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Existence Solutions for a Nonlinear Langevin Fractional q-Difference System in Banach Space

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Abstract: This paper is related to deriving some necessary and appropriate conditions for qualitative results about a class of coupled Langevin fractional q-difference system in Banach space. A combination of the Mönch's fixed-point theorem and measuring of non-compactness is used in our analysis. For the demonstration of our theoretical results, we provide an example.

Keywords: Coupled fractional differential system, quantum calculus, fractional q-derivative, Mönch fixed point theorems, Kuratowski measures of noncompactness, Banach space.

1 Introduction

The subject of fractional calculus (FCs) and q-calculus is one of the various areas of mathematical analysis. In 1910, Jackson [1] introduced the subject of the q-difference equation. After that, In the early 20^{th} century, numerous works on the q-difference equation appeared, especially in [2,3]. In order to find some earlier work on this topic, we refer to [4,5], whereas the conceptual background of q-fractional calculus (q-FCs) is found in [6]. There are countless applications and studies on the q-FCs, such as [7,8,9,10,11,12,13,14,15,16,17,18].

In particular, fractional Langevin differential equations (FLDE) have been one of the most important topics in chemistry, physics and electrical engineering. The Langevin equation (LE) introduced by Langevin in 1908, is found to be an effective tool to describe the evolution of physical phenomena under fluctuating environments [19]. For instance, when the random fluctuation force is supposed to be white noise, the LE accurately captures Brownian motion. The generalized LE can be used to describe a particle's motion if the random fluctuation force is not white noise. The typical LE does not accurately describe the dynamics of systems in complex media. The generalized LE [20,21] which includes the fractal and memory features with a dissipative memory kernel into the LE, is one of many adaptations of LEs that have been proposed to describe dynamical processes in a fractal media. Another potential extension calls for the ordinary derivative in the LE to be changed to a fractional derivative to produce the fractional LE [22,23,24,25,26].

This paper is mainly concerned with the existence results for the following Langevin fractional q-difference equations:

$$\begin{cases} \mathscr{D}_q^{\beta_1}(\mathscr{D}_q^{\alpha_1}+\lambda_1)\omega_1(\varkappa)=\mathscr{F}_1(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa)), \\ \mathscr{D}_q^{\beta_2}(\mathscr{D}_q^{\alpha_2}+\lambda_2)\omega_2(\varkappa)=\mathscr{F}_2(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa)), \end{cases} \varkappa\in\Pi, \tag{1}$$

with the Dirichlet boundary conditions

$$\begin{cases} \omega_1(0)=\gamma_1\ ,\omega_1(1)=\eta_1, \\ \omega_2(0)=\gamma_2\ ,\omega_2(1)=\eta_2. \end{cases} \tag{2}$$

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where $\Pi := [0,1]$, $0 < \alpha_i, \beta_i \leq 1$, $i = 1,2$, and $\mathscr{D}_q$ is the $q-$derivative of the $q-$Caputo type. $\mathscr{F}_i : \Pi \times \mathscr{E}^2 \to \mathscr{E}$ are continuous functions, and $\mathscr{E}$ is a Banach space with norm $\|\cdot\|$, λ_i, $i = 1,2$, are any real numbers.

In the present manuscript, we will discuss the existence findings for the problem (1) which are based on the combination of the Kuratowski measure of noncompactness (KMNC) and the Mönchs fixed point theorem (MFPT). This method was further expanded and utilized in other studies, where it proved to be a very helpful tool for finding solutions to various types of integral equations. The strong MNC was deemed to be first by Banaŝ et al. [27], for more details are found in Akhmerov et al. [28], Alvàrez [29], Benchohra et al [30], Boutiara et al. [31,32,33,34], Mönch [35], Szufla [36].

Our work is designed as. In the next portion, we provide the backgrounds and preliminary lemmas that we will utilize them to prove our outcomes. The main results related to qualitative theory are established in part 3. The section 4 is enriched by constructing a pertinent application. The last portion is concluded with some future directions and remarks.

2 Preliminaries

Some necessary results needed onward for our analysis are given bellow.
Let $C(\Pi, \mathscr{E})$ be the Banach space endowed with norm

$$\|\omega\|_\infty = \sup\{\|\omega(\varkappa)\| : \quad \varkappa \in \Pi\}.$$

Further, $\mathscr{C} := C(\Pi, \mathscr{E}) \times C(\Pi, \mathscr{E})$ the Banach space equipped with the norm

$$\|(\omega, y)\|_{\mathscr{C}} = \|\omega\|_\infty + \|y\|_\infty.$$

Let $L^1(\Pi, \mathscr{E})$ be the space of Bochner-integrable functions endowed with norm

$$\|\omega\|_{L^1} = \int_\Pi |\omega(\varkappa)|\, dt.$$

Now we look at some specifications of the concept of q-FCs.
For $a \in \mathbb{R}$, we set

$$[a]_q = \frac{1-q^a}{1-q}.$$

A q-analogue of the power $((a-b)^n)$ is

$$(a-b)^{(0)} = 1, \qquad (a-b)^{(n)} = \prod_{k=0}^{n-1}(a - bq^k), \quad a,b \in \mathbb{R}, n \in \mathbb{N}.$$

On the whole,

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right), \quad a,b,\alpha \in \mathbb{R}.$$

Definition 1.[37] *The q-gamma function is described by*

$$\Gamma_q(\eta) = \frac{(1-q)^{(\eta-1)}}{(1-q)^{\eta-1}}, \quad \eta \in \mathbb{R} - \{0,-1,-2,\ldots\}.$$

Take note of the q-gamma function fulfills $\Gamma_q(1+\eta) = [\eta]_q \Gamma_q(\xi)$.

