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A Multi-Index Generalized Derivative; Some Introductory Notes

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Abstract: In this work we present a generalized multi-index derivative, which contains as particular cases, with an index, several local derivatives known from the literature (both conformable and non-conformable). Obviously this new development contains many of the desirable properties of integer derivatives. It is worth noting that the multi-index differential operator departs from the classic single-order local derivatives, as in the case of the particular cases presented, which leads to an adequate and useful mathematical tool to generalize widely accepted results, with applications potentials to Physics, fundamental within mathematical simulation.

Keywords: Fractional derivatives and integral, generalized derivative, fractional calculus

1 Introduction

We know that the Fractional Calculus is contemporary of the Ordinary Calculus, classical, of integer order, this area, together with the generalized calculus are areas in expansion and continuous development today. In the last 50 years, they have focused the attention of pure and applied researchers and today they constitute one of the most dynamic areas of Mathematical Sciences [1]. One of the first operators that can be called fractional is the Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$, with $\Re(\alpha) > 0$, defined as follows [2].

Let $\tau_1 < \tau_2$ and $\mathrm{w} \in L^1((\tau_1, \tau_2); \mathbb{R})$. The right and left side Riemann-Liouville fractional integrals of order α, with $\Re(\alpha) > 0$, are defined, respectively, by

$$ {}^{\mathrm{RL}}\mathscr{J}^{\alpha}_{\tau_1+}\mathrm{w}(\mathrm{t}) = \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\mathrm{t}} (\mathrm{t}-\eta)^{\alpha-1}\mathrm{w}(\eta)\,\mathrm{d}\eta, \qquad (1)$$

and

$$ {}^{\mathrm{RL}}\mathscr{J}^{\alpha}_{\tau_2-}\mathrm{w}(\mathrm{t}) = \frac{1}{\Gamma(\alpha)} \int_{\mathrm{t}}^{\tau_2} (\eta-\mathrm{t})^{\alpha-1}\mathrm{w}(\eta)\,\mathrm{d}\eta, \qquad (2)$$

with $\mathrm{t} \in (\tau_1, \tau_2)$. As we know, by manipulating simple algebraic identities, we can follow the idea of fractional differential operators of Riemann-Liouville or Caputo type. Considering $\alpha = 1 + \alpha - 1$ or $\alpha = \alpha - 1 + 1$, respectively.

It is important to note that the global fractional derivatives (e.g., Caputo and Riemann-Liouville) are not collecting mere local information. By contrast, fractional operators keep track of the history of the process being studied; this feature allows modeling the non-local and distributed responses that commonly appear in natural and physical phenomena. On the other side, one has to recognize that these fractional derivatives D^{α} show some drawbacks, this made new formulations and extensions necessary, which resulted in the appearance of the Local or Generalized Fractional Calculus, in the 60s of the last century and that we will see later. An attractive characteristic of this field is that there are numerous fractional operators, and this permits researchers to choose the most appropriate operator for the sake of modeling the problem under investigation, in [3] a fairly complete classification of these fractional operators is presented, with abundant information, on the other hand, in the work [4] some reasons are presented why new operators linked to applications and developments theorists appear every day. These operators had been developed by numerous mathematicians with a barely

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specific formulation, for instance, the Riemann-Liouville (RL), the Weyl, Erdelyi-Kober, Hadamard integrals, and the Liouville and Katugampola fractional operators and many authors have introduced new fractional operators generated from general classical local derivatives.

In addition, Chapter 1 of [5] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, Section 1.4 LIMITATIONS AND STRENGTH OF LOCAL AND FRACTIONAL DERIVATIVES concludes "We can therefore conclude that both the Riemann?Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [6] that, the local fractional operator is not a fractional derivative" (p.24). As we said before, they are new tools that have demonstrated their usefulness and potential in the modeling of different processes and phenomena.

Although local fractional derivatives have been used since the 1960s, it was not until 2014 when they were formalized with the work [7], where a local derivative, called conformable, is defined as follows. Given a function $\mathrm{w}:[0,+\infty)\to\mathbb{R}$, then the conformable fractional derivative of w of order α, with $0<\alpha\leq 1$, is defined by

$$\mathscr{T}_{\alpha}\mathrm{w}(\mathrm{t})=\lim_{\varepsilon\to 0}\frac{\mathrm{w}\left(\mathrm{t}+\varepsilon \mathrm{t}^{(1-\alpha)}\right)-\mathrm{w}(\mathrm{t})}{\varepsilon},\qquad \mathrm{t}>0. \quad (3)$$

*Remark.*If w is α-differentiable in some $0<\alpha\leq 1$, and $\lim_{\mathrm{t}\to 0^+}\mathscr{T}_{\alpha}\mathrm{w}(\mathrm{t})$ exists, then define $\mathscr{T}_{\alpha}\mathrm{w}(0)=\lim_{t\to 0^+}\mathscr{T}_{\alpha}\mathrm{w}(\mathrm{t})$. Additionally we have if w is differentiable then

$$\mathscr{T}_{\alpha}\mathrm{w}(\mathrm{t})=\mathrm{w}'(\mathrm{t})\mathrm{t}^{(1-\alpha)},$$

of the latter we see that if $\alpha\to 1$ we obtain the classical derivative.

