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New Uniqueness Results for Fractional Differential Equations with a Caputo and Khalil Derivatives

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Abstract: This paper in concerned with studying fractional differential problems involving two different derivatives. The first problem which involves the derivatives of Caputo is studied in its exisence and uniqueness of solutions. Then, an example is discussed to show the applicability of the result. In the second part, we use Khalil derivatives and the tanh numerical method to discuss traveling waves phenomena for a generalised partial differential equation. Some applications on beam equations are studied.

Keywords: Caputo derivative, conformable derivative, Euler-Bernoulli beam equation, fixed point, tanh method.

1 Introduction

The study of nonlinear partial differential equations for nonlinear phenomena is an important tool in modeling real word. As applications, we are interested to the beam problems. The nonlinear beam equations can be seen as deflection physical models, and the well known Euler-Bernoulli beam theory is a simplification of the theory of elasticity which provides a tool for calculating the deflection characteristics of beams. In fact, the Euler-Bernoulli beam theory is well established in such a way that engineers are very confident with the determination of the stress field or deflections of the elastic beam. In the field of modern science and engineering, the Euler-Bernoulli beam equation plays an important role in engineering. It is written in the form [1]:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)+\rho A\frac{\partial^2 u}{\partial t^2}=f(u), \tag{1.1}$$

For more information, one can see the above reference.
In case of a beam made of homogeneous material, (1.1) can be reduced to:

$$EI\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}=f(u). \tag{1.2}$$

There is now a substantial literature on traveling waves in nonlinearly supported beams, see McKenna and Walter in [2] to study travelling wave solutions for another type of beam equations:

$$u_{tt}+u_{xxxx}=f(u), \tag{1.3}$$

where $u=u(x,t)$ is the deflection of the roadbed, the x-axis points are in the direction along the bridge and t is time.
In recent years, the fractional differential equations arise in many scientific disciplines, such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to [3,4,5,6,7,8]. The theory on existence and uniqueness of solutions of nonlinear fractional differential equations has attracted the attention of many authors. Fixed point theorems contribute with a substantial and great role in the study of the uniqueness and existence. For some recent results, we refer the interested reader to [9,10].
We need now to note that elastic beams are an essential element which is needed in structural problems; like for instance: aircraft, ships, bridges, and buildings, see [11, 12]. In the sense of mathematical analysis, the deformation and the deflexion of the beam can be analyzed using the ODE [13]:

$$\begin{cases} u^{(4)}(\tau)=g\left(\tau,u(\tau),u''(\tau)\right), & \tau\in(0,1)\\ u(0)=u(1)=u''(0)=u''(1)=0, \end{cases} \tag{1.4}$$

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In this sense, in 2020, the two authors of the paper [14] have been concerned with investigating of the following Riemann-Liouville beam equation:

$$\begin{cases} D^{\alpha}\left(D^{\beta}u\right)(t)=h\left(t,u(t),D^{\beta}u(t)\right), \quad 0<t<1 \\ u(0)=u(1)=D^{\beta}u(0)=D^{\beta}u(1)=0, \end{cases} \tag{1.5}$$

where $\alpha,\beta\in(1,2]$, D^{α} and D^{β} are the Riemann?Liouville derivatives.

Recently, in [15], the author has investigated the following classes of beam problem

$$\begin{cases} \frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+A(x,t)\frac{\partial^{4}u}{\partial x^{4}}=f\left(x,t,u,u_t,u_x,u_{xx},u_{xxx}\right), \\ u(x,0)=f(x), \\ u_t(x,0)=g(x), a\leq x\leq b. \end{cases} \tag{1.6}$$

To rely the two part of our work and to motivate the second part, we note that travelling waves are observed in many areas of sciences and applications. The phenomena of waves can be observed in interaction and convection and also in some natural propagation.
There are many methods to find solutions of travelling waves type, we cite for instance: the first integral method [16,17], the exp-function method [18,19,20,21,22], the (G'/G) expansion method [23,24], and also the tanh method [25,26].
The tanh method is one of most direct method for finding solutions of nonlinear diffusion equations. This method has been presented by Malfliet [27,28] and by also by Wazwaz [29,30] for the computation of exact traveling wave solutions. Its ida is to express the solution of the nonlinear differential equation as a polynomial and it is based on "the balance principle".
We end the historical part of the present paper by citing the work in [31], where M. Rakah et al. have studied the uniqueness of solutions for the following problem:

$$\begin{cases} D^{\alpha}D^{\beta}D^{\gamma}u(t)=\frac{a_1 f(t,u(t),D^{\gamma}u(t))}{K(u(t))} \\ +\frac{a_2 g(t,D^{\gamma}u(t),D^{\gamma}D^{\rho}u(t))+a_3 h(t,u(t))}{K(u(t))}, \\ \\ u(0)+u(1)=\int_0^{\eta}bu(s)ds, 0<\eta<1, \\ D^{\gamma}u(0)+D^{\gamma}u(1)=0, \\ D^{\mu}D^{\mu}u(0)+D^{\mu}D^{\mu}u(1)=0, \\ t\in[0,1], 0<\alpha,\beta,\gamma,\rho,\mu\leq 1, \\ a_1,a_2,a_3\in\mathbb{R}. \end{cases} \tag{1.7}$$