Definition 2.[37] *The q-derivative (q-D) of order $(n \in \mathbb{N})$ of a function $(\mathscr{F} : \Pi \to \mathbb{R})$ is described by $((\mathscr{D}_q^0 \mathscr{F})(\varkappa) = \mathscr{F}(\varkappa))$,*

$$(\mathscr{D}_q \mathscr{F})(\varkappa) := (\mathscr{D}_q^1 \mathscr{F})(\varkappa) = \frac{\mathscr{F}(\varkappa) - \mathscr{F}(qt)}{(1-q)\varkappa}, \quad \varkappa \neq 0, \qquad (\mathscr{D}_q \mathscr{F})(0) = \lim_{\varkappa \to 0} (\mathscr{D}_q \mathscr{F})(\varkappa),$$

and

$$(\mathscr{D}_q^n \mathscr{F})(\varkappa) = (\mathscr{D}_q \mathscr{D}_q^{n-1} \mathscr{F})(\varkappa), \quad \varkappa \in \Pi, n \in \{1,2,\ldots\}.$$

Set $(I_\varkappa := \{tq^n : n \in \mathbb{N}\} \cup \{0\})$.

Definition 3.[37] *The q-integral (q-I) of a function* $(\mathscr{F} : \mathscr{I}_{\varkappa} \to \mathbb{R})$ *is given by*

$$(\mathscr{I}_q\mathscr{F})(\varkappa) = \int_0^{\varkappa} \mathscr{F}(s)\, d_qs = \sum_{n=0}^{\infty} \varkappa(1-q)q^n\mathscr{F}\big(tq^n\big),$$

if the series is convergent. we observe that $((\mathscr{D}_q\mathscr{I}_q\mathscr{F})(\varkappa) = \mathscr{F}(\varkappa))$, *while if* $\mathscr{F}$ *is continuous at 0, then*

$$(\mathscr{I}_q\mathscr{D}_q\mathscr{F})(\varkappa) = \mathscr{F}(\varkappa) - \mathscr{F}(0).$$

Definition 4.[38] *The Riemann Liouville (RL) fractional q-integral (Fq-I) of order* $\alpha \in \mathbb{R}_+$ *of a function* $(\mathscr{F} : \Pi \to \mathbb{R})$ *is given by* $((\mathscr{I}_q^0\mathscr{F})(\varkappa) = \mathscr{F}(\varkappa))$, *and*

$$\big(\mathscr{I}_q^{\alpha}\mathscr{F}\big)(\varkappa) = \int_0^{\varkappa} \frac{(\varkappa - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mathscr{F}(s)\, d_qs, \quad \varkappa \in \Pi.$$

Lemma 1.[17] *For* $\alpha \in \mathbb{R}_+$ *and* $(\lambda \in (-1,\infty))$, *we get*

$$\big(\mathscr{I}_q^{\alpha}(\varkappa - a)^{(\lambda)}\big)(\varkappa) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(\varkappa - a)^{(\lambda+\alpha)}, \quad 0 < a < \varkappa < T.$$

In particular,

$$\big(\mathscr{I}_q^{\alpha}1\big)(\varkappa) = \frac{1}{\Gamma_q(1+\alpha)}\varkappa^{(\alpha)}.$$

Definition 5.[39] *The RL fractional q-derivative (Fq-D) of order* $(\alpha \in \mathbb{R}_+)$ *of a function* $(\mathscr{F} : \Pi \to \mathbb{R})$ *is defined by* $((\mathscr{D}_q^0\mathscr{F})(\varkappa) = \mathscr{F}(\varkappa))$, *and*

$$\big(\mathscr{D}_q^{\alpha}\mathscr{F}\big)(\varkappa) = \big(\mathscr{D}_q^{[\alpha]}\mathscr{I}_q^{[\alpha]-\alpha}\mathscr{F}\big)(\varkappa), \quad \varkappa \in \Pi,$$

where $([\alpha])$ *is the integer part of* α.

Lemma 2.[40] *Let* $\alpha > 0$ *and* $n \in \mathbb{N}$. *Therefore, the following equality holds*

$$({}_{RL}\mathscr{I}_q^{\alpha}\, {}_{RL}\mathscr{D}_q^n\mathscr{F})(\varkappa) = {}_{RL}\mathscr{D}_q^n\, {}_{RL}\mathscr{I}_q^{\alpha}\mathscr{F}(\varkappa) - \sum_{k=0}^{\alpha-1} \frac{\varkappa^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)}(\mathscr{D}_q^k\mathscr{F})(0).$$

Definition 6.[39] *The Caputo Fq-D of order* $(\alpha \in \mathbb{R}_+)$ *of a function* $(\mathscr{F} : \Pi \to \mathbb{R})$ *is given by* $(({}^C\mathscr{D}_q^0\mathscr{F})(\varkappa) = \mathscr{F}(\varkappa))$, *and*

$$\big({}^C\mathscr{D}_q^{\alpha}\mathscr{F}\big)(\varkappa) = \big(\mathscr{I}_q^{[\alpha]-\alpha}\mathscr{D}_q^{[\alpha]}\mathscr{F}\big)(\varkappa), \quad \varkappa \in \Pi.$$

