^2} + q_k^{p-2} \frac{q_{k+1}^{p-2}-q_{k-1}^{p-2}}{2h} \right\} \end{bmatrix} \boxplus \\ &+ \frac{(\Delta t)^\alpha}{2\Gamma(\alpha+3)} \sum_{n=2}^{p} \begin{bmatrix} \delta t_p^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p}-2q_k^{p}-q_{p-1}^{p}}{h^2} + q_k^{p} \frac{q_{k+1}^{p}-q_{k-1}^{p}}{2h} \right\} \\ -2\delta t_{p-1}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-1}-2q_k^{p-1}-q_{k-1}^{p-1}}{h^2} + q_k^{p-1} \frac{q_{k+1}^{p-1}-q_{k-1}^{p-1}}{2h} \right\} \\ -\delta t_{p-2}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-2}-2q_k^{p-2}-q_{k-1}^{p-2}}{h^2} + q_k^{p-2} \frac{q_{k+1}^{p-2}-q_{k-1}^{p-2}}{2h} \right\} \end{bmatrix} \oplus \\ &\times . \end{aligned} \tag{42}$$

Here

$$\boxplus = \begin{bmatrix} (p-n+1)^\alpha(p-n+3+2\alpha) \\ -(p-n)^\alpha(p-n+3+3\alpha) \end{bmatrix}$$

and

$$\oplus = \begin{bmatrix} (p-n+1)^\alpha \begin{bmatrix} 2(p-n)^2+(3\alpha+10)(p-n) \\ 2\alpha^2+9\alpha+12 \end{bmatrix} \\ +(p-n)^\alpha \begin{bmatrix} 2(p-n)^2+(5\alpha+10)(p-n) \\ 6\alpha^2+18\alpha+12 \end{bmatrix} \end{bmatrix}$$

5.2 Using Atangana Baleanu Derivative

In this section we introduce numerical scheme for solving our equation in which we are using Atangana Baleanu Derivative fractal fractional operator.

$$ {}_0^{ABC}D_t^\delta q(z,t) = \kappa \frac{\partial^2 q(z,t)}{\partial z^2} - q(z,t)\frac{\partial q(z,t)}{\partial z} \tag{43}$$

so we can write above equation as

$$ {}_0^{FFM}D_t^{\delta,\alpha} q(z,t) = \widetilde{Q}(z,t). \tag{44}$$

under the condition

$$q(z,0) = \phi(z), q(z,t)|_{\delta\omega} = f(t) \tag{45}$$

Applying Atangana Baleanu Derivative fractal operator on Eq. (44), we have

$$q(z,t)-q(z,0)=\frac{1-\delta}{AB\Gamma(\delta)}\widetilde{Q}(z,t)+\frac{\delta}{AB(\delta)\Gamma(\delta)}\int_0^t \varepsilon^{\delta-1}\widetilde{Q}(z,\varepsilon)(t-\varepsilon)^{\alpha-1}d\varepsilon. \tag{46}$$

when $t_{p+1}=(p+1)\Delta t$ and with $Q(z,\varepsilon)=\delta\varepsilon^{\delta-1}\widetilde{Q}(z,\varepsilon)$ it becomes

$$q(z,t_{p+1})-q(z,0)=\frac{1-\delta}{AB\Gamma(\delta)}\widetilde{Q}(z,t_p)+\frac{\delta}{AB(\delta)\Gamma(\delta)}\int_0^{t_{p+1}} Q(z,\varepsilon)(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon. \tag{47}$$

so

$$q(z,t_{p+1})=q(z,0)+\frac{1-\delta}{AB\Gamma(\delta)}Q(z,t_p)+\frac{\delta}{AB(\delta)\Gamma(\delta)}\sum_{n=2}^{m}\int_{t_n}^{t_{n+1}} Q(z,\varepsilon)(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon. \tag{48}$$

