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Optimal Coordinated Search Problem for a Randomly Located Target

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Abstract: The purpose of this paper is to study the coordinated search problem in which velocity of the searcher is a random variable. We are interested in finding a hole which randomly located on one of two disjoint oil pipelines or a cut in one of two lines of the power cable under sea surface. There are two searchers on every line that start the searching process from the origin of their line. In addition to calculating the expected value of the detection time, we derived the conditions which can minimize this expected value.

Keywords: Coordinated search, expected value, multiple-searchers, optimal search plan

1 Introduction

The study of search problems for any missing target whether this target is stationary or moving, is considered crucial in many of real-world applications, especially when the target is essential, e.g., searching for explosives in a city or searching for fugitive criminal.

The main aim is to find the lost target in the least possible time or cost. When the lost target is on a line, we obtain the so-called linear search problem [1–8]. The coordinated search is one of many search methods, and it was investigated on straight line [9, 10].

The problem of four searchers searching for a target located on two intersecting lines was investigated [11], and the problem of finding a target located on one of n-disjoint lines was described by the quasi-coordinated search problem, in which the motion of every two searchers are independent of the motion of other searchers while the case of moving target was discussed [12], also a quasi-coordinated search method of a lost target was studied [13], in this case the target is randomly moving on one of the two disjoint lines in accordance with a random walk motion and there are two searchers on each line, moving from the zero point at a constant speed, the authors also studied the problem of finding a lost target is random walker on one of two joint lines, the four searchers start from the point of intersection [14].

In the above discussed literature, the path of the searcher is deterministic, but the case where two searchers looking for a target located on an under-water oil tube with their speed represented by a random variable with a certain probability function was studied [15].

In the present paper we will assume that there is only one target present on one of the two lines (each line represents an oil pipeline or a power cable) under the surface of the water, but these two lines are disjoint and the speed of searchers represented by a random variable with a certain probability function The present paper is organized as follows. In section 2, we define search strategy and calculate the expected value of the time to find the lost target. In section 3, we derive the conditions under which can minimize the expected value. Finally, we summarize our results in section 4.

2 Search Plan

Let X be the position of randomly located target on one of two disjoint lines L_1 and L_2 under sea surface which follow a distribution function, the lines are divided into stop points c_{ij}, d_{ij} where $i = 1,2$ for first and second lines respectively, each line divided into two parts, right par c_{ij}, $i = 1,2$, $j = 0,1,2,\cdots$ and left part d_{ij}, $i = 1,2$, $j = 0,1,2,\cdots$. We have four searchers S_1, S_2, S_3 and S_4, the searchers S_1 and S_2 start together and looking for the target from the point $c_{10} = d_{10} = 0$ on the first line L_1, while searchers S_3 and S_4 start together and looking for the target from point $c_{20} = d_{20} = 0$ on the second line L_2.

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The searchers S_1 and S_3 search in the right part of the lines, while S_2 and S_4 search in the left part, with random velocity v_k, $k = 1,2,3,4$, two searchers in each line will use the modern communication technology to relate with each other instead of return to the origin point, thus there is no standby and return time.

All the searchers start at the same moment following the search plan denoted by $\phi : R \to R^+$ which completely defined by sequences $a = \{a_{ij}, i = 1,2; j = 1,2,\cdots\}$ and $b = \{b_{ij}, i = 1,2; j = 1,2,\cdots\}$.

Let the search plan be defined by $\phi = (a,b) \in \Phi$ where Φ is the set of all search plans. Let

$$\alpha_i = \inf\{x; F(x) > 0\}$$
and (1)
$$\beta_i = \sup\{x; F(x) < 1\},\ i = 1,2,$$

where $F(x)$ is the distribution function of the target's position, α_i is the minimum value of b_{ij} and β_i is the maximum value of a_{ij}.

The searchers S_1, S_2 search on the first line L_1 and S_3, S_4 search on the second line L_2, the searchers S_1, S_3 search in the right part of the two lines and S_2, S_4 search the other parts.

S_1 will search in the right part of the first line as following:

I- Starts from the origin and goes to the positive part of L_1 with distance $a_{11} = c_{11} - c_{10}$ such that $0 < a_{11} < H_{11}^2$ and sends a report to the ship (located at the starting point and receives communications from searchers which use under water audio signals instead of electromagnetic waves) whether the target is found or not.

II- He waits the reply from the ship to decide whether to complete search process or not. If the reply to not complete the searching, then the target has been found by the searcher S_2.

III- Otherwise, S_1 goes to the positive part of L_1 with distance $a_{12} = c_{12} - c_{11}$, then sends a report to the ship and so on until the target gets detected. The searcher S_2 will search in the left part in the same way, but with distance $b_{11} = d_{11} - d_{10}$ such that $0 < b_{11} < \tilde{H}_{11}^2$, $b_{12} = d_{12} - d_{11}$ such that $0 < b_{12} < \tilde{H}_{12}^2$.