Lemma 3.[39] *Let* $(\alpha \in \mathbb{R}_+)$. *Then*

$$\big(\mathscr{I}_q^{\alpha C}\mathscr{D}_q^{\alpha}\mathscr{F}\big)(\varkappa) = \mathscr{F}(\varkappa) - \sum_{k=0}^{[\alpha]-1} \frac{\varkappa^k}{\Gamma_q(1+k)}\big(\mathscr{D}_q^k\mathscr{F}\big)(0).$$

In particular, if $(\alpha \in (0,1))$, *then*

$$\big(\mathscr{I}_q^{\alpha C}\mathscr{D}_q^{\alpha}\mathscr{F}\big)(\varkappa) = \mathscr{F}(\varkappa) - \mathscr{F}(0).$$

Lemma 4.[41] *Let* $\alpha, \beta \geq 0$ *and let* $\mathscr{F}$ *is function on* $[0,1]$. *Then*

$$\begin{aligned}\mathscr{I}_q^{\alpha}\mathscr{I}_q^{\beta}\mathscr{F}(\varkappa) =& \mathscr{I}_q^{\alpha+\beta}\mathscr{F}(\varkappa) = \mathscr{I}_q^{\beta}\mathscr{I}_q^{\alpha}\mathscr{F}(\varkappa),\\ \mathscr{D}_q^{\alpha}\mathscr{I}_q^{\alpha}\mathscr{F}(\varkappa) =& \mathscr{F}(\varkappa).\end{aligned}$$

Let us now recall some basic facts about the notion of KMNC.

Definition 7.[28,27] *Let* $\mathscr{E}$ *is Banach space and* $\Omega_{\mathscr{E}}$ *the bounded subsets of* $\mathscr{E}$. *The KMNC is the map* $\mu : \Omega_{\mathscr{E}} \to [0,\infty]$ *defined by*

$$\mu(\mathscr{T}) = \inf\{\varepsilon > 0 : \mathscr{T} \subseteq \cup_{i=1}^n \mathscr{T}_i \text{ and } diam(\mathscr{T}_i) \leq \varepsilon\};\text{ here } \mathscr{T} \in \Omega_{\mathscr{E}}.$$

Proposition 1.[28, 27] *The KMNC fulfills the following:*

1. $\mu(\mathscr{T})=0 \Leftrightarrow \overline{\mathscr{T}}$ *is compact* ($\mathscr{T}$ *is relatively compact).*
2. $\mu(\mathscr{T})=\mu(\overline{\mathscr{T}})$.
3. $\mathscr{R}\subset\mathscr{T}\Rightarrow \mu(\mathscr{R})\leq\mu(\mathscr{T})$.
4. $\mu(\mathscr{R}+\mathscr{T})\leq\mu(\mathscr{R})+\mu(\mathscr{T})$
5. $\mu(c\mathscr{T})=|c|\mu(\mathscr{T}); c\in\mathbb{R}$.
6. $\mu(conv\mathscr{T})=\mu(\mathscr{T})$.

$\overline{\mathscr{T}}$ *and* $conv\mathscr{T}$ *indicate, respectively, the closure and convex hull of the bounded set* $\mathscr{T}$*. For more details see* [28, 27].

A set $\mathscr{V}$ of functions $v:\varPi\to\mathscr{E}$, represented by

$$\mathscr{V}(\varkappa)=\{v(\varkappa): v\in\mathscr{V}\}, \varkappa\in\varPi,$$

and

$$\mathscr{V}(\varPi)=\{v(\varkappa): v\in\mathscr{V}, \varkappa\in\varPi\}.$$

Now, let's recall Mönch's fixed point theorem (MFPT) and an essential lemma.

Theorem 1.[35, 36, 42] *Let* $\mathscr{D}$ *is closed, bounded and convex subset of a Banach space such that* $0\in\mathscr{D}$*, and let* $\mathscr{N}$ *is continuous mapping of* $\mathscr{D}$ *into itself. If*

$$\mathscr{V}=\overline{conv}\mathscr{N}(\mathscr{V}) \;\; or \;\; \mathscr{V}=\mathscr{N}(\mathscr{V})\cup\{0\}\Rightarrow\mu(\mathscr{V})=0 \tag{3}$$

holds for each subset $\mathscr{V}$ *of* $\mathscr{D}$*, then* $\mathscr{N}$ *has a fixed point.*

Lemma 5.[36] *Let* $\mathscr{D}$ *is a closed, bounded and convex subspace of* $C(\varPi,\mathscr{E})$*,* $\mathscr{G}:\varPi\times\varPi$ *is continuous function and* $\mathscr{F}:\varPi\times\mathscr{E}\longrightarrow\mathscr{E}$ *is function which fulfills the Caratheodory conditions, and Assume there exists* $p\in L^1(\varPi,\mathbb{R}^+)$ *such that, for any* $\varkappa\in\varPi$ *and any bounded set* $\mathscr{T}\subset\mathscr{E}$*, we get*

$$\lim_{h\to 0^+}\mu(\mathscr{F}(\mathscr{I}_{\varkappa,h}\times\mathscr{T}))\leq p(\varkappa)\mu(\mathscr{T}); \text{ here } \mathscr{I}_{\varkappa,h}=[\varkappa-h,\varkappa]\cap\varPi.$$

If $\mathscr{V}$ *is an equicontinuous subset of* $\mathscr{D}$*, then*

$$\mu\left(\left\{\int_{\varPi}\mathscr{G}(s,\varkappa)\mathscr{F}(s,\omega(s))ds:\omega\in\mathscr{V}\right\}\right)\leq\int_{\varPi}\|G(\varkappa,s)\|p(s)\mu(\mathscr{V}(s))ds.$$

3 Main results

This section focuses on the existence of solutions to system (1)-(2).