The aim of the first part of this paper to "extend in a certain sense" the above cited work and to study the following differential problem:

$$\begin{cases} D^{\alpha}D^{\beta}D^{\gamma}u(t)=\frac{\eta_1 f(t,u(t),D^{\gamma}u(t))}{S(t,u(t),D^{\gamma}u(t))} \\ +\frac{\eta_2 g(t,u(t),D^{\rho}u(t))+\eta_3 h(t,u(t))}{S(t,u(t),D^{\gamma}u(t))}, \\ \\ u(0)=A_0, \\ u(1)=A_1, \\ D^{\gamma}u(0)=\int_0^{\theta}u(s)ds, 0<\theta<1, \\ t\in J, \end{cases} \tag{1.8}$$

where $J=[0,1]$, $0<\alpha,\beta,\gamma,\rho\leq 1$, $\alpha+\beta\notin]0,1),\beta+\gamma\notin]0,1),\gamma+\rho\notin]0,1)$, the functions $f,g:I\times\mathbb{R}^2\to\mathbb{R}$, $h:J\times\mathbb{R}\to\mathbb{R}$, and $S:\mathbb{R}^2\to\mathbb{R}_+^*$ are continuous, the operators $D^{\alpha},D^{\beta},D^{\gamma},D^{\rho}$ are the derivatives in the sense of Caputo, and the constants η_1,η_1,η_1 are reals.
We prove an existence and uniqueness result, then we discuss an illustrative example.
In the second part of our paper, we will use the tanh method to find new traveling wave solutions of the following problem:

$$T_t^{2\alpha}u+T_x(G(u)T_x^{3\beta}u)+T_x(H(u)T_x^{\beta}u)=F(u), \tag{1.9}$$

where T_x^{β},T_t^{α} are the conformable fractional derivatives, with $0<\alpha,\beta\leq 1$ and f,G,H are given functions.
We think it is important for the reader to know that the above considered problem generalises the work in [31] since the function S is more general than the function K which depends only on the unknown function $u(t)$. On the other hand, it is clear that the second part of the present work is concerned with new travelling wave solutions that have applications in physics; the interested reader is invited to see the two examples of applications that are studied in the present work. New traveling waves are obtained using Khalil derivatives. For more information on Tanh method and Khalil approach, one can consult the paper [32].

2 Preliminaries

We need to introduce the Caputo derivatives. For more details, we refer to the reference, see [33,34]:

Definition 2.1. Let $\alpha>0$, and $f:J\longmapsto\mathbb{R}$ be a continuous function. The Riemann-Liouville integral of order $\alpha>0$ is defined by:

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,$$

where $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du$.

Definition 2.2. For a function $f \in C^n(J,\mathbb{R})$ and $n-1 < \alpha \leq n$, the Caputo fractional derivative is defined by:

$$D^\alpha f(t) = I^{n-\alpha}\frac{d^n}{dt^n}(f(t)) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds.$$

The following lemmas are also important to be cited.

Lemma 2.1. Given $n \in \mathbb{N}^*$ and $n-1 < \alpha < n$, then the set of solutions of $D^\alpha y(t) = 0$ is given by

$$y(t) = \sum_{i=0}^{n-1} k_i t^i, k_i \in \mathbb{R}.$$

Lemma 2.2. Taking $n \in \mathbb{N}^*$ and $n-1 < \alpha < n$, then, we have

$$I^\alpha D^\alpha y(t) = y(t) + k_0 + k_1 t + ... + k_n t^n, k_i \in \mathbb{R}.$$

The following result gives us a relation between the integral form and the differential problem given in (1.8).

Lemma 2.3. Let H a continuous function over J. Then, the problem

$$\begin{cases} D^\alpha D^\beta D^\gamma u(t) = H(t), \\ u(0) = A_0, \\ u(1) = A_1, \\ D^\gamma u(0) = \int_0^\theta u(s)ds, 0 < \theta < 1. \end{cases} \tag{2.1}$$

is equivalent to the following integral representation:

$$\begin{aligned} u(t) = {} & J^{\alpha+\beta+\gamma}H(t) + \left[\frac{E_5 t^\gamma - E_1 t^{\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma)}\right] \\ & \times \int_0^1 (1-s)^{\alpha+\beta+\gamma-1}H(s)ds \\ & + \left[\frac{E_2 t^{\beta+\gamma} - E_2 t^\gamma}{\Gamma(\alpha+\beta+\gamma+1)}\right]\int_0^\theta (1-s)^{\alpha+\beta+\gamma}H(s)ds \\ & + t^\gamma[A_0E_6 + A_1E_7] - t^{\beta+\gamma}[A_0E_3 + A_1E_4] + A_0, \end{aligned} \tag{2.2}$$