IV- Similarly, the searchers S_3 and S_4 do the same thing on the line L_2.

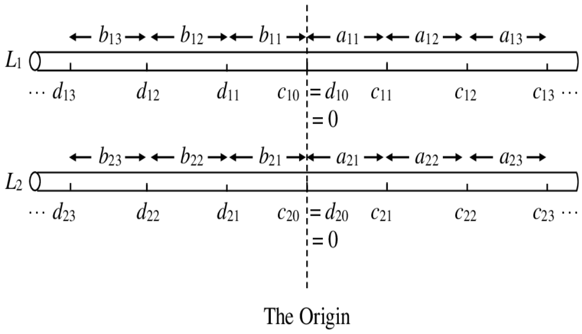

Fig. 1: Coordinated search technique for finding a target

We consider any searcher has a random velocity with probability density function

$$h(v) = |v - v_0|\,\delta(v^2 - v_0), \quad -\infty \le v \le \infty, \tag{2}$$

where δ is dirac delta function and v_0 is the initial velocity, see [15].

We consider the length of the largest distance which traveled by S_1 and S_3 before the i^{th} connection in the right part be H_{1j} and H_{2j} respectively, (in the left part be $\tilde{H}_{1j}$ and $\tilde{H}_{2j}$ for the searchers S_2 and S_4 respectively), the distance which traveled by the searchers S_1, S_3 are random and given by $0 < a_{ij} \le H_{ij}^2$, $i = 1,2$; $j = 1,2,3,\cdots,n$ in the right part where $a_{ij} = c_{ij} - c_{i(j-1)}$ (and $0 < b_{ij} \le \tilde{H}_{ij}^2$ in the left part where $b_{ij} = d_{ij} - d_{i(j-1)}$) we consider the p. d. f. of the random distance see [15] is given by:

$$f_1(a_{ij}) = \frac{1}{H_{ij}\sqrt{a_{ij}}} - \frac{1}{H_{ij}^2},\quad 0 < a_{ij} \le H_{ij}^2, \qquad i = 1,2,;\ j = 1,2,3,\cdots,n \tag{3}$$

$$f_2(b_{ij}) = \frac{1}{\tilde{H}_{ij}\sqrt{b_{ij}}} - \frac{1}{\tilde{H}_{ij}^2},\quad 0 < b_{ij} \le \tilde{H}_{ij}^2, \qquad i = 1,2,;\ j = 1,2,3,\cdots,n \tag{4}$$

The searchers wish to minimize the expected time to detect the target. Let X be a random variable representing the target's location. Since the searchers are moving with random velocity, then they are moving on the line at different distances, depending on the probability that the target is on the line. So, if their exist $\Gamma_i \ge 0$, $i = 1,2$ is a random variable has a known distribution with expected value $E(\Gamma_i)$ then we can consider $\tilde{H}_{ij} = H_{ij} + E(\Gamma_i)$. So, (4) become

$$f_2(b_{ij}) = \frac{1}{(H_{ij} + E(\Gamma_i))\sqrt{b_{ij}}} - \frac{1}{(H_{ij} + E(\Gamma_i))^2}, \tag{5}$$

where $0 < b_{ij} \le (H_{ij} + E(\Gamma_i))^2$. Let γ be a measure of probability caused by the target position and

$\gamma(x,y) = F(y) - F(x)$. Also, let $\Omega(\Phi)$ be the detection time by one of the searchers.

Theorem 2.1. The expected value of detection time is given by

$$\begin{aligned}E[\Omega(\phi)] =& \frac{1}{6}\sum_{j=0}^{\infty}[H^2_{1(j+1)}(\gamma(\alpha_1,d_{1j})) - (H_{1(j+1)} \\ &+ E(\Gamma_1))^2(\gamma(c_{1j},\beta_1)] + [H^2_{2(j+1)}(\gamma(\alpha_2,d_{2j})) \\ &- (H_{2(j+1)} + E(\Gamma_2))^2(\gamma(c_{2j},\beta_2))]. \qquad (6)\end{aligned}$$

Proof. Since, the searchers S_1 and S_2 each one, moves the first line with different and random distance (have pdfs (3) and (5) in the two parts of first line) according the probability of the existence of the target on the probability of the existence of the target on the first line. Then, the expected value of the distance in the right is:

$$E(a_{1j}) = \int_0^{H^2_{1j}} a_{1j}\left(\frac{1}{H_{1j}\sqrt{a_{1j}}} - \frac{1}{H^2_{1j}}\right)da_{1j} = \frac{H^2_{1j}}{6},$$

and in the left part is,

$$\begin{aligned}E(b_{1j}) = &\int_0^{(H_{1j}+E(\Gamma_1))^2} b_{1j}\left(\frac{1}{(H_{1j}+E(\Gamma_1))\sqrt{b_{1j}}}\right. \\ &\left. - \frac{1}{(H_{1j}+E(\Gamma_1))^2}\right)db_{1j} = \frac{(H_{1j}+E(\Gamma_1))^2}{6},\end{aligned}$$

also we assume that the expected value of the velocity is $E(v) = \pm 1$, then the expected value of detection time in the right part is $\Omega_{11} = \frac{E(a_{1j})}{+1} = \frac{H^2_{1j}}{6}$, (in the left part is $\Omega_{12} = \frac{E(b_{1j})}{-1} = -\frac{(H_{1j}+E(\Gamma_1))^2}{6}$).