Inserting the polynomial we have

$$\begin{aligned}q^{p+1}&=q^0+\frac{1-\delta}{AB\Gamma(\delta)}Q(z,t_p)+\frac{\delta}{AB(\delta)\Gamma(\delta)}\\&\times\sum_{n=2}^{m}\int_{t_n}^{t_{n+1}}\left\{\begin{array}{l}Q(z,\varepsilon_{p-2})+\frac{Q(z,\varepsilon_{p-1})-Q(z,\varepsilon_{p-2})}{\Delta t}(\varepsilon-t_{p-2})\\+\frac{Q(z,\varepsilon_p)-2Q(z,\varepsilon_{p-1})+Q(z,\varepsilon_{p-2})}{2(\Delta t)^2}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})\end{array}\right\}\\&\times(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon.\end{aligned} \tag{49}$$

Thus we have

$$\begin{aligned}q^{p+1}&=q^0+\frac{1-\delta}{AB\Gamma(\delta)}Q(z,t_p)+\frac{\delta}{AB(\delta)\Gamma(\delta)}\\&\times\sum_{n=2}^{m}\int_{t_n}^{t_{n+1}}\left\{\begin{array}{l}\int_{t_n}^{t_{n+1}}Q(z,\varepsilon_{p-2})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\\+\int_{t_n}^{t_{n+1}}\frac{Q(z,\varepsilon_{p-1})-Q(z,\varepsilon_{p-2})}{\Delta t}(\varepsilon-t_{p-2})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\\+\int_{t_n}^{t_{n+1}}\frac{Q(z,\varepsilon_p)-2Q(z,\varepsilon_{p-1})+Q(z,\varepsilon_{p-2})}{2(\Delta t)^2}\\\times(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\end{array}\right\}.\end{aligned} \tag{50}$$

Now, at the point z_k, we set

$$\begin{aligned}q_k^{p+1}&=q^0+\frac{1-\delta}{AB\Gamma(\delta)}Q(z,t_p)+\frac{\delta}{AB(\delta)\Gamma(\delta)}\sum_{n=2}^{p}Q(z,\varepsilon_{p-2})\int_{t_n}^{t_{n+1}}(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\\&+\frac{\delta}{AB(\delta)\Gamma(\delta)}\sum_{n=2}^{p}\frac{Q(z_k,\varepsilon_{p-1})-Q(z_k,\varepsilon_{p-2})}{\Delta t}\int_{t_n}^{t_{n+1}}(\varepsilon-t_{p-2})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon\\&+\frac{\delta}{AB(\delta)\Gamma(\delta)}\sum_{n=2}^{p}\frac{Q(z_k,\varepsilon_p)-2Q(z_k,\varepsilon_{p-1})+Q(z_k,\varepsilon_{p-2})}{2(\Delta t)^2}\\&\times\int_{t_n}^{t_{n+1}}(\varepsilon-t_{p-2})(\varepsilon-t_{p-1})(t_{p+1}-\varepsilon)^{\alpha-1}d\varepsilon.\end{aligned} \tag{51}$$

Putting the calculations for the above integrals into Eq. (51), we obtain

$$\begin{aligned}q_k^{p+1}&=q_k^0+\frac{1-\delta}{AB\Gamma(\delta)}Q(z,t_p)+\frac{\delta(\Delta t)^\alpha}{AB(\delta)\Gamma(\alpha+1)}\sum_{n=2}^{p}Q(z_k,\varepsilon_{p-2})[(p-n+1)^\alpha-(p-n)^\alpha]\\&+\frac{\delta(\Delta t)^\alpha}{AB(\delta)\Gamma(\alpha+2)}\sum_{n=2}^{p}Q(z_k,\varepsilon_{p-1})-Q(z_k,\varepsilon_{p-2})\left[\begin{array}{c}(p-n+1)^\alpha(p-n+3+2\alpha)\\-(p-n)^\alpha(p-n+3+3\alpha)\end{array}\right]\\&+\frac{\delta(\Delta t)^\alpha}{AB(\delta)\Gamma(\alpha+3)}\sum_{n=2}^{p}Q(z_k,\varepsilon_p)-2Q(z_k,\varepsilon_{p-1})+Q(z_k,\varepsilon_{p-2})\\&\times\left[\begin{array}{c}(p-n+1)^\alpha\left[\begin{array}{c}2(p-n)^2+(3\alpha+10)(p-n)\\2\alpha^2+9\alpha+12\end{array}\right]\\+(p-n)^\alpha\left[\begin{array}{c}2(p-n)^2+(5\alpha+10)(p-n)\\6\alpha^2+18\alpha+12\end{array}\right]\end{array}\right].\end{aligned} \tag{52}$$

