Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/080401

Inverse Problem of Determining an Order of the Riemann-Liouville Time-Fractional Derivative

Shavkat Alimov[1] *and Ravshan Ashurov*[2,*]

[1] Department of Mathematics, National University of Uzbekistan, Tashkent, Uzbekistan
[2] Laboratory of Differential Equations and their Applications, Institute of Mathematics of the Academy of Sciences, Tashkent, Uzbekistan

Received: 2 Nov. 2021, Revised: 18 Dec. 2021, Accepted: 20 Mar. 2022
Published online: 1 Oct. 2022

Abstract: It is considered the inverse problem of identification the order ρ of the fractional Riemann - Liouville derivative in time in the abstract subdiffusion equation, the elliptical part of which is a self-adjoint positive operator with a discrete spectrum. It is proved that the norm $||u(t)||$ of the solution at a fixed $t = t_0$ restores uniquely the order ρ. At the same time, an interesting effect was discovered: for sufficiently large t, the norm $||u(t)||$, considered as a function of ρ, is monotolically decreasing. A number of examples of concrete subdiffusion equations are discussed.

Keywords: Identification of order of derivative, Fourier's method, inverse problem, Riemann-Liouville derivatives, subdiffusion equations.

1 Main results

It has long been known that for modeling subdiffusion (anomalous or slow diffusion), it is necessary to use differential equations of fractional order $\rho \in (0,1)$. But in this case, unlike integer-order differential equations, the order ρ of the fractional derivative is often unknown, and there are no specific instruments for measuring it. The problem of determining this parameter is interesting not only theoretically, but also necessary for solving initial-boundary value problems and studying the properties of solutions. Such a problem in the mathematical literature is called the inverse problem of identification the order of a fractional derivative. The paper [1] by Li, Liu, Yamamoto surveys works on such inverse problems.

In this paper, we deal with order inversion in the subdiffusion equation with a fractional Riemann-Liouville derivative with respect to time.

Let H be a separable Hilbert space, $(\cdot,\cdot)$-the inner product, and $||\cdot||$-the norm of the space H. Further, let A be a self-adjoint positive operator acting on H. Assume that A has a compact inverse and denote by $\{v_k\}$ its system of complete orthonormal eigenfunctions, and by λ_k-the set of countable non-negative eigenvalues: $0 < \lambda_1 \leq \lambda_2 \cdots$.

For vector-valued functions (or simply functions) $h : \mathbb{R}_+ \to H$, fractional integrals and derivatives are defined in the same way as for scalar functions (see, e.g., [2]), while known properties and formulas are preserved. For example, if $h(t)$ is a function defined on $[0,\infty)$, then its fractional integral of order $\nu < 0$ has the form

$$\partial_t^\nu h(t) = \frac{1}{\Gamma(-\nu)} \int_0^t \frac{h(\xi)}{(t-\xi)^{\nu+1}} d\xi, \quad t > 0,$$

here it is assumed that the integral exists and $\Gamma(\nu)$ is the gamma function of Euler. The fractional derivative in the sense of Riemann - Liouville is defined as

$$\partial_t^\rho h(t) = \frac{d}{dt} \partial_t^{\rho-1} h(t), \; 0 < \rho < 1.$$

* Corresponding author e-mail: ashurovr@gmail.com

In order to get the definition of the Caputo fractional derivative, it is necessary to swap differentiation and fractional integration in this definition:

$$D_t^\rho h(t) = \partial_t^{\rho-1} \frac{d}{dt} h(t).$$

It should be noted that in the case of $\rho = 1$ both introduced fractional derivatives coincide with the ordinary derivative: $\partial_t h(t) = D_t h(t) = \frac{d}{dt} h(t)$.

Suppose that $\rho \in (0,1]$ is a fixed number, and let $C((a,b);H)$ be the set of continuous vector functions $u(t)$ of $t \in (a,b)$. Consider the Cauchy type problem:

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = 0, \quad 0 < t \leq T; \\ \lim\limits_{t\to 0} \partial_t^{\rho-1} u(t) = \varphi, \end{cases} \tag{1}$$

where φ is a given vector in H. If $\rho = 1$, then the initial condition has the form $u(0) = \varphi$.

Problem (1) is called *the forward problem*.