If the target lies in $]c_{10},c_{11}]$, then

$$\Omega_{12} = -\frac{(H_{11}+E(\Gamma_1))^2}{6}.$$

If the target lies in $]c_{11},c_{12}]$, then

$$\Omega_{12} = -\left[\frac{(H_{11}+E(\Gamma_1))^2 + (H_{12}+E(\Gamma_1))^2}{6}\right].$$

If the target lies in $]c_{12},c_{13}]$, then

$$\begin{aligned}\Omega_{12} = &-[(H_{11}+E(\Gamma_1))^2 + (H_{12}+E(\Gamma_1))^2 + (H_{13} \\ &+ E(\Gamma_1))^2]/6.\end{aligned}$$

If the target lies in $]c_{1(k-1)},c_{1k}]$, then

$$\begin{aligned}\Omega_{12} = &-[(H_{11}+E(\Gamma_1))^2 + (H_{12}+E(\Gamma_1))^2 + \cdots + \\ &+ (H_{1k}+E(\Gamma_1))^2]/6.\end{aligned}$$

Similarly, if the target lies in $[d_{1k},d_{1(k-1)}[$, then

$$\Omega_{11} = \left[\frac{H^2_{11}+H^2_{12}+\cdots+H^2_{1k}}{6}\right],$$

and so on.

$$\begin{aligned}E(\Omega_1(\phi)) = &-\left[\frac{(H_{11}+E(\Gamma_1))^2}{6}\right][\gamma(c_{10},c_{11})] \\ &-\left[\frac{(H_{11}+E(\Gamma_1))^2+(H_{12}+E(\Gamma_1))^2}{6}\right] \\ &[\gamma(c_{11},c_{12})] - [[(H_{11}+E(\Gamma_1))^2 + (H_{12} \\ &+E(\Gamma_1))^2 + \cdots + (H_{1k}+E(\Gamma_1))^2]/6] \\ &[\gamma(c_{1(k-1)},c_{1k})] - \cdots + \frac{H^2_{11}}{6}[\gamma(d_{11},d_{10}] \\ &+ \frac{H^2_{11}+H^2_{12}}{6}[\gamma(d_{12},d_{11})] + \cdots \\ &+ \left[\frac{H^2_{11}+H^2_{12}+\cdots+H^2_{1k}}{6}\right][\gamma(d_{1k},d_{1(k-1)})] \\ = &\frac{H^2_{11}}{6}[(\gamma(d_{11},d_{10}) + \gamma(d_{12},d_{11}) + \cdots \\ &+ \gamma(d_{1k},d_{1(k-1)})] + \frac{H^2_{12}}{6}[(\gamma(d_{12},d_{11}) \\ &+ \cdots + \gamma(d_{1k},d_{1(k-1)})] + \cdots + \frac{H^2_{1k}}{6}[\gamma(d_{1k}, \\ &d_{1(k-1)})] + \cdots - \frac{(H_{11}+E(\Gamma_1))^2}{6}[\gamma(c_{10},c_{11}) \\ &+ \gamma(c_{11},c_{12}) + \cdots + \gamma(c_{1(k-1)},c_{1k})] \\ &- \frac{(H_{12}+E(\Gamma_1))^2}{6}[\gamma(c_{11},c_{12}) + \cdots \\ &+ \gamma(c_{1(k-1)},c_{1k})] - \cdots - \frac{(H_{1k}+E(\Gamma_1))^2}{6} \\ &[\gamma(c_{1(k-1)},c_{1k})] - \cdots \\ = &\frac{H^2_{11}}{6}[\gamma(\alpha_1,d_{10})] + \frac{H^2_{12}}{6}[\gamma(\alpha_1,d_{11})] + \cdots \\ &+ \frac{H^2_{1k}}{6}[\gamma(\alpha_1,d_{1(k-1)})] + \cdots \\ &- \frac{(H_{11}+E(\Gamma_1))^2}{6}[\gamma(c_{10},\beta_1)] \\ &- \frac{(H_{12}+E(\Gamma_1))^2}{6}[\gamma(c_{11},\beta_1)] - \cdots \\ &- \frac{(H_{1k}+E(\Gamma_1))^2}{6}[\gamma(c_{1(k-1)},\beta_1)] - \cdots \\ = &\frac{1}{6}\sum_{j=0}^{\infty}(H^2_{1(j+1)}[\gamma(\alpha_1,d_{1j})] - (H_{1(j+1)} \\ &+ E(\Gamma_1))^2[\gamma(c_{1j},\beta_1)].\end{aligned}$$