Replacing this $Q(z_k,\varepsilon)$ in our equation, we obtain

$$
\begin{aligned}
q_k^{p+1} &= q_k^0 + \frac{1-\delta}{AB\Gamma(\delta)} \delta t_p^{\delta-1} \left\{ \kappa \frac{q_{k+1}^p - 2q_k^p - q_{k-1}^p}{h^2} - q_k^p \frac{q_{k+1}^p - q_{k-1}^p}{2h} \right\} \\
&+ \frac{\delta(\Delta t)^\alpha}{AB(\delta)\Gamma(\alpha+1)} \sum_{n=2}^{p} \delta t_{p-2}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-2} - 2q_k^{p-2} - q_{k-1}^{p-2}}{h^2} - q_k^{p-2} \frac{q_{k+1}^{p-2} - q_{k-1}^{p-2}}{2h} \right\} \\
&\times [(p-n+1)^\alpha - (p-n)^\alpha] + \frac{\delta(\Delta t)^\alpha}{AB(\delta)\Gamma(\alpha+2)} \\
&\times \sum_{n=2}^{p} \left[\begin{array}{c} \delta t_{p-1}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-1} - 2q_k^{p-1} - q_{k-1}^{p-1}}{h^2} + q_k^{p-1} \frac{q_{k+1}^{p-1} - q_{k-1}^{p-1}}{2h} \right\} \\ -\delta t_{p-2}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-2} - 2q_k^{p-2} - q_{k-1}^{p-2}}{h^2} + q_k^{p-2} \frac{q_{k+1}^{p-2} - q_{k-1}^{p-2}}{2h} \right\} \end{array} \right] \boxplus \\
&+ \frac{\delta(\Delta t)^\alpha}{AB(\delta)\Gamma(\alpha+3)} \sum_{n=2}^{p} \left[\begin{array}{c} \delta t_p^{\delta-1} \left\{ \kappa \frac{q_{k+1}^p - 2q_k^p - q_{p-1}^p}{h^2} + q_k^p \frac{q_{k+1}^p - q_{k-1}^p}{2h} \right\} \\ -2\delta t_{p-1}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-1} - 2q_k^{p-1} - q_{k-1}^{p-1}}{h^2} + q_k^{p-1} \frac{q_{k+1}^{p-1} - q_{k-1}^{p-1}}{2h} \right\} \\ +\delta t_{p-2}^{\delta-1} \left\{ \kappa \frac{q_{k+1}^{p-2} - 2q_k^{p-2} - q_{k-1}^{p-2}}{h^2} + q_k^{p-2} \frac{q_{k+1}^{p-2} - q_{k-1}^{p-2}}{2h} \right\} \end{array} \right] \oplus
\end{aligned} \tag{53}
$$

6 Numerical Estimation

In this section, we present the numerical graphical representation of the fractal fractional Burger differential equation. We have made use of the model with the Caputo differential operator and the numerical scheme that was suggested by Atangana and Toufik where the Newton's polynomial interpolation is used. The numerical simulation are given for different values of fractal-fractional orders. The used initial conditions are 1, 10 and 1000 respectively. The figures are showing the behavior of fractal fractional Viscous Burger's equation.