Definition 1. *A solution of the forward problem is called a function $u(t)$ such that $\partial_t^\rho u(t), Au(t) \in C((0,T];H)$ and satisfying the Cauchy problem (1) for all $t \in (0,T]$.*

As usual, denote by $E_{\rho,\mu}(t)$ the Mittag-Leffler function:

$$E_{\rho,\mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + \mu)}.$$

For the forward problem (1) we have the following result:

Theorem 1. *For any $\varphi \in H$ the forward problem has a unique solution and this solution has the form*

$$u(t) = \sum_{k=1}^{\infty} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)(\varphi, v_k) v_k. \tag{2}$$

A number of authors have studied the forward problem (1) for various operators A. We will only mention a few of these works. The case of the subdiffusion equation with $Au = u_{xx}$ has been thoroughly studied (see, e.g. the monographs by A.A. Kilbas et al. [3] and A.V. Pskhu [4]). Gorenflo, Luchko and Yamamoto [5] studied the subdiffusion equations in Sobelev spaces and Kubica and Yamamoto [6] considered the forward problem for subdifussion equations having time-dependent coefficients. The multidimensional case ($x \in \mathbb{R}^N$) was studied, for example, in [3], [7] - [10], while in [3], [7] - [9] the authors considered the Laplace operator instead of differential expression u_{xx}, and the author of [10] investigated pseudo-differential operators with constant coefficients defined in the entire space $\mathbb{R}^N$.

A result similar to the above, for equation (1) with the Caputo derivative, was obtained by M. Ruzhansky et al. [11]. In the case when A is an arbitrary elliptic differential operator, this theorem is proved in [12].

Obviously solution (2) depends on $\rho \in (0,1]$. Now suppose that the order of the fractional derivative ρ is unknown and consider the inverse problem: is it possible to uniquely determine this parameter ρ if as additional information we have the norm

$$W(t_0,\rho) = ||u(t_0)||^2 = d_0 \tag{3}$$

at a fixed time instant $t_0 > 0$?

We will call *the inverse problem* the problem (1) together with the over-determination condition (3).

Let us fix a number $\rho_0 \in (0,1)$ and consider the inverse problem for $\rho \in [\rho_0, 1]$.

Definition 2. *A solution of the inverse problem is a pair $\{u(t),\rho\}$ of a solution $u(t)$ of the forward problem and a parameter $\rho \in [\rho_0,1]$ satisfying the over-determination condition (3).*

Lemma 1. *There exists a number $T_0 = T_0(\rho_0,\lambda_1)$ such that for all $t_0 \geq T_0$ and for an arbitrary $\varphi \in H$ function $W(t_0,\rho)$ monotonically decreases in $\rho \in [\rho_0,1]$.*

Let us formulate the main result of the work:

Theorem 2. *Let $t_0 \geq T_0$ and $\varphi \in H$. Then for the inverse problem to have a solution $\{u(t),\rho\}$ it is necessary and sufficient that condition*

$$W(t_0,1) \leq d_0 \leq W(t_0,\rho_0)$$

be satisfied.

It should be noted that in a review article by Z. Li et al. [1] (p. 440) in the section ”Conclusions and open problems” the question was raised: if additional information about the solution at a fixed time point is specified as ”observation data”, is it possible to unambiguously recover the order of fractional derivatives? Obviously, the formulated Theorem 2 gives a positive answer to this question, since in this theorem the value $W(t,\rho)$ at a fixed time t_0 is taken as the ”observation data”.

The considering inverse problem, in view of its importance for applications, has been studied by many specialists (see the review article [1] and the bibliography therein, [13] -[26]). It should be noted that in all previously known papers, the authors considered the following additional condition

$$u(x_0,t) = h(t),\ 0<t<T, \tag{4}$$

at the observation point $x_0 \in \overline{\Omega}$. But, as it turned out, this condition guarantees only the uniqueness of the solution of the inverse problem (see [13] - [16]). In this regard, it should be emphasized that, as Theorem 2 states, condition (3), in contrast to condition (4), guarantees both the uniqueness and existence of a solution.

Hatano et al. [17] considered the subdiffusion equation $\partial_t^\rho u = \triangle u$ with initial function $\varphi \in C_0^\infty(\Omega)$ and homogeneous Dirichlet boundary condition (see also [18]). The authors proved that if $\triangle\varphi(x_0) \neq 0$, then

$$\rho = \lim_{t\to 0}\left[t\partial_t u(x_0,t)[u(x_0,t)-\varphi(x_0)]^{-1}\right].$$

For the best of our knowledge, only the paper [19] by J. Janno considers the problem of the existence of a solution to the inverse problem. By specifying an over-determination boundary condition $Bu(\cdot,t) = h(t), 0<t<T$, the author succeeded in proving an existence theorem for recovering the integral operator kernel in the considered equation and also the order ρ, $0<\rho<1$ of the Caputo derivative.