Definition 8. *By a solution of the coupled system* (1)-(2) *we mean a coupled measurable functions* $(\omega_1,\omega_2)\in\mathscr{C}$ *such that* $\omega_i(0)=\gamma_i, \omega_i(1)=\eta_i, i=1,2$*, and the FDE* $\mathscr{D}_q^{\beta_i}(\mathscr{D}_q^{\alpha_i}+\lambda_i)\omega_i(\varkappa)=\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa))$ *are fulfilled on* $\varPi$*.*

The following auxiliary lemmas are needed in order to prove the existence of solutions for the system (1)-(2).

Lemma 6. *For* $i=1,2$*,* $\mathscr{F}_i:\varPi\times\mathscr{E}^2\to\mathscr{E}$ *are continuous functions. Then the system* (1)-(2) *is equivalent to the integral equation (IE)*

$$\begin{aligned}\omega_i(\varkappa)&=\mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa))-\lambda_i\mathscr{I}_q^{\alpha_i}\omega_i(\varkappa)\\&+\varkappa^{\alpha_i}\left\{\eta_i-\gamma_i-\mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(1,\omega_1(1),\omega_2(1))+\lambda_i\mathscr{I}_q^{\alpha}\omega_i(1)\right\}+\gamma_i,\end{aligned} \tag{4}$$

if and only if ω_i*,* $i=1,2$ *is a solution of the FBVP*

$$\mathscr{D}_q^{\beta_i}(\mathscr{D}_q^{\alpha_i}+\lambda_i)\omega_i(\varkappa)=\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa)), \qquad \varkappa\in\varPi, i=1,2, \tag{5}$$

$$\omega_i(0)=\gamma_i, \qquad \omega_i(1)=\eta_i, i=1,2. \tag{6}$$

Proof. Assume that ω satisfies (5). Then by applying Lemma 1, Lemma 3 and Lemma 4, we can reduce the system (5)-(6) to an equivalent IE

$$\omega_i(\varkappa) = \mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa)) - \lambda_i\mathscr{I}_q^{\alpha_i}\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa)) + c_0\frac{\varkappa^{\alpha_i}}{\Gamma_q(\alpha_i+1)} + c_1. \tag{7}$$

Applying the BCs (6), we get

$$\omega_i(0) = c_1,$$

$$\omega_i(1) = \mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa))(1) - \lambda_i\mathscr{I}_q^{\alpha_i}\omega_i(1) + \frac{c_0}{\Gamma_q(\alpha_i+1)} + c_1.$$

So, we have

$$c_1 = \gamma_i,$$

$$\mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa))(1) - \lambda_i\mathscr{I}_q^{\alpha_i}\omega_i(1) + \frac{c_0}{\Gamma_q(\alpha_i+1)} + \gamma_i = \eta_i.$$

Consequently,

$$c_1 = \gamma_i$$

$$c_0 = \Gamma_q(\alpha_i+1)\left\{\eta_i - \gamma_i - \mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa))(1) + \lambda_i\mathscr{I}_q^{\alpha_i}\omega_i(1)\right\}.$$

Finally, we obtain the solution (4)

$$\begin{aligned}\omega_i(\varkappa) &= \mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa))(\varkappa) - \lambda_i\mathscr{I}_q^{\alpha_i}\omega_i(\varkappa)\\ &\quad + \varkappa^{\alpha_i}\left\{\eta_i - \gamma_i - \mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa))(1) + \lambda_i\mathscr{I}_q^{\alpha_i}\omega_i(1)\right\} + \gamma_i,\end{aligned}$$

which completes the proof.

Our further hypotheses are as follows.
($\mathscr{H}_1$) For any $i=1,2$, $\mathscr{F}_i : \Pi\times\mathscr{E}^2 \to \mathscr{E}$ fulfills the Caratheodory conditions;
($\mathscr{H}_2$) There exists $p_i, q_i \in C(\Pi,\mathbb{R}^+)$, such that,

$$\|\mathscr{F}_i(\varkappa,\omega)\| \le p_i(\varkappa)\|\omega_1\| + p_i(\varkappa)\|\omega_2\|, \text{ for } \varkappa\in\Pi \text{ and every } \omega_i\in\mathscr{E}, i=1,2.$$