Now we will search on the second line according to the probability of existence of the target on the second line, then the expected value of the distance in the right is:

$$E(a_{2j}) = \int_0^{H^2_{2j}} a_{2j}\left(\frac{1}{H_{2j}\sqrt{a_{2j}}} - \frac{1}{H^2_{2j}}\right)da_{2j} = \frac{H^2_{2j}}{6},$$

and in the left part is:

$$E(b_{2j}) = \frac{(H_{2j}+E(\Gamma_2))^2}{6},$$

also, we assume that the expected value of the velocity is

$$E(v) = \pm 1,$$

then the expected value of the detection time in the right part is $\Omega_{21} = \frac{E(a_{2j})}{+1} = \frac{H_{2j}^2}{6}$, (in the left part is $\Omega_{22} = \frac{E(b_{2j})}{-1} = -\frac{(H_{2j}+E(\Gamma_2))^2}{6}$). If the target lies in $]c_{20}, c_{21}]$ then

$$\Omega_{22} = -\frac{(H_{21}+E(\Gamma_2))^2}{6}.$$

If the target lies in $]c_{21}, c_{22}]$ then

$$\Omega_{22} = -\left[\frac{(H_{21}+E(\Gamma_2))^2+(H_{22}+E(\Gamma_2))^2}{6}\right].$$

If the target lies in $[c_{2(k-1)}, c_{2k}]$ then

$$\Omega_{22} = -[(H_{21}+E(\Gamma_2))^2+(H_{22}+E(\Gamma_2))^2+\cdots + (H_{2k}+E(\Gamma_2))^2]/6.$$

Similarly, if the target lies in $[d_{2k}, d_{2(k-1)}[$, then $\Omega_{21} = \left[\frac{H_{21}^2+H_{22}^2+\cdots+H_{2k}^2}{6}\right]$ and so on.

$$\begin{aligned}
E(\Omega_2(\phi)) = &-\left[\frac{(H_{21}+E(\Gamma_2))^2}{6}\right][\gamma(c_{20},c_{21})] \\
&-\left[\frac{(H_{21}+E(\Gamma_2))^2+(H_{22}+E(\Gamma_2))^2}{6}\right] \\
&[\gamma(c_{21},c_{22})]-[[(H_{21}+E(\Gamma_2))^2+(H_{22} \\
&+E(\Gamma_2))^2+\cdots+(H_{2k}+E(\Gamma_2))^2]/6] \\
&[\gamma(c_{2(k-1)},c_{2k})]-\cdots+\frac{H_{21}^2}{6}[\gamma(d_{21},d_{20}] \\
&+\frac{H_{21}^2+H_{22}^2}{6}[\gamma(d_{22},d_{21})]+\cdots \\
&+\left[\frac{H_{21}^2+H_{22}^2+\cdots+H_{2k}^2}{6}\right][\gamma(d_{2k},d_{2(k-1)})] \\
= &\frac{H_{21}^2}{6}[(\gamma(d_{21},d_{20})+\gamma(d_{22},d_{21})+\cdots \\
&+\gamma(d_{2k},d_{2(k-1)})]+\frac{H_{22}^2}{6}[(\gamma(d_{22},d_{21}) \\
&+\cdots+\gamma(d_{2k},d_{2(k-1)})]+\frac{H_{2k}^2}{6}[\gamma(d_{2k}, \\
&d_{2(k-1)})]+\cdots-\frac{(H_{21}+E(\Gamma_2))^2}{6}[\gamma(c_{20},c_{21}) \\
&+\gamma(c_{21},c_{22})+\cdots+\gamma(c_{2(k-1)},c_{2k})] \\
&-\frac{(H_{22}+E(\Gamma_2))^2}{6}\gamma[(c_{21},c_{22})+\cdots \\
&+\gamma(c_{2(k-1)},c_{2k})]-\cdots-\frac{(H_{2k}+E(\Gamma_2))^2}{6} \\
&[\gamma(c_{2(k-1)},c_{2k})]-\cdots \\
= &\frac{H_{21}^2}{6}[\gamma(\alpha_2,d_{20})]+\frac{H_{22}^2}{6}[\gamma(\alpha_2,d_{21})]+\cdots \\
&+\frac{H_{2k}^2}{6}[\gamma(\alpha_2,d_{2(k-1)})]+\cdots \\
&-\frac{(H_{21}+E(\Gamma_2))^2}{6}[\gamma(c_{20},\beta_2)] \\
&-\frac{(H_{22}+E(\Gamma_2))^2}{6}[\gamma(c_{21},\beta_2)]-\cdots \\
&-\frac{(H_{2k}+E(\Gamma_2))^2}{6}[\gamma(c_{2(k-1)},\beta_2)]-\cdots \\
= &\frac{1}{6}\sum_{j=0}^{\infty}(H_{2(j+1)}^2[\gamma(\alpha_2,d_{2j})]-(H_{2(j+1)} \\
&+E(\Gamma_2))^2[\gamma(c_{2j},\beta_2))],
\end{aligned}$$