where, we put

$$\begin{aligned} \phi_1 &= \Gamma(\beta+\gamma+1), \\ \phi_2 &= \Gamma(\gamma+1), \\ \phi_3 &= \frac{\Gamma(\gamma+2)}{\theta^{\gamma+1} - \Gamma(\gamma+2)}, \\ \phi_4 &= \frac{\theta^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)}, \\ F_1 &= \frac{\phi_2}{\phi_1+\phi_2-\phi_4}, \\ E_1 &= 1 - \frac{\phi_1 F_1 \phi_4}{\phi_2}, \\ E_2 &= \frac{F_1\phi_3}{\phi_2}, \\ E_3 &= \frac{\phi_2 - \phi_1 F_1 \theta + \phi_1 F_1 \phi_4}{\phi_1\phi_2}, \\ E_4 &= \frac{\phi_1 F_1 \theta - \phi_2}{\phi_1\phi_2}, \\ E_5 &= \frac{\phi_1 F_1 \phi_4}{\phi_2}, \\ E_6 &= \frac{F_1\theta - F_1\phi_4}{\phi_2}, \\ E_7 &= \frac{F_1\phi_1}{\phi_2}. \end{aligned}$$

Proof. Using Lemma 4, we write

$$D^\beta D^\gamma u(t) = J^\alpha H(t) - c_0,$$

and

$$D^\gamma u(t) = J^{\alpha+\beta}H(t) - c_0 J^\beta(1) - c_1,$$

are valid.
Hence,

$$u(t) = J^{\alpha+\beta+\gamma}H(t) - c_0\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} - c_1\frac{t^\gamma}{\Gamma(\gamma+1)} - c_2. \tag{2.3}$$

Therefore,

$$\begin{cases} -c_2 = A_0, \\ J^{\alpha+\beta+\gamma}H(1) - \frac{c_0}{\Gamma(\beta+\gamma+1)} - \frac{c_1}{\Gamma(\gamma+1)} - c_2 = A_1, \\ -c_1 = J^{\alpha+\beta+\gamma+1}H(\theta) - c_0\frac{\theta^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)} \\ \quad - c_1\frac{\theta^{\gamma+1}}{\Gamma(\gamma+2)} - c_2\theta. \end{cases}$$

By solving the above system, we get

$$c_2 = -A_0,$$

$$c_1 = F_1\phi_3 J^{\alpha+\beta+\gamma+1}H(\theta) - F_1\phi_4\phi_1 J^{\alpha+\beta+\gamma}H(1)$$

$$+F_1A_0(\theta-\phi_4)+F_1\phi_1A_1,$$

$$c_0 = (\phi_1 - \frac{\phi_1^2F_1\phi_4}{\phi_2})J^{\alpha+\beta+\gamma}H(1) - \frac{\phi_1F_1\phi_3}{\phi_2}$$

$$\times J^{\alpha+\beta+\gamma+1}H(\theta) - A_0(\frac{\phi_2-\phi_1F_1\theta+\phi_1F_1\phi_4}{\phi_2})$$

$$+A_1(\frac{\phi_1F_1\theta-\phi_2}{\phi_2}).$$

Inserting the values of c_0, c_1 and c_2 in (2.3), we achieve the proof.

Let us now consider the following notions:

$X := \{x \in C(J,\mathbb{R}), D^\gamma x \in C(J,\mathbb{R}), D^\rho x \in C(J,\mathbb{R})\}$,

and

$$\|x\|_X = \|x\|_\infty + \|D^\gamma x\|_\infty + \|D^\rho x\|_\infty,$$

where,

$$\|x\|_\infty = \sup_{t\in I}|x(t)|,$$
$$\|D^\gamma x\|_\infty = \sup_{t\in I}|D^\gamma x(t)|,$$
$$\|D^\rho x\|_\infty = \sup_{t\in I}|D^\rho x(t)|.$$

Then, we consider the nonlinear operator $T : X \to X$

$$Tu(t) = \frac{1}{\Gamma(\alpha+\beta+\gamma)}\int_0^t (1-s)^{\alpha+\beta+\gamma-1}H_u(s)ds$$

$$+\frac{[E_5t^\gamma - E_1t^{\beta+\gamma}]}{\Gamma(\alpha+\beta+\gamma)}\int_0^1 (1-s)^{\alpha+\beta+\gamma-1}H_u(s)ds \quad (2.4)$$

$$+\frac{[E_2t^{\beta+\gamma} - E_2t^\gamma]}{\Gamma(\alpha+\beta+\gamma+1)}\int_0^\theta (1-s)^{\alpha+\beta+\gamma}H_u(s)ds$$

$$+t^\gamma[A_0E_6+A_1E_7] - t^{\beta+\gamma}[A_0E_3+A_1E_4]+A_0.$$

with

$$H_u(s) = \frac{\eta_1 f(s,u(s),D^\gamma u(s)) + \eta_2 g(s,u(s),D^\rho u(s))}{S(s,u(s),D^\gamma u(s))} + \frac{\eta_3 h(s,u(s))}{S(s,u(s),D^\gamma u(s))}.$$

3 Main Results

3.1 Part 1: Existence of Exactly One Solution

In this subsection, we note that we need to work with the following hypotheses: $(A1)$: The functions f,g defined on $J\times\mathbb{R}^2$, h defined on $J\times\mathbb{R}$, and non-negative function S defined on $J\times\mathbb{R}^2$, all these functions are supposed continuous..