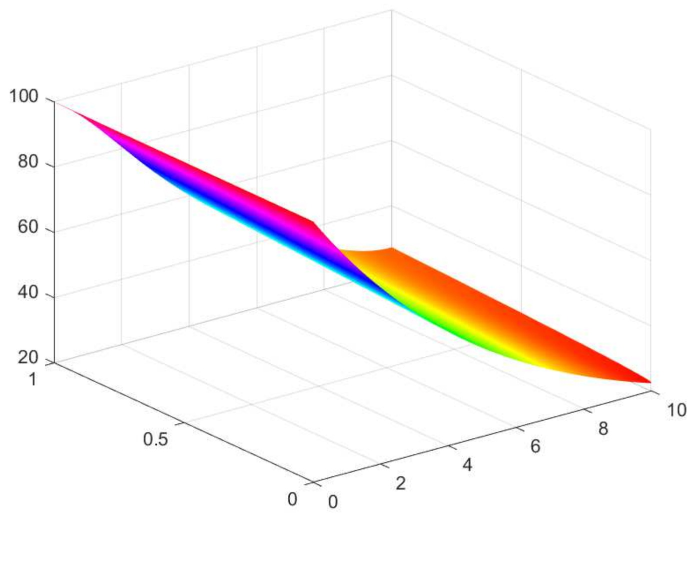

Fig. 1: Numerical Estimation For $\phi = 1.0$

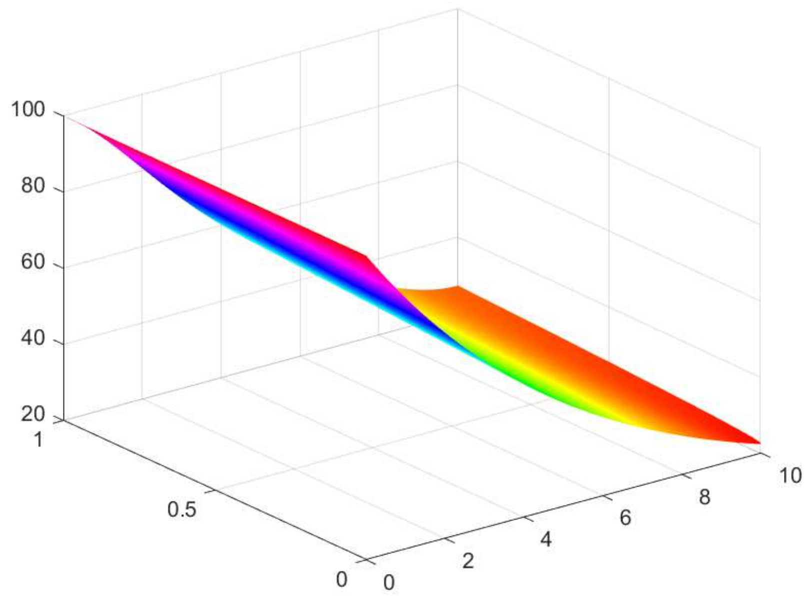

Fig. 2: Numerical Estimation For $\phi = 0.98$

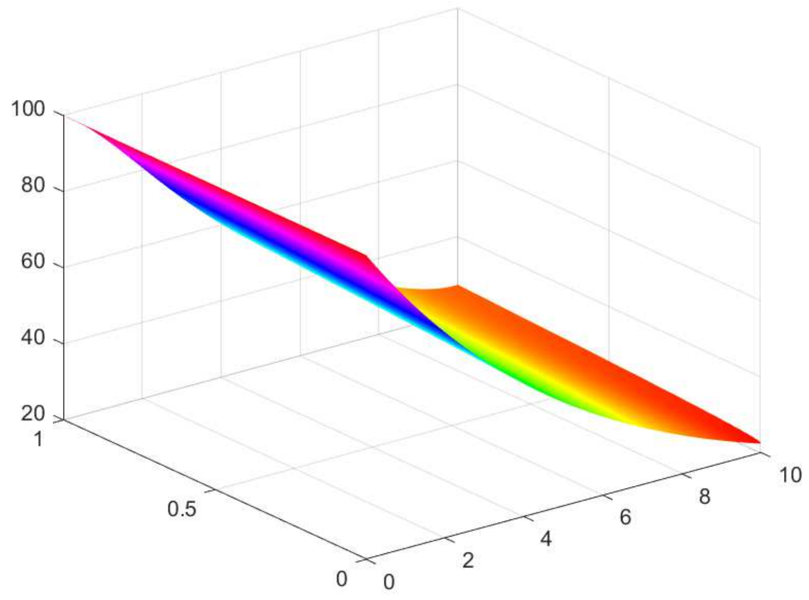

Fig. 3: Numerical Estimation For $\phi = 0.95$

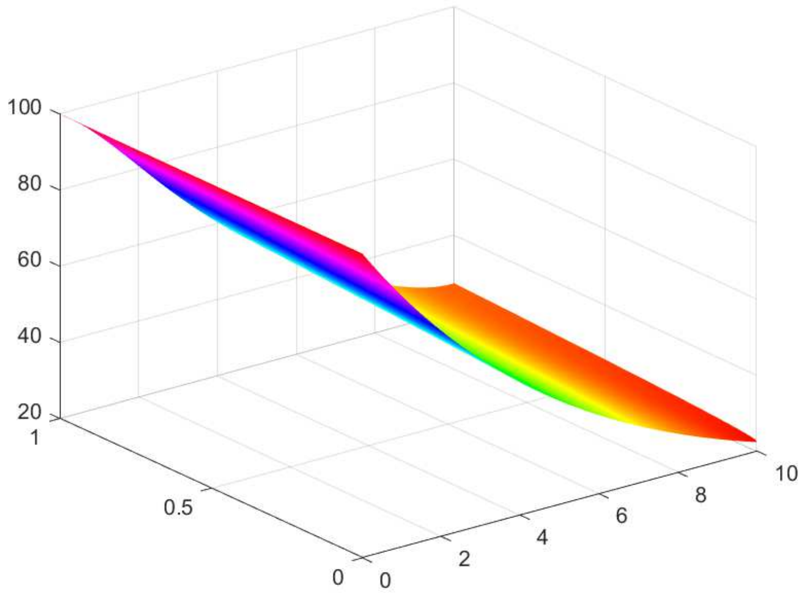

Fig. 4: Numerical Estimation For $\phi = 0.93$

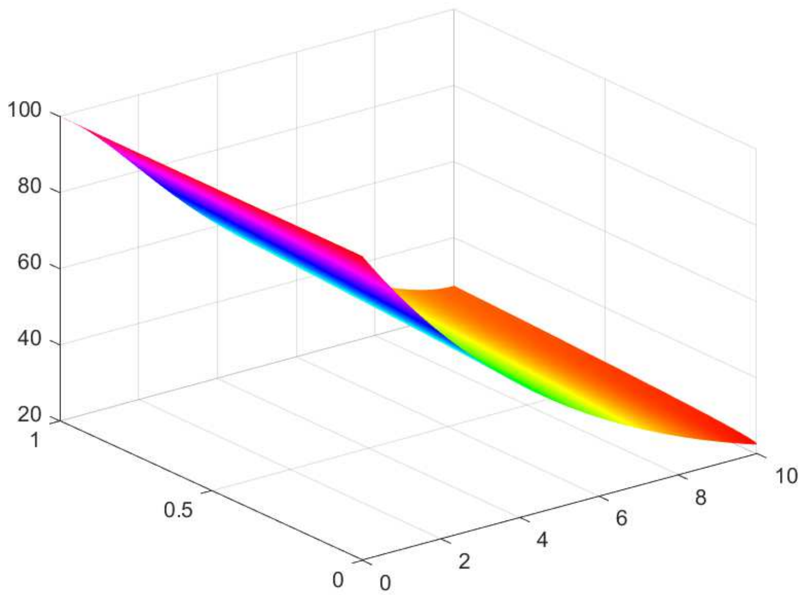