$$\therefore E(\Omega(\phi)] = E(\Omega_1(\phi))+E(\Omega_2(\phi)) \quad \Rightarrow$$

$$\begin{aligned}
E[\Omega(\phi)] = &\frac{1}{6}\sum_{j=0}^{\infty}[H_{1(j+1)}^2(\gamma(\alpha_1,d_{1j}))-(H_{1(j+1)} \\
&+E(\Gamma_1))^2(\gamma(c_{1j},\beta_1)]+[H_{2(j+1)}^2 \\
&(\gamma(\alpha_2,d_{2j}))-(H_{2(j+1)} \\
&+E(\Gamma_2))^2(\gamma(c_{2j},\beta_2))].
\end{aligned}$$

Since $0 < c_{ij}-c_{i(j-1)} \le H_{ij}^2$, then if there exist $\varepsilon_i \ge 0$, one can get

$$H_{ij}^2 = (c_{ij}-c_{i(j-1)}+\varepsilon_i)^2.$$

Hence we can put equation (6) in this form to find the optimal search path

$$\begin{aligned}
E[\Omega(\phi)] = &\frac{1}{6}\sum_{j=0}^{\infty}[(c_{1(j+1)}-c_{1j}+\varepsilon_1)^2\{\gamma(\alpha_1,d_{1j})\} \\
&-(c_{1(j+1)}-c_{1j}+\varepsilon_1+E(\Gamma_1))^2\{\gamma(c_{1j},\beta_1)\}] \\
&[(c_{2(j+1)}-c_{2j}+\varepsilon_2)^2\{\gamma(\alpha_2,d_{2j})\}-(c_{2(j+1)} \\
&-c_{2j}+\varepsilon_2+E(\Gamma_2))^2\{\gamma(c_{2j},\beta_2)\}]. \qquad (7)
\end{aligned}$$

Also, if their exist $\xi_i \ge 0$ one can get

$$\tilde{H}_{ij}^2 = (d_{ij}-d_{i(j-1)}+\xi_i)^2.$$

Compensate for $H_{ij} = \tilde{H}_{ij}-E(\Gamma_i)$, $H_{ij} = d_{ij}-d_{i(j-1)}-\xi_i-E(\Gamma_i)$, in (6) we get:

$$\begin{aligned}
E[\Omega(\phi)] = &\frac{1}{6}\sum_{j=0}^{\infty}[(d_{1(j+1)}-d_{1j}-\xi_1-E(\Gamma_1))^2 \\
&\{\gamma(\alpha_1,d_{1j})\}-(d_{1(j+1)}-d_{1j}+\xi_1)^2 \\
&\{\gamma(c_{1j},\beta_1)\}]+[(d_{2(j+1)}-d_{2j}-\xi_2 \\
&-E(\Gamma_2))^2\{\gamma(\alpha_2,d_{2j})\}-(d_{2(j+1)}-d_{2j} \\
&+\xi_2)^2\{\gamma(c_{2j},\beta_2)\}]. \qquad (8)
\end{aligned}$$

In this work, we don't need to find the necessary and sufficient conditions that explain the existence of optimal search plan because the target has bounded asymmetric distribution, we will get the optimal values of the points c_{ij} and d_{ij} which afford the optimal path to get the target by using (7) and (8).

3 Optimal search path for bounded asymmetric target distribution.

In this section our main interest is to minimize $E(\Omega(\phi))$, this happens when we find the optimal values of $\{a_{ij}; i = 1,2, j \geq 1\}$ and $\{b_{ij}; i = 1,2, j \geq 1\}$ that give the optimal search path for target's location distribution from class M. If M' is a subclass of M for which only one element and if a^* and b^* are optimal values of a and b; respectively, then the optimal search path will be in M'. It is noted that the search path depends on the target distribution $F(x)$, and the values of a and b that searchers used, and they are two unknown factors, and because the value of a depend on the value of $\{c_{ij}; i = 1,2, j \geq 0\}$ and $\{d_{ij}; i = 1,2, j \geq 0\}$, we will find the optimal values $\{c^*_{ij}; i = 1,2, j \geq 0\}$ and $\{d^*_{ij}; i = 1,2, j \geq 0\}$.