($\mathscr{H}_3$) For any $\varkappa\in\Pi$ and every bounded measurable sets $\mathscr{T}_i\subset\mathscr{E}$, i=1,2, we get

$$\lim_{h\to 0^+}\mu(\mathscr{F}_i(\mathscr{I}_{\varkappa,h}\times\mathscr{T}_1,\mathscr{T}_2),0) \le p_1(\varkappa)\mu(\mathscr{T}_1) + q_1(\varkappa)\mu(\mathscr{T}_2),$$

and

$$\lim_{h\to 0^+}\mu(0,\mathscr{F}_i(\mathscr{I}_{\varkappa,h}\times\mathscr{T}_1,\mathscr{T}_2)) \le p_2(\varkappa)\mu(\mathscr{T}_1) + q_2(\varkappa)\mu(\mathscr{T}_2),$$

where μ is the KMNC and $\mathscr{I}_{\varkappa,h} = [\varkappa-h,\varkappa]\cap\Pi$.
Set

$$p_i^* = \sup_{\varkappa\in\Pi} p_i(\varkappa) \quad \textit{and} \quad q_i^* = \sup_{\varkappa\in\Pi} q_i(\varkappa), i=1,2.$$

Theorem 2. *Suppose that ($\mathscr{H}_1$)-($\mathscr{H}_3$) hold. If*

$$\Lambda < 1, \tag{8}$$

with

$$\begin{aligned}\Lambda :&= \textstyle\sum_{i=1}^2(\Lambda_i)\\ &= \textstyle\sum_{i=1}^2\left\{\frac{2(p_i^*+q_i^*)\rho}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{2|\lambda_i|\rho}{\Gamma_q(\alpha_i+1)}\right\},\end{aligned}$$

Then the system (1)-(2) *possesses a solution on* Π.

Proof. Transform the problem (1) into a FP problem. We consider the operator $\mathscr{G}:\mathscr{C}\to\mathscr{C}$ formulated by

$$\mathscr{G}\omega_i(\varkappa)=\mathscr{I}_q^{\alpha_i+\beta_i}h(\varkappa)-\lambda_i\mathscr{I}_q^{\alpha_i}h(\varkappa)+\varkappa^{\alpha_i}\left\{\eta_i-\gamma_i-\mathscr{I}_q^{\alpha_i+\beta_i}h(1)+\lambda_i\mathscr{I}_q^{\alpha_i}h(1)\right\}+\gamma_i. \tag{9}$$

Evidently, the FPs $\mathscr{G}$ are solutions of the system (1)-(2). Let

$$\rho\geq\frac{\eta_i}{1-\Lambda},\qquad i=1,2. \tag{10}$$

and consider

$$\mathscr{D}_\rho=\{\omega_i\in\mathscr{C},i=1,2:\|(\omega_1,\omega_2)\|\leq\rho\}.$$

It's clear, the subset $\mathscr{D}_\rho$ is convex, bounded and closed. We shall show that $\mathscr{G}$ fulfills the conditions of MFPT. We will present the proof in three claims.

Claim 1: $\mathscr{G}$ is sequentially continuous:
Let $\{\omega_{1,n},\omega_{2,n}\}_n$ be a sequence such that $(\omega_{1,n},\omega_{2,n})\to(\omega_1,\omega_2)$ in $\mathscr{C}$. Then for any $\varkappa\in\Pi$,

$$\begin{aligned}
\|(\mathscr{G}\omega_{i,n})(\varkappa)-(\mathscr{G}\omega_i)(\varkappa)\|\leq{}&\mathscr{I}_q^{\alpha_i+\beta_i}\|\mathscr{F}_i(s,\omega_{1,n}(s),\omega_{2,n}(s))-\mathscr{F}_i(s,\omega_1(s),\omega_2(s))\|(\varkappa)\\
&+|\lambda_i|\mathscr{I}_q^{\alpha_i}\|\omega_{i,n}(s)-\omega_i(s)\|(\varkappa)\\
&+\varkappa^{\alpha_i}\mathscr{I}_q^{\alpha_i+\beta_i}\|\mathscr{F}_i(s,\omega_{1,n}(s),\omega_{2,n}(s))-\mathscr{F}_i(s,\omega_1(s),\omega_2(s))\|(1)\\
&+\varkappa^{\alpha_i}|\lambda_i|\mathscr{I}_q^{\alpha_i}\|\omega_{i,n}(s)-\omega_i(s)\|(1),\\
\leq{}&\left\{2\mathscr{I}_q^{\alpha_i+\beta_i}(1)\right\}\|\mathscr{F}_i(s,\omega_{1,n}(s),\omega_{2,n}(s))-\mathscr{F}_i(s,\omega_1(s),\omega_2(s))\|\\
&+\left\{2|\lambda_i|\mathscr{I}_q^{\alpha_i}(1)\right\}\|\omega_{i,n}(s)-\omega_i(s)\|.
\end{aligned}$$

Since for any $i=1,2$, the function $\mathscr{F}_i$ fulfills $(\mathscr{H}_1)$, we obtain $\mathscr{F}_i(\varkappa,\omega_{1,n}(\varkappa),\omega_{2,n}(\varkappa))$ converge uniformly (CU) to $\mathscr{F}_i(\varkappa,\omega_1(\varkappa),\omega_2(\varkappa))$.
Therefore, the Lebesgue dominated convergence theorem (LDCT) implies that $(\mathscr{G}(\omega_{1,n},\omega_{2,n}))(\varkappa)$ CU to $(\mathscr{G}(\omega_1,\omega_{2,}))(\varkappa)$. Hence$(\mathscr{G}(\omega_{1,n},\omega_{2,n}))\to(\mathscr{G}(\omega_1,\omega_{2,}))$. Then $\mathscr{G}:\mathscr{D}_\rho\to\mathscr{D}_\rho$ is sequentially continuous.
Claim 2: $\mathscr{G}(\mathscr{D}_\rho)\subseteq\mathscr{D}_\rho$:
Take $\omega_i\in\mathscr{D}_\rho$, i=1,2, by $(\mathscr{H}_2)$, we get, for any $\varkappa\in\Pi$ and suppose that $(\mathscr{G}(\omega_i))(\varkappa)\neq0$, i=1,2.