We also note the following recent papers, where various inverse problems for the subdiffusion equations are investigated. In the papers by Z. Li, Z. Zhang [20] and R. Ashurov, Yu. Fayziev [21] the authors investigated the inverse problem of simultaneously recovering the source function and the order of the fractional time derivative. In [20] only the uniqueness is proved, while in [21] both the existence and uniqueness of a solution to the inverse problem are proved. The next two papers [22] and [23] are devoted to the study of the inverse problem of recovering both the orders of fractional derivatives with respect to space and time variables. Again, in paper [22] only the uniqueness is proved, and in paper [23] both the existence and uniqueness are proved. It should also be noted that the elliptic part of the subdiffusion equation considered in [23] has a continuous spectrum. The authors of [24] discuss for a homogeneous subdifussion equation, elliptical part of which is an arbitrary second order operator, similar issues discussed in the present paper. Work [25] is, as it were, a continuation of work [24]: here the elliptic part is a differential operator of arbitrary order, and the subdiffusion equation is inhomogeneous. Due to the inhomogeneity of the equation, the unknown order of the fractional derivative also depends on the right-hand side of the equation. As additional information for the inverse problem, the authors of both papers considered the value at a fixed time of the solution projection onto the first eigenfunction. Note that the methods proposed in these papers, in contrast to the present paper, are applicable only when the first eigenvalue of the corresponding elliptic operator is equal to zero. We also note the article [26], in which a result similar to Theorem 2 was proved for the subdiffusion equation with the Caputo derivative. The method applied in work [26] is similar to that used in the present work, and it is based on the study of the monotonicity of the Mittag-Leffler function $E_{\rho,\rho}(t)$ with respect to the parameter ρ. It should be especially noted that this property of the Mittag-Leffler function has not been studied by other authors. As it turned out, this method can be applied to solve inverse problems of determining the order of the fractional derivative for various fractional differential equations. So in work [27] this method was applied to determine the vector order of fractional derivatives for systems of pseudo-differential equations, and in work [28] - for equations of mixed type. The authors of [29] applied a slightly modified version of this method to solve a similar inverse problem for the fractional wave equation.

In conclusion, we give the following remarks:

1) Any differential operator can be taken as the operator A if it has a complete orthonormal system of eigenfunctions. The operators discussed in M. Ruzhansky et al. [11], including the fractional Sturm-Liouville operators, harmonic and anharmonic oscillators, and fractional Laplacians are examples of such operators.

Note that the work [11] is devoted to the study of another type of inverse problems, namely, the inverse problem of determining the source function.

2) The results of this work can also be applied to systems of fractional differential equations. To do this, it suffices to take $\mathbb{R}^N$ as the Hilbert space and N-dimensional symmetric quadratic matrix $\{a_{i,j}\}$ as the operator A.

3) Further, one can consider various functions as $W(t,\rho)$. The following functions $W(t,\rho) = ||Au(t)||^2$ and $W(t,\rho) = (u,\varphi)$ can be cited as an example.

2 Forward problem

This section is devoted to the proof of Theorem 1.

We note at once that the function (2) satisfies formally the conditions of problem (1) (see e.g., [30], p. 173 and [31]). To show that it is indeed a solution of problem (1), according to Definition 1, it is necessary to prove that $Au(t) \in C((0,T];H)$ and $\partial_t^\rho u(t) \in C((0,T];H)$.

To do this we need the following asymptotic estimate of the Mittag-Leffler function (see, for example, [32], p. 136)

$$|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad t > 0. \tag{5}$$

Let

$$S_j(t) = \sum_{k=1}^{j} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)(\varphi, v_k) v_k.$$

Then

$$AS_j(t) = \sum_{k=1}^{j} \lambda_k t^{\rho-1} E_\rho(-\lambda_k t^\rho)(\varphi, v_k) v_k.$$

Apply the Parseval equality and estimate (5) to get

$$||AS_j(t)||^2 = \sum_{k=1}^{j} |\lambda_k t^{\rho-1} E_\rho(-\lambda_k t^\rho)(\varphi, v_k)|^2 \leq C t^{-2} ||\varphi||^2$$

(here we used the inequality $\lambda t^\rho (1+\lambda t^\rho)^{-1} < 1$). Hence, $Au(t) \in C((0,T];H)$.