From now we assume that the target distribution $F(x)$ be known and regular (I.e., $F(x)$ is absolutely continuous with positive density $f(x)$) and $E(X) < \infty$. In order to obtain the optimal values $\{c^*_{ij}; i = 1,2,\cdots, j \geq 0\}$ we must solve this non-linear program problem (NLP):

NLP$_{(1)}$

$$\min_{c_{1j}}(c_{1j}-c_{1(j-1)}+\varepsilon_1)^2[\gamma(\alpha_1,d_{1(j-1)})]-(c_{1j}-c_{1(j-1)}+\varepsilon_1+E(\Gamma_1))^2[\gamma(c_{1(j-1)},\beta_1)]+(c_{1(j+1)}-c_{1j}+\varepsilon_1)^2[\gamma(\alpha_1,d_{1j})]-(c_{1(j+1)}-c_{1j}+\varepsilon_1+E(\Gamma_1))^2[\gamma(c_{1j},\beta_1)]$$

sub. to:

$$\frac{c_{1j}-c_{1(j-1)}+\varepsilon_1}{c_{1j}-c_{1(j-1)}}\geq 1,\quad \frac{c_{1(j+1)}-c_{1j}+\varepsilon_1}{c_{1(j+1)}-c_{1j}}\geq 1,$$
$$c_{1j}-c_{1(j-1)}>0,\qquad c_{1(j+1)}-c_{1j}>0,$$
$$\frac{c_{1j}-c_{1(j-1)}+\varepsilon_1+E(\Gamma_1)}{d_{1j}-d_{1(j-1)}}\geq 1,$$
$$\frac{c_{1(j+1)}-c_{1j}+\varepsilon_1+E(\Gamma_1)}{d_{1(j+1)}-d_{1j}}\geq 1,$$
$$d_{1j}-d_{1(j-1)}>0,\qquad d_{1(j+1)}-d_{1j}>0,$$
$$\varepsilon_1\geq 0,\qquad \Gamma_1\geq 0,$$

Definition 2.1. (Optimal solution). For the previous NLP $c^* \in R$ are said to be optimal, if found $c \in R$ such that $g(c^*) \leq g(c), \forall c \in R$.
Then the previous NLP$_{(1)}$ take the form (on the first line):

NLP$_{(2)}$

$$\min_{c_{1j}}(c_{1j}-c_{1(j-1)}+\varepsilon_1)^2[\gamma(\alpha_1,d_{1(j-1)})]-(c_{1j}-c_{1(j-1)}+\varepsilon_1+E(\Gamma_1))^2[\gamma(c_{1(j-1)},\beta_1)]+(c_{1(j+1)}-c_{1j}+\varepsilon_1)^2[\gamma(\alpha_1,d_{1j})]-(c_{1(j+1)}-c_{1j}+\varepsilon_1+E(\Gamma_1))^2[\gamma(c_{1j},\beta_1)]$$

sub. to:

$$1-\frac{c_{1j}-c_{1(j-1)}+\varepsilon_1}{c_{1j}-c_{1(j-1)}}\leq 0,$$
$$1-\frac{c_{1(j+1)}-c_{1j}+\varepsilon_1}{c_{1(j+1)}-c_{1j}}\leq 0,$$
$$c_{1(j-1)}-c_{1j}<0,\qquad c_{1j}-c_{1(j+1)}<0,$$
$$1-\frac{c_{1j}-c_{1(j-1)}+\varepsilon_1+E(\Gamma_1)}{d_{1j}-d_{1(j-1)}}\leq 0,$$
$$1-\frac{c_{1(j+1)}-c_{1j}+\varepsilon_1+E(\Gamma_1)}{d_{1(j+1)}-d_{1j}}\leq 0,$$
$$d_{1(j-1)}-d_{1j}<0,\qquad d_{1j}-d_{1(j+1)}<0,$$
$$-\varepsilon_1\leq 0,\qquad -\Gamma_1\leq 0.$$

from the Kuhn-Tucker conditions, we get

$$2(c_{1j}-c_{1(j-1)}+\varepsilon_1)[\gamma(\alpha_1,d_{1(j-1)})]-2(c_{1j}-c_{1(j-1)}+\varepsilon_1+E(\Gamma_1))[\gamma(c_{1(j-1)},\beta_1)]-2(c_{1(j+1)}-c_{1j}+\varepsilon_1)[\gamma(\alpha_1,d_{1j})]+2(c_{1(j+1)}-c_{1j}+\varepsilon_1+E(\Gamma_1))[\gamma(c_{1j},\beta_1)]+(c_{1(j+1)}-c_{1j}+\varepsilon_1+E(\Gamma_1))^2f(c_{1j})+u_1\big(0-\big[2(c_{1j}-c_{1(j-1)}+\varepsilon_1)(c_{1j}-c_{1(j-1)})-(c_{1j}-c_{1(j-1)}+\varepsilon_1)\big]/((c_{1j}-c_{1(j-1)})^2)\big)+u_2\big(0-\big[2(c_{1(j+1)}-c_{1j}+\varepsilon_1)(c_{1(j+1)}-c_{1j})-(c_{1(j+1)}-c_{1j}+\varepsilon_1)\big]/(c_{1(j+1)}-c_{1j})^2)\big)+u_3\left(0-\frac{1}{d_{1j}-d_{1(j-1)}}\right)+u_4\left(0-\frac{1}{d_{1(j+1)}-d_{1j}}\right)+u_5(-1)+u_6(1)=0, \tag{9}$$