$$\begin{aligned}
\|(\mathscr{G}\omega_i)(\varkappa)\|\leq{}&\mathscr{I}_q^{\alpha_i+\beta_i}\|\mathscr{F}_i(s,\omega_1(s),\omega_2(s))\|(\varkappa)-\lambda\mathscr{I}_q^{\alpha}\|\omega_i\|(\varkappa)\\
&+\varkappa^{\alpha_i}\left\{\eta_i-\gamma_i-\mathscr{I}_q^{\alpha_i+\beta_i}\|\mathscr{F}_i(s,\omega_1(s),\omega_2(s))\|(1)+\lambda_i\mathscr{I}_q^{\alpha}\|\omega_i\|(1)\right\}+\gamma_i,\\
\leq{}&\mathscr{I}_q^{\alpha_i+\beta_i}\left[\|\omega_1\|p_i(s)+\|\omega_2\|q_i(s)\right](\varkappa)-\lambda_i\mathscr{I}_q^{\alpha_i}\|\omega_i\|(\varkappa)\\
&+\varkappa^{\alpha_i}\left\{\eta_i-\gamma_i-\mathscr{I}_q^{\alpha_i+\beta_i}\left[\|\omega_1\|p_i(s)+\|\omega_2\|q_i(s)\right](1)+\lambda_i\mathscr{I}_q^{\alpha_i}\|\omega_i\|(1)\right\}+\gamma_i,\\
\leq{}&(p_i^*+q_i^*)\rho\left\{\mathscr{I}_q^{\alpha_i+\beta_i}(1)(\varkappa)+\varkappa^{\alpha_i}\mathscr{I}_q^{\alpha_i+\beta_i}(1)(1)\right\}\\
&+\rho\left\{|\lambda_i|\mathscr{I}_q^{\alpha_i}(1)(\varkappa)+\varkappa^{\alpha_i}|\lambda_i|\mathscr{I}_q^{\alpha_i}(1)(1)\right\}+\varkappa^{\alpha_i}(\eta_i-\gamma_i)+\gamma_i,\\
\leq{}&\left\{\frac{2(p_i^*+q_i^*)\rho}{\Gamma_q(\alpha_i+\beta_i+1)}\right\}+\left\{\frac{2|\lambda_i|\rho}{\Gamma_q(\alpha_i+1)}\right\}+\eta_i\\
\leq{}&RM_i+\eta_i.
\end{aligned}$$

Hence we get

$$\|(\mathscr{G}(\omega_1,\omega_2))\|_{\mathscr{C}}\leq\sum_{i=1}^{2}(\rho\Lambda_i+\eta_i)\leq\rho. \tag{11}$$

Claim 3: $\mathscr{G}(\mathscr{D}_\rho)$ is equicontinuous :
By Claim 2, obviously that $\mathscr{G}(\mathscr{D}_\rho)\subset\mathscr{C}$ is bounded. For the equicontinuity of $\mathscr{G}(\mathscr{D}_\rho)$, let $\varkappa_1,\varkappa_2\in\Pi$, $\varkappa_1<\varkappa_2$ and

$\omega \in \mathscr{D}_\rho$, so $\mathscr{G}\omega(\varkappa_2) - \mathscr{G}\omega(\varkappa_1) \neq 0$. Then

$$\begin{aligned}
\|\mathscr{G}\omega_i(\varkappa_2) - \mathscr{G}\omega_i(\varkappa_1)\| &\leq |\mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(s,\omega_1(s),\omega_2(s))(\varkappa_2) - \mathscr{I}_q^{\alpha_i+\beta_i}\mathscr{F}_i(s,\omega_1(s),\omega_2(s))(\varkappa_1)| \\
&+ |\lambda_i|(\mathscr{I}_q^{\alpha_i}|\omega_i(s)|(\varkappa_2) - \mathscr{I}_q^{\alpha_i}|\omega_i(s)|(\varkappa_2)) \\
&+ (\varkappa_2^{\alpha_i} - \varkappa_2^{\alpha_i})\left\{\eta_i - \gamma_i - \mathscr{I}_q^{\alpha_i+\beta_i}|\mathscr{F}_i(s,\omega_1(s),\omega_2(s))|(1) + \lambda_i \mathscr{I}_q^{\alpha_i}|\omega_i|(1)\right\}, \\
&\leq (p_i^* + q_i^*)\rho|\mathscr{I}_q^{\alpha_i+\beta_i}(1)(\varkappa_2) - \mathscr{I}_q^{\alpha_i+\beta_i}(1)(\varkappa_1)| + \rho|\lambda_i|(\mathscr{I}_q^{\alpha_i}|(1)|(\varkappa_2) - \mathscr{I}_q^{\alpha_i}|(1)|(\varkappa_1)) \\
&+ (\varkappa_2^{\alpha_i} - \varkappa_1^{\alpha_i})\left\{\eta_i - \gamma_i - \mathscr{I}_q^{\alpha_i+\beta_i}|\mathscr{F}_i(s,\omega_1(s),\omega_2(s))|(1) + \lambda_i \mathscr{I}_q^{\alpha_i}|\omega_i|(1)\right\}, \\
&\leq \frac{\rho|\lambda_i|)}{\Gamma_q(\alpha_i+1)}\{(\varkappa_2^{\alpha_i} - \varkappa_1^{\alpha_i}) + 2(\varkappa_2 - \varkappa_1)^{\alpha_i}\} \\
&+ \frac{\rho(p_i^* + q_i^*)}{\Gamma_q(\alpha_i+\beta_i+1)}\left\{(\varkappa_2^{\alpha_i+\beta_i} - \varkappa_1^{\alpha_i+\beta_i}) + 2(\varkappa_2 - \varkappa_1)^{\alpha_i+\beta_i}\right\} \\
&+ (\varkappa_2^{\alpha_i} - \varkappa_1^{\alpha_i})\left\{\eta_i - \gamma_i - \mathscr{I}_q^{\alpha_i+\beta_i}|\mathscr{F}_i(s,\omega_1(s),\omega_2(s))|(1) + \lambda_i \mathscr{I}_q^{\alpha_i}|\omega_i|(1)\right\}.
\end{aligned}$$