Since $\partial_t^\rho S_j(t) = -AS_j(t)$ (see equation in (1), then from the arguments above, we have $\partial_t^\rho u(t) \in C((0,T];H)$.

To prove the uniqueness of the solution, we assume, as usual, that there are two solutions $u_1(t)$ and $u_2(t)$. Then the difference $u(t) = u_1(t) - u_2(t)$ satisfies the homogeneous problem:

$$\partial_t^\rho u(t) + Au(t) = 0, \quad t > 0; \tag{6}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(t) = 0. \tag{7}$$

Set

$$w_k(t) \; = \; (u(t), v_k).$$

For any $k \in \mathbb{N}$ equation (6) implies

$$\partial_t^\rho w_k(t) = (\partial_t^\rho u(t), v_k) = -(Au(t), v_k) = -(u(t), Av_k) = -\lambda_k w_k(t).$$

Therefore, we arrive at the Cauchy problem (see (7)):

$$\partial_t^\rho w_k(t) + \lambda_k w_k(t) = 0, \quad t > 0; \quad \lim_{t\to 0} \partial_t^{\rho-1} w_k(t) = 0.$$

It is known (see, for example, [30], p. 173 and [31]) that this problem has a unique solution: $w_k(t) = 0$, $t \in (0,\infty)$, for all $k \geq 1$. This means that the solution of the forward problem is unique.

Therefore, Theorem 1 is proved completely.

3 Inverse problem

Lemma 2. *Let $0 < \rho_0 < 1$. Then there is a number $T_0 = T_0(\rho_0, \lambda_1)$ such that functions $e_\lambda(\rho) = t_0^{\rho-1} E_{\rho,\rho}(-\lambda t_0^\rho)$ are positive and monotonically decrease in $\rho \in [\rho_0, 1]$ for all $t_0 \geq T_0$ and $\lambda \geq \lambda_1$.*

Proof. Let $0 < \beta < \pi$ and $\delta(1;\beta)$ stand for a contour oriented by non-decreasing $\arg\zeta$ and consisting of the following parts: 1) the ray $\arg\zeta = -\beta$, $|\zeta| \geq 1$, 2) the arc $-\beta \leq \arg\zeta \leq \beta$ of the circle $|\zeta| = 1$, and 3) the ray $\arg\zeta = \beta$, $|\zeta| \geq 1$. It is obvious that in this case the complex ζ-plane is divided into two unbounded parts: $G^{(+)}(1;\beta)$ to the right of $\delta(1;\beta)$ in orientation and $G^{(-)}(1;\beta)$ - to the left of it.

Let $\beta = \frac{3\pi}{4}\rho$, $\rho \in [\rho_0, 1)$. Since $-\lambda t_0^\rho \in G^{(-)}(1;\beta)$, then (see [32], formula (2.29), p. 135)

$$t_0^{\rho-1} E_{\rho,\rho}(-\lambda t_0^\rho) = -\frac{1}{\lambda^2 t_0^{\rho+1}\Gamma(-\rho)} + \frac{1}{2\pi i\rho\lambda^2 t_0^{\rho+1}} \int\limits_{\delta(1;\beta)} \frac{e^{\zeta^{1/\rho}}\zeta^{\frac{1}{\rho}+1}}{\zeta+\lambda t_0^\rho} d\zeta = f_1(\rho) + f_2(\rho). \tag{8}$$

To show the validity of the lemma, it suffices to prove that $\frac{d}{d\rho}e_\lambda(\rho) < 0$ for all $\rho \in [\rho_0, 1)$, since the positiveness of $e_\lambda(\rho)$ is a consequence of the inequality $e_\lambda(1) = e^{-\lambda t} > 0$.