$$u_1\left(1-\frac{c_{1j}-c_{1(j-1)}+\varepsilon_1}{c_{1j}-c_{1(j-1)}}\right)=0, \tag{10}$$

$$u_2\left(1-\frac{c_{1(j+1)}-c_{1j}+\varepsilon_1}{c_{1(j+1)}-c_{1j}}\right)=0, \tag{11}$$

$$u_3\left(1-\frac{c_{1j}-c_{1(j-1)}+\varepsilon_1+E(\Gamma_1)}{d_{1j}-d_{1(j-1)}}\right)=0, \tag{12}$$

$$u_4\left(1-\frac{c_{1(j+1)}-c_{1j}+\varepsilon_1+E(\Gamma_1)}{d_{1(j+1)}-d_{1j}}\right)=0, \tag{13}$$

$$u_5(c_{1(j-1)}-c_{1j})=0, \tag{14}$$

$$u_6(c_{1j}-c_{1(j+1)})=0. \tag{15}$$

There are many cases to solve the equations (9)-(15), but he case which gives the optimal value of $\{c_{1j}; j \geq 0\}$ is the case: $u_1 = u_2 = \cdots = u_6 = 0$. Consequently, the optimal value of $c_{1(j+1)}$ is given after solving the equation,

$$c_{1(j+1)}^2 f(c_{1j}) - c_{1(j+1)}[2\gamma(\alpha_1, d_{1j}) - 2\gamma(c_{1j}, \beta_1) + 2f(c_{1j})(c_{1j} + \varepsilon_1 + E(\Gamma_1))] = \theta_1, \quad (16)$$

where

$$\theta_1 = -2(c_{1j} - c_{1(j-1)} + \varepsilon_1)[\gamma(\alpha_1, d_{1(j-1)})] + 2(c_{1j} - c_{1(j-1)} + \varepsilon_1 + E(\Gamma_1))[\gamma(c_{1(j-1)}, \beta_1)] + 2(-c_{1j} + \varepsilon_1)[\gamma(\alpha_1, d_{1j})] - 2(-c_{1j} + \varepsilon_1 + E(\Gamma_1))[\gamma(c_{1j}, \beta_1)] - (c_{1j} + \varepsilon_1 + E(\Gamma_1))^2 f(c_{1j}).$$

Also, by solving the NLP$_{(3)}$ we get the optimal values of d_{1j}

NLP$_{(3)}$

$$\min_{d_{1j}} (d_{1j} - d_{1(j-1)} + \xi_1 - E(\Gamma_1))^2[\gamma(\alpha_1, d_{1(j-1)})] - (d_{1j} - d_{1(j-1)} + \xi_1)^2[\gamma(c_{1(j-1)}, \beta_1)] + (d_{1(j+1)} - d_{1j} + \xi_1 - E(\Gamma_1))^2[\gamma(\alpha_1, d_{1j})] - (d_{1(j+1)} - d_{1j} + \xi_1)^2[\gamma(c_{1j}, \beta_1)]$$

sub. to:

$$1 - \frac{d_{1j} - d_{1(j-1)} + \xi_1 - E(\Gamma_1)}{d_{1j} - d_{1(j-1)}} \le 0,$$
$$1 - \frac{d_{1(j+1)} - d_{1j} + \xi_1 - E(\Gamma_1)}{d_{1(j+1)} - d_{1j}} \le 0,$$
$$1 - \frac{d_{1j} - d_{1(j-1)} + \xi_1}{d_{1j} - d_{1(j-1)}} \le 0,$$
$$1 - \frac{d_{1(j+1)} - d_{1j} + \xi_1}{d_{1(j+1)} - d_{1j}} \le 0,$$
$$c_{1(j-1)} - c_{1j} < 0, \qquad c_{1j} - c_{1(j+1)} < 0,$$
$$d_{1(j-1)} - d_{1j} < 0, \qquad d_{1j} - d_{1(j+1)} < 0,$$
$$-\xi_1 \le 0, \qquad -\Gamma_1 \le 0.$$