As $\varkappa_1 \to \varkappa_2$, the right hand side of the above inequality tends to zero.
This means that $\mathscr{G}(\mathscr{D}_\rho) \subset \mathscr{D}_\rho$.

Finally we show that the implication (3) holds:
Let $\mathscr{V} \subset \mathscr{D}_\rho$ such that $\mathscr{V} = \overline{conv}(\mathscr{G}(\mathscr{V}) \cup \{(0,0)\})$. Since $\mathscr{V}$ is bounded and equicontinuous, and therefore the function $v \to v(\varkappa) = \mu(\mathscr{V}(\varkappa))$ is continuous on Π. By $(\mathscr{H}_2)$, and the properties of μ, for any $\varkappa \in \Pi$, we get

$$\begin{aligned}
v(\varkappa) &\leq \mu(\mathscr{G}(\mathscr{V})(\varkappa) \cup \{(0,0)\})) \leq \mu((\mathscr{G}\mathscr{V})(\varkappa)), \\
&\leq \mu(\{((\mathscr{N}_1 v_1)(\varkappa),(\mathscr{N}_2 v_2)(\varkappa) : (v_1,v_2) \in \mathscr{V}\}), \\
&\leq 2\mathscr{I}_q^{\alpha_1+\beta_1}\mu(\{(\{(\mathscr{F}_1(s,v_1(s),v_2(s))(\varkappa));0) : (v_1,v_2) \in \mathscr{V}\}) \\
&+2|\lambda_1|\mathscr{I}_q^{\alpha_1}\mu(\{(v_1(s),0) : (v_1,0) \in \mathscr{V}\}) \\
&+2\mathscr{I}_q^{\alpha_2+\beta_2}\mu(\{(0,\mathscr{F}_2(s,v_1(s),v_2(s))) : (v_1,v_2) \in \mathscr{V}\}) \\
&+2|\lambda_2|\mathscr{I}_q^{\alpha_2}\mu(\{(0,v_2(s)) : (0,v_2) \in \mathscr{V}\}), \\
&\leq 2\mathscr{I}_q^{\alpha_1+\beta_1}[p_1(s)\mu(\{(v_1(s),0) : (v_1,0) \in \mathscr{V}\}) \\
&+q_1(s)\mu(\{(0,v_2(s)) : (0,v_2) \in \mathscr{V}\})] \\
&+2|\lambda_1|\mathscr{I}_q^{\alpha_1}\mu(\{(v_1(s),0) : (v_1,0) \in \mathscr{V}\}) \\
&+2\mathscr{I}_q^{\alpha_2+\beta_2}[p_2(s)\mu(\{(v_1(s),0) : (v_1,0) \in \mathscr{V}\}) \\
&+q_2(s)\mu(\{(0,v_2(s)) : (0,v_2) \in \mathscr{V}\})] \\
&+2|\lambda_2|\mathscr{I}_q^{\alpha_2}\mu(\{(0,v_2(s)) : (0,v_2) \in \mathscr{V}\}).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu(\mathscr{V}(\varkappa)) \leq& 2\mathscr{I}_q^{\alpha_1+\beta_1}(p_1(s)+q_1(s)) \times \mu(\mathscr{V}(s)) \\
& 2|\lambda_1|\mathscr{I}_q^{\alpha_1}((1)(s)) \times \mu(\mathscr{V}(s)) \\
& + 2\mathscr{I}_q^{\alpha_2+\beta_2}(p_2(s)+q_2(s)) \times \mu(\mathscr{V}(s)) \\
& 2|\lambda_2|\mathscr{I}_q^{\alpha_2}((1)(s)) \times \mu(\mathscr{V}(s)).
\end{aligned}$$

Hence

$$\begin{aligned}
\mu(\mathscr{V}(\varkappa)) \leq& \left\{\frac{2(p_1^*+q_2^*)}{\Gamma_q(\alpha_1+\beta_1+1)} + \frac{2|\lambda_1|}{\Gamma_q(\alpha_1+1)}\right\}\sup_{\varkappa \in I}\mu(\mathscr{V}(\varkappa)) \\
&+ \left\{\frac{2(p_2^*+q_2^*)}{\Gamma_q(\alpha_2+\beta_2+1)} + \frac{2|\lambda_2|}{\Gamma_q(\alpha_2+1)}\right\}\sup_{\varkappa \in I}\mu(\mathscr{V}(\varkappa)).
\end{aligned}$$