We first estimate the derivative $f_1'(\rho)$. Let $\Psi(\rho)$ be the logarithmic derivative of the gamma function (see [33]):

$$\Psi(\rho) = \big(\ln\Gamma(\rho)\big)' = \frac{\Gamma'(\rho)}{\Gamma(\rho)}.$$

Then

$$f_1'(\rho) = \frac{\ln t_0 - \Psi(-\rho)}{\lambda^2 t_0^{\rho+1}\Gamma(-\rho)}.$$

Since

$$\frac{1}{\Gamma(-\rho)} = -\frac{\rho}{\Gamma(1-\rho)} = -\frac{\rho(1-\rho)}{\Gamma(2-\rho)}, \quad \Psi(-\rho) = \Psi(1-\rho) + \frac{1}{\rho} = \Psi(2-\rho) + \frac{1}{\rho} - \frac{1}{1-\rho},$$

then

$$f_1'(\rho) = \frac{1}{\lambda^2 t_0^{\rho+1}} \frac{\rho(1-\rho)[\Psi(2-\rho) - \ln t_0] + 1 - 2\rho}{\Gamma(2-\rho)} = -\frac{f_{11}(\rho)}{\lambda^2 t_0^{\rho+1}\Gamma(2-\rho)}. \tag{9}$$

The known (see [33]) estimate $\Psi(2-\rho) < 1-\gamma$, where $\gamma \approx 0,57722$ is the Euler-Mascheroni constant, leads to

$$f_{11}(\rho) > \rho(1-\rho)[\ln t_0 - (1-\gamma)]) + 2\rho - 1.$$

For $t_0 = e^{1-\gamma}e^{2/\rho}$ one has $\rho(1-\rho)[\ln t_0 - (1-\gamma)]) + 2\rho - 1 = 1$. Therefore, $f_{11}(\rho) \geq 1$, if $t_0 \geq T_0$ and

$$T_0 = e^{1-\gamma}e^{2/\rho_0}. \tag{10}$$

Thus, by virtue of (9), for all such t_0 we arrive at

$$f_1'(\rho) \leq -\frac{1}{\lambda^2 t_0^{\rho+1}}. \tag{11}$$

We turn to the estimate of derivative $f_2'(\rho)$. Let $F(\zeta,\rho)$ be the integrand in (8):

$$F(\zeta,\rho) = \frac{1}{2\pi i\rho\lambda^2 t_0^{\rho+1}} \cdot \frac{e^{\zeta^{1/\rho}}\zeta^{1/\rho+1}}{\zeta+\lambda t_0^\rho}.$$

In order to take into account the dependence of the integration domain $\delta(1;\beta)$ on ρ while differentiating the function $f_2'(\rho)$, we represent integral (8) as

$$f_2(\rho) = f_{2+}(\rho) + f_{2-}(\rho) + f_{21}(\rho),$$

where

$$f_{2\pm}(\rho) = e^{\pm i\beta} \int\limits_1^\infty F(se^{\pm i\beta}, \rho)\, ds,$$

$$f_{21}(\rho) = i\int\limits_{-\beta}^{\beta} F(e^{iy},\rho)e^{iy}dy = i\beta \int\limits_{-1}^{1} F(e^{i\beta s},\rho)e^{i\beta s}ds.$$

Let us consider the function $f_{2+}(\rho)$. Since $\beta=\frac{3\pi}{4}\rho$ and $\zeta=s\,e^{i\beta}$, then

$$e^{\zeta^{1/\rho}}=e^{\frac{1}{2}(i-1)s^{\frac{1}{\rho}}}.$$

For the derivative of $f_{2+}(\rho)$ one has

$$f'_{2+}(\rho)=I\cdot\int\limits_1^\infty\frac{e^{\frac{i-1}{2}s^{1/\rho}}s^{\frac{1}{\rho}+1}e^{2ia\rho}\big[\frac{1}{\rho^2}(\frac{1-i}{2}s^{\frac{1}{\rho}}-1)\ln s+2ia-\frac{1}{\rho}-\ln t_0-\frac{ias\,e^{ia\rho}+\lambda t_0^\rho\ln t_0}{s\,e^{ia\rho}+\lambda t_0^\rho}\big]}{s\,e^{ia\rho}+\lambda t_0^\rho}ds,$$

where $I=e^{ia}(2\pi i\rho\lambda^2t_0^{\rho+1})^{-1}$ and $a=\frac{3\pi}{4}$. By virtue of the inequality $|s\,e^{ia\rho}+\lambda t_0^\rho|\geq\lambda t_0^\rho$ we arrive at

$$|f'_{2+}(\rho)|\leq\frac{C}{\rho\lambda^3t_0^{2\rho+1}}\int\limits_1^\infty e^{-\frac{1}{2}s^{1/\rho}}s^{\frac{1}{\rho}+1}\big[\frac{1}{\rho^2}s^{1/\rho}\ln s+\ln t_0\big]ds.$$