$(A2)$: There exist non-negative continuous functions $\zeta_1(t),\zeta_2(t),\lambda_1(t),\lambda_2(t),\psi(t)$, such that for any $t\in J$, $u_1,u_2,v_1,v_2\in\mathbb{R}$,

$$\left|\frac{f(t,u_1,u_2)}{S(t,u_1,u_2)} - \frac{f(t,v_1,v_2)}{S(t,v_1,v_2)}\right| \le \Sigma_{i=1}^2 \zeta_i(t)|u_i-v_i|,$$

$$\left|\frac{g(t,u_1,u_2)}{S(t,u_1,u_2)} - \frac{g(t,v_1,v_2)}{S(t,v_1,v_2)}\right| \le \Sigma_{i=1}^2 \lambda_i(t)|u_i-v_i|,$$

$$\left|\frac{h(t,u_1)}{S(t,u_1,u_2)} - \frac{h(t,v_1)}{S(t,v_1,v_2)}\right| \le \Sigma_{i=1}^2 \psi_i(t)|u_i-v_i|.$$

It is to note that we take:

$$\kappa = Max(\sup_{t\in I}|\zeta_1(t)|, \sup_{t\in I}|\zeta_2(t)|),$$
$$\Lambda = Max(\sup_{t\in I}|\lambda_1(t)|, \sup_{t\in I}|\lambda_2(t)|),$$
$$\Psi = Max(\sup_{t\in I}|\psi_1(t)|, \sup_{t\in I}|\psi_2(t)|).$$

Further, we consider the quantities:

$$\Sigma_1 = (|\eta_1|\kappa+|\eta_2|\Lambda+|\eta_3|\Psi)\Big[\frac{1+|E_5|+|E_1|}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{2|E_2|}{\Gamma(\alpha+\beta+\gamma+2)}\Big].$$

$$\Sigma_2 = (|\eta_1|\kappa+|\eta_2|\Lambda+|\eta_3|\Psi)\Big[\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{|E_5\phi_2|+|\frac{E_1\phi_1}{\Gamma(\beta+1)}|}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{|\frac{E_2\phi_1}{\Gamma(\beta+1)}|+|E_2\phi_2|}{\Gamma(\alpha+\beta+\gamma+2)}\Big].$$

$$\Sigma_3 = (|\eta_1|\kappa+|\eta_2|\Lambda+|\eta_3|\Psi)\Big[\frac{1}{\Gamma(\alpha+\beta+\gamma-\rho+1)} + \frac{|\frac{E_5\phi_2}{\Gamma(\gamma-\rho+1)}|+|\frac{E_1\phi_1}{\Gamma(\beta+\gamma-\rho+1)}|}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{|\frac{E_2\phi_1}{\Gamma(\beta+\gamma-\rho+1)}|+|\frac{E_2\phi_2}{\Gamma(\gamma-\rho+1)}|}{\Gamma(\alpha+\beta+\gamma+2)}\Big].$$

Theorem 3.1. Assume that $(A_1),(A_2)$ are satisfied. Then, the problem (1.8) has a unique solution, provided that $\Sigma < 1$, where $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$.

Proof. We begin this proof by showing that T satisfies the Banach contraction principle. For $(u,v) \in X^2$, we can write

$$\|Tu - Tv\|_\infty \leq (|\eta_1|\kappa + |\eta_2|\Lambda + |\eta_3|\Psi) \times \Big[\frac{1+|E_5|+|E_1|}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{2|E_2|}{\Gamma(\alpha+\beta+\gamma+2)}\Big]\|u-v\|_X. \quad (2.5)$$

We have also

$$D^\gamma Tu(t) = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (1-s)^{\alpha+\beta-1}H_u(s)ds + \frac{[E_5\phi_2 - \frac{E_1\phi_1 t^\beta}{\Gamma(\beta+1)}]}{\Gamma(\alpha+\beta+\gamma)}\int_0^1 (1-s)^{\alpha+\beta+\gamma-1}H_u(s)ds + \frac{[\frac{E_2\phi_1 t^\beta}{\Gamma(\beta+1)} - E_2\phi_2]}{\Gamma(\alpha+\beta+\gamma+1)}\int_0^\theta (1-s)^{\alpha+\beta+\gamma}H_u(s)ds + \phi_2[A_0E_6 + A_1E_7] - \frac{\phi_1 t^\beta}{\Gamma(\beta+1)}[A_0E_3 + A_1E_4].$$