Applying Kuhn-Tucker conditions, and by soling the following equations, we get the optimal values of $d_{1(j+1)}$,

$$d_{1(j+1)}^2 f(d_{1j}) + d_{1(j+1)}[-2\gamma(\alpha_1, d_{1j}) - 2\gamma(c_{1j}, \beta_1) - 2(d_{1j} + \xi_1 + E(\Gamma_1)] f(d_{1j}) = \Lambda_1,$$

where

$$\Lambda_1 = -2(d_{1j} - d_{1(j-1)} + \xi_1 - E(\Gamma_1))[\gamma(\alpha_1, d_{1(j-1)})] + 2(d_{1j} - d_{1(j-1)} + \xi_1)[\gamma(c_{1(j-1)}, \beta_1)] + 2(-d_{1j} + \xi_1 - E(\Gamma_1))[\gamma(\alpha_1, d_{1j})] - 2(-d_{1j} + \xi_1)[\gamma(c_{1j}, \beta_1)] - (d_{1j} + \xi_1 + E(\Gamma_1))^2 f(d_{1j}), \quad (17)$$

by solving the equations (16) and (17) we can get the optimal search path on the first line by the same way in order to obtain the optimal values $\{c_{2j}^*, j \ge 0\}$ and $\{d_{2j}^*, j \ge 0\}$ we must solve the NLP$_{(4)}$ and NLP$_{(5)}$

NLP$_{(4)}$

$$\min_{c_{2j}} (c_{2j} - c_{2(j-1)} + \varepsilon_2)^2[\gamma(\alpha_2, d_{2(j-1)})] - (c_{2j} - c_{2(j-1)} + \varepsilon_2 + E(\Gamma_2))^2[\gamma(c_{2(j-1)}, \beta_2)] + (c_{2(j+1)} - c_{2j} + \varepsilon_2)^2[\gamma(\alpha_2, d_{2j})] - (c_{2(j+1)} - c_{2j} + \varepsilon_2 + E(\Gamma_2))^2[\gamma(c_{2j}, \beta_2)]$$

sub. to:

$$1 - \frac{c_{2j} - c_{2(j-1)} + \varepsilon_2}{c_{2j} - c_{2(j-1)}} \le 0,$$
$$1 - \frac{c_{2(j+1)} - c_{2j} + \varepsilon_2}{c_{2(j+1)} - c_{2j}} \le 0,$$
$$c_{2(j-1)} - c_{2j} < 0, \qquad c_{2j} - c_{2(j+1)} < 0,$$
$$1 - \frac{c_{2j} - c_{2(j-1)} + \varepsilon_2 + E(\Gamma_2)}{d_{2j} - d_{2(j-1)}} \le 0,$$
$$1 - \frac{c_{2(j+1)} - c_{2j} + \varepsilon_2 + E(\Gamma_2)}{d_{2(j+1)} - d_{2j}} \le 0,$$
$$d_{2(j-1)} - d_{2j} < 0, \qquad d_{2j} - d_{2(j+1)} < 0,$$
$$-\varepsilon_2 \le 0, \qquad -\Gamma_2 \le 0,$$

NLP$_{(5)}$

$$\min_{d_{2j}} (d_{2j} - d_{2(j-1)} + \xi_2 - E(\Gamma_2))^2[\gamma(\alpha_2, d_{2(j-1)})] - (d_{2j} - d_{2(j-1)} + \xi_2)^2[\gamma(c_{2(j-1)}, \beta_2)] + (d_{2(j+1)} - d_{2j} + \xi_2 - E(\Gamma_2))^2[\gamma(\alpha_2, d_{2j})] - (d_{2(j+1)} - d_{2j} + \xi_2)^2[\gamma(c_{2j}, \beta_2)]$$

sub. to:

$$1 - \frac{d_{2j} - d_{2(j-1)} + \xi_1 - E(\Gamma_2)}{d_{2j} - d_{2(j-1)}} \le 0,$$
$$1 - \frac{d_{2(j+1)} - d_{2j} + \xi_2 - E(\Gamma_2)}{d_{2(j+1)} - d_{2j}} \le 0,$$
$$1 - \frac{d_{2j} - d_{2(j-1)} + \xi_2}{d_{2j} - d_{2(j-1)}} \le 0,$$
$$1 - \frac{d_{2(j+1)} - d_{2j} + \xi_2}{d_{2(j+1)} - d_{2j}} \le 0,$$
$$c_{2(j-1)} - c_{2j} < 0, \qquad c_{2j} - c_{2(j+1)} < 0,$$
$$d_{2(j-1)} - d_{2j} < 0, \qquad d_{2j} - d_{2(j+1)} < 0,$$
$$-\xi_2 \le 0, \qquad -\Gamma_2 \le 0.$$

Since, we have two lines that do not intersect, then

$$\min(\Omega(\phi)) = \min(\Omega_1(\phi) + \min(\Omega_2(\phi).$$

4 Conclusion and future work

The coordinated search method is discussed to find a randomly located target on one of two disjoint lines. The target's position has a known probability distribution. the expected value of the detection time until one of the searchers return on the origin is obtained. Also, the conditions that give the optimal detection distances to minimize this expected value is discussed.

In the future work, this technique will be extended on the plane when the target's distribution is bounded in each searching step.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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