Lemma 3. *Let* $0<\rho\leq1$ *and* $m\in\mathbb{N}$. *Then*

$$J(\rho)=\frac{1}{\rho}\int\limits_1^\infty e^{-\frac{1}{2}s^{\frac{1}{\rho}}}s^{\frac{m}{\rho}+1}ds\leq C_m.$$

Proof. Let us make a change of variables:

$$r=s^{\frac{1}{\rho}}\quad s=r^\rho,\quad ds=\rho r^{\rho-1}dr.$$

Then

$$J(\rho)=\int\limits_1^\infty e^{-\frac{1}{2}r}r^{m-1+2\rho}dr\leq\int\limits_1^\infty e^{-\frac{1}{2}r}r^{m+1}dr=C_m.$$

Lemma 3 is proved.

Application of this lemma gives (note, $\frac{1}{\rho}\ln s<s^{\frac{1}{\rho}}$, provided $s\geq1$)

$$|f'_{2+}(\rho)|\leq\frac{C}{\lambda^3t_0^{2\rho+1}}\big[\frac{C_3}{\rho}+C_1\ln t_0\big]\leq\frac{C}{\lambda^3t_0^{2\rho+1}}\big[\frac{1}{\rho}+\ln t_0\big].$$

One has the same estimate for $f'_{2-}(\rho)$.
Finally consider $f_{21}(\rho)$. We have

$$f'_{21}(\rho)=\frac{a}{2\pi i\lambda^2t_0^{\rho+1}}\cdot\int\limits_{-1}^1\frac{e^{e^{ias}}e^{ias}e^{2ia\rho s}\big[2ias-\ln t_0-\frac{iase^{ia\rho s}+\lambda t_0^\rho\ln t_0}{e^{ia\rho s}+\lambda t_0^\rho}\big]}{e^{ia\rho s}+\lambda t_0^\rho}ds.$$

Therefore,

$$|f'_{21}(\rho)|\leq C\frac{\ln t_0}{\lambda^3t_0^{2\rho+1}}.$$

Estimates of $f'_{2\pm}$ and f'_{21}, and estimate (11) leads to

$$\frac{d}{d\rho}e_\lambda(\rho)\leq-\frac{1}{\lambda^2t_0^{\rho+1}}+C\frac{1/\rho+\ln t_0}{\lambda^3t_0^{2\rho+1}}.$$

Therefore, this derivative is negative provided

$$t_0^{\rho_0}>\frac{C}{\lambda_1}\left(\frac{1}{\rho_0}+\ln t_0\right).$$

Hence, one can specify a number $T_0=T_0(\rho_0,\lambda_1)$ (see also (10)) such that for all $t_0\geq T_0$

$$\frac{d}{d\rho}\big[t_0^{\rho-1}E_{\rho,\rho}(-\lambda t_0^\rho)\big]<0,\quad\lambda\geq\lambda_1,\quad\rho\in[\rho_0,1].$$

Lemma 2 is proved.

Note

$$W(t,\rho) = ||u(t)||^2 = \sum_{k=1}^{\infty} |(\varphi, v_k)|^2 |t^{\rho-1} E_\rho(-\lambda_k t^\rho)|^2.$$

Therefore, it is easy to see that Lemma 1 is a consequence of Lemma 2. In turn, Theorem 2 follows from Lemma 1.

Acknowledgements
The authors are grateful for the financial support Ministry of Innovative Development of the Republic of Uzbekistan, Grant No. F-FA-2021-424.