By (A_2), we obtain

$$\|D^\gamma Tu - D^\gamma Tv\|_\infty \leq (|\eta_1|\kappa + |\eta_2|\Lambda + |\eta_3|\Psi) \times \Big[\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{|E_5\phi_2| + |\frac{E_1\phi_1}{\Gamma(\beta+1)}|}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{|\frac{E_2\phi_1}{\Gamma(\beta+1)}| + |E_2\phi_2|}{\Gamma(\alpha+\beta+\gamma+2)}\Big]\|u-v\|_X. \quad (2.6)$$

On other hand, we have

$$D^\rho Tu(t) = \frac{1}{\Gamma(\alpha+\beta+\gamma-\rho)}\int_0^t (1-s)^{\alpha+\beta+\gamma-\rho-1} \times H_u(s)ds + \frac{[\frac{E_5\phi_2 t^{\gamma-\rho}}{\Gamma(\gamma-\rho+1)} - \frac{E_1\phi_1 t^{\beta+\gamma-\rho}}{\Gamma(\beta+\gamma-\rho+1)}]}{\Gamma(\alpha+\beta+\gamma)} \times \int_0^1 (1-s)^{\alpha+\beta+\gamma-1}H_u(s)ds + \frac{[\frac{E_2\phi_1 t^{\beta+\gamma-\rho}}{\Gamma(\beta+\gamma-\rho+1)} - \frac{E_2\phi_2 t^{\gamma-\rho}}{\Gamma(\gamma-\rho+1)}]}{\Gamma(\alpha+\beta+\gamma+1)}\int_0^\theta (1-s)^{\alpha+\beta+\gamma} \times H_u(s)ds + \frac{\phi_2 t^{\gamma-\rho}}{\Gamma(\gamma-\rho+1)}[A_0E_6 + A_1E_7] - \frac{\phi_1 t^{\beta+\gamma-\rho}}{\Gamma(\beta+\gamma-\rho+1)}[A_0E_3 + A_1E_4].$$

By (A_2), we obtain

$$\|D^\rho Tu - D^\rho Tv\|_\infty \leq (|\eta_1|\kappa + |\eta_2|\Lambda + |\eta_3|\Psi) \times \Big[\frac{1}{\Gamma(\alpha+\beta+\gamma-\rho+1)} + \frac{|\frac{E_5\phi_2}{\Gamma(\gamma-\rho+1)}| + |\frac{E_1\phi_1}{\Gamma(\beta+\gamma-\rho+1)}|}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{|\frac{E_2\phi_1}{\Gamma(\beta+\gamma-\rho+1)}| + |\frac{E_2\phi_2}{\Gamma(\gamma-\rho+1)}|}{\Gamma(\alpha+\beta+\gamma+2)}\Big] \times \|u-v\|_X. \quad (2.7)$$

From (2.5), (2.6) and (2.7), we get

$$\|Tu_1 - Tu_2\|_X \leq (\Sigma_1 + \Sigma_2 + \Sigma_3)\|u_1 - u_2\|_X.$$

We conclude that T is contraction. As a consequence of Banach fixed point theorem, we deduce that T has a unique fixed point which is a solution of (1.8).

An Example

As illustrative example for the first part of our results, we consider the following problem

$$\begin{cases} D^{\frac{6}{10}}D^{\frac{8}{10}}D^{\frac{7}{10}}u(t)=\dfrac{f(t,u(t),D^{\frac{7}{10}}u(t))}{S(t,u(t),D^{\frac{7}{10}}u(t))} \\ +\dfrac{\frac{1}{2}g(t,u(t),D^{\frac{65}{100}}u(t))+5h(t,u(t))}{S(t,u(t),D^{\frac{7}{10}}u(t))}, \\ u(0)=2, \\ u(1)=3, \\ D^{\frac{7}{10}}u(0)=\int_0^{0.6}u(s)ds, \end{cases} \quad (2.8)$$

where

$$f(t,u,v)=\frac{2|u|+|u|e^{t+u+v}}{(e^t+200)(|u|+1)}+\frac{|v|(2+e^{t+u+v})}{(75+t^2)(2+|v|)},$$

$$g(t,u,v)=\frac{|u|(2+e^{t+u+v})}{(t^4+40)(3+|u|)}+\frac{2e^t sinv+e^{2t+u+v}sinv}{\pi(200+t^2)},$$

$$h(t,u)=\frac{|u|(2+e^{t+u+D^{\frac{7}{10}}u(t)})}{10(t^3+30)(1+|u|)},$$

$$S(t,u,v)=2+e^{t+u+v},$$

and
$\alpha=\frac{6}{10},\ \beta=\frac{8}{10},\ \gamma=\frac{7}{10},\ \rho=\frac{65}{100},\ \theta=0.6,$

$\Sigma_1=0.0302,\ \ \Sigma_2=0.0457,\ \ \Sigma_3=0.0447,$

$\Sigma=\Sigma_1+\Sigma_2+\Sigma_3=0.1206.$

The conditions of Theorem 6 hold. Therefore, the problem (2.8) has thus a unique solution on $[0,1]$.