This means that

$$\sup_{\varkappa\in\Pi}\mu(\mathscr{V}(\varkappa)) \leq \Lambda \sup_{\varkappa\in\Pi}\mu(\mathscr{V}(\varkappa)).$$

By (8) it follows that $\sup_{\varkappa\in\Pi}\mu((\mathscr{V}(\varkappa)) = 0$, that is $\mu(\mathscr{V}(\varkappa)) = 0$ for any $\varkappa \in \Pi$, and then $\mathscr{V}(\varkappa)$ is relatively compact in $\mathscr{E}$. In view of the Ascoli-Arzela theorem, $\mathscr{V}$ is relatively compact in $\mathscr{D}_\rho$. Now, from Theorem 1, we conclude that $\mathscr{G}$ has a FP, which is a solution of the system (1)-(2).

4 Applications

The purpose of this section is to illustrate our outcomes with an example.
Let $\mathscr{E} = l^1 = \{\omega = (\omega_1, \omega_2, ..., \omega_n, ...) : \sum_{n=1}^{\infty} |\omega_n| < \infty\}$ with

$$\|\omega\|_{\mathscr{E}} = \sum_{n=1}^{\infty} |\omega_n|,$$

Consider the nonlinear Langevin $\frac{1}{4}$-FE:

$$\begin{cases} \mathscr{D}_{1/4}^{1/4}\left(\mathscr{D}_{1/4}^{1/3} - \frac{1}{27}\right)\omega(\varkappa) = \frac{\sqrt{3}|\omega|\cos^2(2\pi\varkappa)}{3(27-\varkappa)} + \frac{\sqrt{2}\pi|y|}{(7\pi-\varkappa)^2}\left(\frac{|y|}{|y|+3}+1\right), \\ \varkappa \in \Pi = [0,1], \\ \mathscr{D}_{1/4}^{1/2}\left(\mathscr{D}_{1/4}^{2/3} - \frac{2}{37}\right)\omega(\varkappa) = \frac{\sqrt{2}\pi|\omega|}{4(4\pi-\varkappa)^2}\left(\frac{|\omega|}{|\omega|+3}+1\right) + \frac{|y|\sin^2(2\pi\varkappa)}{(10-\varkappa)^2}, \\ \varkappa \in \Pi = [0,1], \\ \omega(0) = \frac{3}{4}, \quad \omega(1) = \frac{1}{4}, \quad \omega(0) = \frac{2}{3}, \quad \omega(1) = \frac{5}{2}. \end{cases} \tag{12}$$

Here

$$\begin{aligned} &\alpha_1 = 1/4, && \alpha_2 = 1/2, && \beta_1 = 1/3, && \beta_2 = 2/3, \\ &\gamma_1 = 3/4, && \gamma_2 = 1/4, && \lambda_1 = 1/27, && \lambda_2 = 2/37, \\ &\eta_1 = 2/3, && \eta_2 = 5/2, && q = 1/4. \end{aligned}$$

with

$$\mathscr{F}_1(\varkappa,\omega) = (((\sin\varkappa+1)e^{-\varkappa})/24)(\omega^2/(1+|\omega|))$$

It's clear, $\mathscr{F}_1$ is continuous. For every $\omega \in \mathscr{E}$ and $\varkappa \in [0,1]$, we obtain

$$|\mathscr{F}_1(\varkappa,\omega_1,\omega_2)| \leq \frac{\sqrt{3}}{81}|\omega_1| + \frac{\sqrt{2}}{49\pi}|\omega_2|,$$

and

$$|\mathscr{F}_2(\varkappa,\omega_1,\omega_2)| \leq \frac{\sqrt{2}}{64\pi}|\omega_1| + \frac{1}{100}|\omega_2|,$$

Hence, $(\mathscr{H}_2)$ is fulfilled with $p_1^* = \frac{\sqrt{3}}{81}$, $q_1^* = \frac{\sqrt{2}}{49\pi}$, $p_2^* = \frac{\sqrt{2}}{64\pi}$ and $q_2^* = \frac{1}{100}$. We will demonstrate that condition (8) holds with $\Pi = [0,1]$. Indeed,

$$\left\{\frac{2(p_1^*+q_1^*)}{\Gamma_q(\alpha_1+\beta_1+1)} + \frac{2|\lambda_1|}{\Gamma_q(\alpha_1+1)}\right\} + \left\{\frac{2(p_2^*+q_2^*)}{\Gamma_q(\alpha_2+\beta_2+1)} + \frac{2|\lambda_2|}{\Gamma_q(\alpha_2+1)}\right\} \simeq 0.6758 < 1.$$

Then, from Theorem 2 we conclude that the coupled system (12) possesses a solution on Π.

5 Conclusion

In this reported work, we have developed some adequate results for qualitative analysis of solution to fractional order in the frame of Caputo q-Ds with Dirichlet BCs in Banach space. The existence results are established for the suggested problem. Our perspective is based on properties of q-Caputo's derivatives and applying of MFPT combined with the technique of KMNC. Even though the method used to prove existence results for the problem at hand is a standard one, its presentation in the present framework is novel. In addition, an example is provided as an illustration of the present work.

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