References

[1] Z. Li, Y. Liu and M. Yamamoto, *Inverse problems of determining parameters of the fractional partial differential equations*, Machado Ed. DeGruyter, 2019.
[2] C. Lizama, *Abstract linear fractional evolution equations*, Machado Ed. DeGruyter, 2019.
[3] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
[4] A. V. Pskhu, *Fractional partial differential equations* (in Russian), M. Nauka, 2005.
[5] R. Gorenflo, Y. Luchko and M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces, *Fract. Calc. Appl. Anal.* **18**, 799-820 (2015).
[6] A. Kubica and M. Yamamoto, Initial-boundary value problems for fractional diffusion equations with time-dependent coefficients, *Fract. Calc. Appl. Anal.* **21**, 276-311 (2018).
[7] O. P. Agrawal, Solution for a fractional diffusion-wave equation defined in a bounded domain, *Nonlin. Dynam.* **29**, 145-155 (2002).
[8] A. V. Pskhu, Initial-value problem for a linear ordinary differential equation of noninteger order, *Sbornik: Math.* **202**, 571-582 (2011).
[9] A. V. Pskhu, Green function of the first doundary-value problem for the fractional diffusion wave equation in a multidimentiona rectangular domain, Achievements of sciences and technology, *Mod. Math. Appl. (in Russian)* **167**, 52-61 (2019).
[10] S. Umarov, *Introduction to fractional and pseudo-differential equations with singular symbols*, Springer, 2015.
[11] M. Ruzhansky, N. Tokmagambetov and B. Torebek, Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations, *J. Inv. Ill-Possed Prob.* **27**, 891-911 (2019).
[12] R. R. Ashurov and O. T. Muhiddinova, Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator, *Lobachevski J. Math.* **42**, 517-525 (2021).
[13] J. Cheng, J. Nakagawa, M. Yamamoto and T. Yamazaki, Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation, *Inverse Prob.* **4**, 1-25 (2009).
[14] S. Tatar and S. Ulusoy, A uniqueness result for an inverse problem in a space-time fractional diffusion equation, *Electron. J. Differ. Equ.* **257**, 1-9 (2013).
[15] Z. Li and M. Yamamoto, Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation, *Appl. Anal.* **94**, 570-579 (2015).
[16] Z. Li, Y. Luchko and M. Yamamoto, Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem, *Comput. Math. Appl.* **73**, 1041-1052 (2017).
[17] Y. Hatano, J. Nakagawa, S. Wang and M. Yamamoto, Determination of order in fractional diffusion equation, *J. Math-for-Ind.* **5A**, 51-57 (2013).
[18] M. D'Ovidio, P. Loreti, A. Momenzadeh and S. S. Ahrabi, Determination of order in linear fractional differential equations, *Calc. Appl. Anal.* **21**, 937-948 (2018).
[19] J. Janno, Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time-fractional diffusion equation, *Electr. J. Differ. Equ.* **216**, 1-28 (2016).
[20] Z. Li and Z. Zhang, Unique determination of fractional order and source term in a fractional diffusion equation from sparse boundary data, (2020).
[21] R. R. Ashurov and Y. E. Fayziev, Determination of fractional order and source term in subdifussion equations, *Euras. Math. J.* **13**, 19-31 (2022).
[22] M. Yamamoto, Uniqueness in determining the order of time and spatial fractional derivatives, (2020).
[23] R. R. Ashurov and R. Zunnunov, Initial-boundary value and inverse problems for subdiffusion equations in R^N, *Fract. Differ. Calc.* **10**, 291-306 (2020).
[24] R. R. Ashurov and S. R. Umarov, Determination of the order of fractional derivative for subdiffusion equations, (2020).
[25] R. R. Ashurov and Y. E. Fayziev, Uniqueness and existence for inverse problem of determining an order of time-fractional derivative of subdiffusion equation, *Lobachevski J. Math.* **42**, 508-516 (2021).
[26] S. A. Alimov and R. R. Ashurov, Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation, *J. Inv. Ill-Posed Probl.* **28**, 651-658 (2020).
[27] R. R. Ashurov and S. R. Umarov, An inverse problem of determining orders of systems of fractional pseudo-differential equations, arXiv:2107.08830 [math.AP].

[28] R. R. Ashurov and R. Zunnunov, Inverse problem for determining the order of the fractional derivative in mixed-type equations, *Lobachevskii J. Math.* **42**, 2714–2729 (2021).
[29] R. R. Ashurov and Y. E. Fayziev, Inverse problem for determining the order of a fractional derivative in a wave equation, *Math. Notes* **110**, 824-836 (2021).
[30] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogozin, *Mittag-Leffler functions, related topics and applications*, Springer, 2014.
[31] R. Ashurov and Y. Fayziev, On construction of solutions of linear fractional differential equations with constant coefficients, *Uzbek Math. J.* **3**, 3-21 (2017).
[32] M. M. Dzherbashian [=Djrbashian], *Integral transforms and representation of functions in the complex domain* (in Russian), M. Nauka, 1966.
[33] H. Bateman, *Higher transcendental functions*, McGraw-Hill, 1953.