3.2 Part 2: Traveling Wave Solutions by Tanh Method

In this section, we are interested in using the tanh method to solve the conformable problem:

$$T_t^{2\alpha}u+T_x(G(u)T_x^{3\beta}u)+T_x(H(u)T_x^{\beta}u)=F(u), \quad (1)$$

where T_x^{β},T_t^{α} are the conformable fractional derivative, with $0<\alpha,\beta\leq 1$ and $f:\mathbb{R}^2\to\mathbb{R}$ is a given function.
Note that (1.9) when $G(u)=1$, $H(u)=0$ and $\alpha=\beta=1$ transforms into the Euler-Bernoulli beam equation:

$$u_{tt}+u_{xxxx}=F(u). \quad (3.1)$$

To be able to study the above conformable problem, we need to introduce the following preliminaries, see [35,36]:

3.2.1 Conformable Derivative and Its Properties

In this subsection, we provide a definition of the conformable derivative and its important properties as established by Khalil et al.[37]

Definition 3.2.1. Let $f:(0,\infty)\to\mathbb{R}$. Then, the conformable fractional derivative of order α is defined by

$$(T^{\alpha}f)(t)=\frac{\partial^{\alpha}f(t,x)}{\partial t^{\alpha}}=\lim_{\varepsilon\to 0}\left(\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}\right),\quad t>0,\quad 0<\alpha\leq 1.$$

It is to note that when $\alpha=1$, the above formula is reduced to the standard derivative or order one.

Definition 3.2.2. The conformable fractional integral of a function $f:(0,\infty)\to\mathbb{R}$ of order α is defined as

$$(I^{\alpha}f)(t)=\int_0^t\tau^{\alpha-1}f(\tau)d\tau,\quad 0<\alpha\leq 1.$$

The following properties are needed.

$$I^{\alpha}T^{\alpha}f(t)=f(t)-f(0)$$

and

$$(T^{\alpha}f)(t)=t^{1-\alpha}\frac{df(t)}{dt}.$$

3.2.2 Description of tanh method

Let us discuss the following nonlinear conformable equation

$$F\left(u,T_t^{\alpha}u,T_x^{\beta}u,T_t^{2\alpha}u,T_t^{\alpha}(T_x^{\beta}u),T_x^{2\beta}u,...\right)=0, \quad (3.2)$$

where $T_t^{\alpha}u$ is the conformable fractional derivative of u of order $\alpha, 0<\alpha\leq 1$.
Introducing the new wave variable, namely

$$\xi=\frac{k}{\alpha}t^{\alpha}+\frac{\omega}{\beta}x^{\beta}, \quad (3.3)$$

where k and ω are constants.
So, we can rewrite the above FPDE in the following nonlinear ordinary differential equation:

$$G\left(U,U',U'',U''',...\right)=0, \quad (3.4)$$

where the prime denotes the derivation with respect to ξ.
We then introduce a new independent variable,

$$Y=tanh(\xi), \quad (3.5)$$

leads to the change of derivatives:

$$
\begin{aligned}
\frac{d}{d\xi} &= \left(1-Y^2\right)\frac{d}{dY},\\
\frac{d^2}{d\xi^2} &= -2Y\left(1-Y^2\right)\frac{d}{dY}+\left(1-Y^2\right)^2\frac{d^2}{dY^2},\\
\frac{d^3}{d\xi^3} &= 2\left(1-Y^2\right)\left(3Y^2-1\right)\frac{d}{dY}-6Y\left(1-Y^2\right)^2\frac{d^2}{dY^2}\\
&\quad +\left(1-Y^2\right)^3\frac{d^3}{dY^3},\\
\frac{d^4}{d\xi^4} &= -8Y\left(1-Y^2\right)\left(3Y^2-2\right)\frac{d}{dY}+4\left(1-Y^2\right)^2\\
&\quad \times\left(9Y^2-2\right)\frac{d^2}{dY^2}-12Y\left(1-Y^2\right)^3\frac{d^3}{dY^3}\\
&\quad +\left(1-Y^2\right)^4\frac{d^4}{dY^4}.
\end{aligned}
\tag{3.6}
$$

In the context of tanh function method, we use

$$u(x,t)=U(\xi)=F(Y)=\sum_{i=0}^{m}a_iY^i, \tag{3.7}$$

where m is a positive integer determined by the balancing procedure in the resulting nonlinear ODE in F. Thus, we have an algebraic system of equations from which the constants $k,\omega,a_i(i=0,\cdots,m)$ are obtained and determine the function U, hence we get the exact solutions of $(3,2)$.