Fig. 5: Numerical Estimation For $\phi = 0.9$

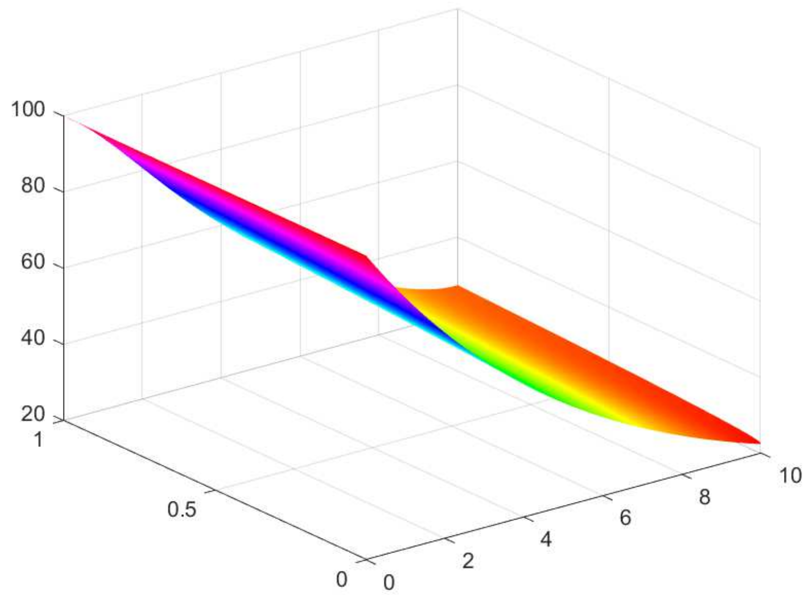

Fig. 6: Numerical Estimation For $\phi = 0.88$

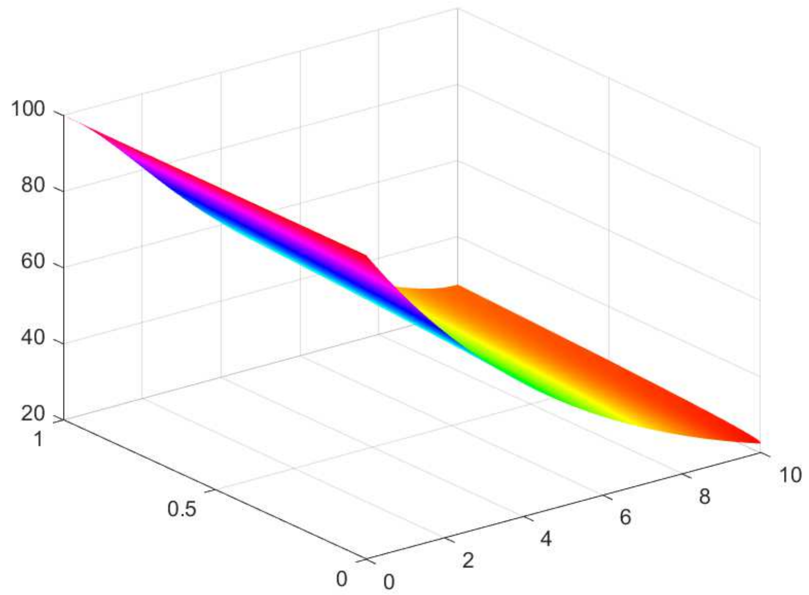

Fig. 7: Numerical Estimation For $\phi = 0.85$

7 Conclusion

We offer the fractal-fractional derivative solution to the Burger equation. The Atangana-Baleanu and Caputo fractal fractional derivative. A numerical approach based on the Newton polynomial was used to the model. The implicit finite difference scheme is linear in nature. The acquired outcomes furnish us with essential insights to enhance our understanding of the dynamics of the Burger Equation. Researchers may gain some understanding from this for several uses in practical medical trials in the future. We offer a few simulations to demonstrate the model's usefulness.

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