3.2.3 Examples for Finding Traveling Wave Solutions

To demonstrate the power of the tanh method, some of well known nonlinear equations will be examined

Example 3.2.1. The oscillations and motion of waves of the elastic beams on elastic foundation scan be described by means of the following equation [38,39] ($G(u)=1$, $H(u)=0$):

$$T_t^{2\alpha}u(x,t)+T_x^{4\beta}u(x,t)=(u(x,t))^3-cu(x,t). \tag{3.8}$$

Using (3.3), to change (3.8) into the following nonlinear ODE

$$k^2U_{\zeta\zeta}+\omega^4U_{\zeta\zeta\zeta\zeta}=U^3-cU, \tag{3.9}$$

Substituting $(3,6)$ and (3.7) into (3.9),we can get

$$
\begin{aligned}
&(k^2+c_1\omega^2)\left[-2Y(1-Y^2)\frac{dF}{dY}+(1-Y^2)^2\frac{d^2F}{dY^2}\right]\\
&+\omega^4\left[-8Y(1-Y^2)(3Y^2-2)\frac{dF}{dY}+4(1-Y^2)^2\right.\\
&\times(9Y^2-2)\frac{d^2F}{dY^2}-12Y(1-Y^2)^3\frac{d^3F}{dY^3}\\
&\left.+(1-Y^2)^4\frac{d^4F}{dY^4}\right]=F^3-cF.
\end{aligned}
\tag{3.10}
$$

To determine the parameter m we usually balance $Y^8\frac{d^4F}{dY^4}$ with F^3. This in turn gives

$$8+m-4=3m$$

so that $m=2$. This gives the solution in the form

$$F(Y)=a_0+a_1Y+a_2Y^2. \tag{3.11}$$

Substituting $(3,11)$ into (3.10), we can get

$$
\begin{aligned}
&(k^2+c_1\omega^2)(1-Y^2)\left[-2Y(a_1+2a_2Y)+2a_2(1-Y^2)\right]\\
&+\omega^4(1-Y^2)\left[-8Y(3Y^2-2)(a_1+2a_2Y)\right.\\
&\left.+8a_2(1-Y^2)(9Y^2-2)\right]-(a_0+a_1Y+a_2Y^2)^3\\
&+c(a_0+a_1Y+a_2Y^2)=0.
\end{aligned}
\tag{3.12}
$$

Then, we have the system:

$$
\begin{cases}
-16\omega^4a_2+2k^2a_2-a_0^3+ca_0=0,\\
16\omega^4a_1-2k^2a_1-3a_0^2a_1+ca_1=0,\\
136\omega^4a_2-8k^2a_2-3a_2a_0^2-3a_1^2a_0+ca_2=0,\\
-40\omega^4a_1+2k^2a_1-6a_0a_1a_2-a_1^3=0,\\
-240\omega^4a_2+6k^2a_2-3a_0a_2^2-3a_1^2a_2=0,\\
24\omega^4a_1-3a_1a_2^2=0,\\
120\omega^4a_2-a_2^3=0.
\end{cases}
$$

We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of (3.8) as follows:

Case 1.

$$a_0=\frac{\sqrt{30c}}{4},a_1=0,a_2=\frac{\sqrt{30c}}{4},$$

$$u(x,t)=\frac{\sqrt{30c}}{4}+\frac{\sqrt{30c}}{4}\tanh^2(\xi). \tag{3.13}$$

Case 2.

$$a_0=\frac{\sqrt{30c}}{4},a_1=0,a_2=-\frac{\sqrt{30c}}{4},$$

$$u(x,t)=\frac{\sqrt{30c}}{4}-\frac{\sqrt{30c}}{4}\tanh^2(\xi). \tag{3.14}$$

Case 3.

$$a_0=-\frac{\sqrt{30c}}{4},a_1=0,a_2=-\frac{\sqrt{30c}}{4},$$

$$u(x,t) = -\frac{\sqrt{30c}}{4} - \frac{\sqrt{30c}}{4}\tanh^2(\xi). \quad (3.15)$$

Case 4.

$$a_0 = -\frac{\sqrt{30c}}{4}, a_1 = 0, a_2 = \frac{\sqrt{30c}}{4},$$

$$u(x,t) = -\frac{\sqrt{30c}}{4} + \frac{\sqrt{30c}}{4}\tanh^2(\xi). \quad (3.16)$$

Case 5.

$$a_0 = -\frac{\sqrt{30c}}{4}, a_1 = 0, a_2 = \frac{\sqrt{30c}}{4},$$

$$u(x,t) = \sqrt{c}. \quad (3.17)$$

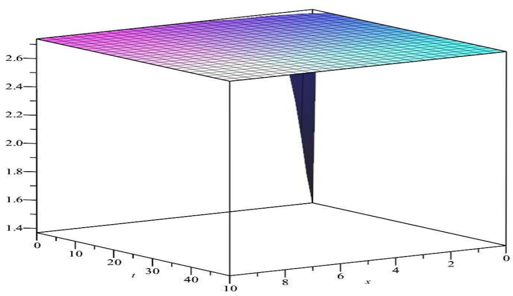

Fig. 1: *3D plot of traveling wave solution (case 1.) of (3.13) sketched within the intervals* $0 \leq x \leq 10$ *and* $0 \leq t \leq 50$.

Example 3.2.2. We now consider the nonlinear beam equation [40] ($G(u) = 1, H(u) = 1$):

$$T_t^{2\alpha}u(x,t) + T_x^{4\beta}u(x,t) + c_1 T_x^{2\beta}u(x,t) + c_2 u(x,t) + (u(x,t))^2 = 0. \quad (3.18)$$

Using (3.3), to change (3.18) into the following nonlinear ODE

$$(k^2 + c_1\omega^2)U_{\zeta\zeta} + \omega^4 U_{\zeta\zeta\zeta\zeta} + c_2 U + U^2 = 0, \quad (3.19)$$

Substituting (3,6) and (3.7) into (3.19), we can get

$$(k^2 + c_1\omega^2)\Big[-2Y(1-Y^2)\frac{dF}{dY} + (1-Y^2)^2\frac{d^2F}{dY^2}\Big] + \omega^4\Big[-8Y \times (1-Y^2)(3Y^2-2)\frac{dF}{dY} + 4(1-Y^2)^2(9Y^2-2)\frac{d^2F}{dY^2} - 12Y(1-Y^2)^3\frac{d^3F}{dY^3} + (1-Y^2)^4\frac{d^4F}{dY^4}\Big] + c_2 F + F^2 = 0. \quad (3.20)$$

To determine the parameter m we usually balance $Y^8\frac{d^4F}{dY^4}$ with F^2. This in turn gives

$$8 + m - 4 = 2m$$

so that $m = 4$. This gives the solution in the form

$$F(Y) = a_0 + a_1 Y + a_2 Y^2 + a_3 Y^3 + a_4 Y^4. \quad (3.21)$$

Substituting $(3,21)$ into (3.20), we can get

$$(k^2 + c_1\omega^2)(1-Y^2)\Big[-2Y(a_1 + 2a_2Y + 3a_3Y^2 + 4a_4Y^3) + (2a_2 + 6a_3Y + 12a_4Y^2)(1-Y^2)\Big] + \omega^4(1-Y^2) \times \Big[-8Y(3Y^2-2)(a_1 + 2a_2Y + 3a_3Y^2 + 4a_4Y^3) + 4(2a_2 + 6a_3Y + 12a_4Y^2)(1-Y^2)(9Y^2-2) - 12Y \times (1-Y^2)^2(6a_3 + 24a_4Y) + (1-Y^2)^3(24a_4)\Big] + c_2(a_0 + a_1Y + a_2Y^2 + a_3Y^3 + a_4Y^4) + (a_0 + a_1Y + a_2Y^2 + a_3Y^3 + a_4Y^4)^2 = 0. \quad (3.22)$$

We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of (3.18) as follows:

Case 1.

$$a_0 = \frac{c_2}{2}, a_1 = 0, a_2 = -\frac{3c_2}{2}, a_3 = 0, a_4 = 0,$$

$$u(x,t) = \frac{c_2}{2} - \frac{3c_2}{2}\tanh^2(\xi). \quad (3.23)$$

Case 2.

$$a_0 = -\frac{3c_2}{2}, a_1 = 0, a_2 = \frac{3c_2}{2}, a_3 = 0, a_4 = 0,$$

$$u(x,t) = -\frac{3c_2}{2} + \frac{3c_2}{2}\tanh^2(\xi). \quad (3.24)$$

Case 3.

$$a_0 = -c_2, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0,$$

$$u(x,t) = -c_2. \quad (3.25)$$

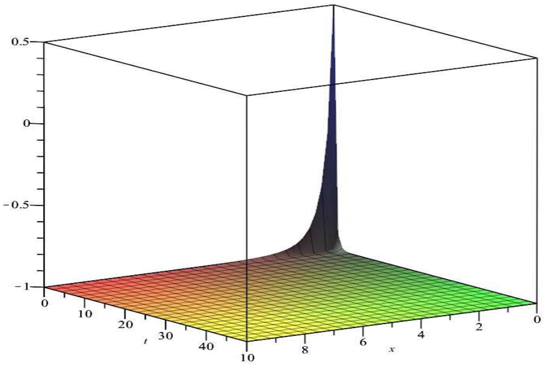

Fig. 2: *3D plot of traveling wave solution (case 1.) of (3.23) sketched within the intervals* $0 \leq x \leq 10$ *and* $0 \leq t \leq 50$.

4 Conclusion

We have first proposed a more general fractional Caputo type problem which has been given by (1.8). Then, we have used Banach contraction principle to discuss an existence and uniqueness result. In the second part of the present paper, we have been concerned with Khalil derivatives to obtain some new travelling wave solutions for two interesting fractional differential equations of beam type. Some graphs on the obtained traveling waves have been showed. The results have been obtained by means of tanh method.

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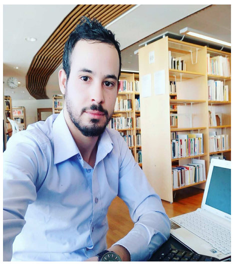

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