Later, in 1918, the authors in [8] presented a fractional local derivative of a new type [9]. Let $\mathrm{w}:[0,+\infty)\to\mathbb{R}$ a function. Then the $\mathscr{N}$-derivative of w of order α is defined by

$$\mathscr{N}_1^{\alpha}\mathrm{w}(\mathrm{t})=\lim_{\varepsilon\to 0}\frac{\mathrm{w}\left(\mathrm{t}+\varepsilon e^{\mathrm{t}^{-\alpha}}\right)-\mathrm{w}(\mathrm{t})}{\varepsilon},$$

forall $\mathrm{t}>0$, $\alpha\in(0,1)$. If w is $\alpha-$differentiable in some $(0,\tau_1)$, and $\lim_{\mathrm{t}\to 0^+}\mathscr{N}_1^{(\alpha)}\mathrm{w}(\mathrm{t})$ exists, then define

$$\mathscr{N}_1^{(\alpha)}\mathrm{w}(0)=\lim_{\mathrm{t}\to 0^+}\mathscr{N}_1^{(\alpha)}\mathrm{w}(\mathrm{t}).$$

Lemma 1.*Let $\mathrm{w}:[0,+\infty)\to\mathbb{R}$ be differentiable, then*

$$\mathscr{N}_1^{\alpha}\mathrm{w}(\mathrm{t})=e^{\mathrm{t}^{-\alpha}}\mathrm{w}'(\mathrm{t}). \quad (4)$$

*Remark.*The authors justify the "non-conformable" term with which they named it, since from (4) we obtain that when $\alpha\to 1$ the ordinary derivative is not obtained and, therefore, the slope of the tangent line to the curve at the point is not maintained.

A generalized derivative was defined in [10] in the following way. Let $\mathrm{w}:[0,+\infty)\to\mathbb{R}$, $\alpha\in(0,1)$ and $\mathrm{V}(\mathrm{t},\alpha)$ be some absoluty continuous function on $I\times(0,1]$. Then, the $\mathscr{N}$-derivative of w of order α is defined by

$$\mathscr{N}_{\mathrm{V}}^{\alpha}\mathrm{w}(\mathrm{t})=\lim_{\varepsilon\to 0}\frac{\mathrm{w}(\mathrm{t}+\varepsilon\mathrm{V}(\mathrm{t},\alpha))-\mathrm{w}(\mathrm{t})}{\varepsilon},\qquad \mathrm{t}>0. \quad (5)$$

Here we will use some cases of V defined using the $\mathfrak{E}_{\tau_1,\tau_2}(.)$, the classic definition of Mittag–Leffler function with $\mathfrak{R}(\tau_1)$, $\mathfrak{R}(\tau_2)>0$. Also we consider $\mathfrak{E}_{\tau_1,\tau_2}(.)_k$ is the k-th term of $\mathfrak{E}_{\tau_1,\tau_2}(.)$. If w is α-differentiable in some $0<\alpha\leq 1$, and $\lim_{\mathrm{t}\to 0^+}\mathscr{N}_{\mathrm{V}}^{\alpha}\mathrm{w}(\mathrm{t})$ exists, then define

$$\mathscr{N}_{\mathrm{V}}^{\alpha}\mathrm{w}(0)=\lim_{\mathrm{t}\to 0^+}\mathscr{N}_{\mathrm{w}}^{\alpha}\mathrm{w}(\mathrm{t}).$$

*Remark.*This generalized derivative has proven its usefulness in various applications, to get an idea we recommend consulting [11,12,13,14].

*Remark.*If the kernel of the previous definition is $\mathrm{V}\equiv 1$ the classical derivative is obtained. By other hand, if we replace ε with $\varepsilon\mathrm{V}(\mathrm{t},\alpha)$ in (5) then

$$\mathscr{N}_{\mathrm{V}}^{\alpha}\mathrm{w}(\mathrm{t})=\lim_{\varepsilon\to 0}\frac{\mathrm{w}(\mathrm{t}+\varepsilon)-\mathrm{w}(\mathrm{t})}{\varepsilon}\mathrm{V}(\mathrm{t},\alpha),$$

if w is differentiable then $\mathscr{N}_{\mathrm{V}}^{\alpha}\mathrm{w}(\mathrm{t})=\mathrm{w}'(\mathrm{t})\mathrm{V}(\mathrm{t},\alpha)$.

The following is a result that distinguishes local derivatives from global classical ones.

Theorem 1.*Let $\alpha\in(0,1]$, $\acute{\mathrm{w}}$ $\mathscr{N}$-differentiable at $\mathrm{t}>0$ and w differentiable at $\acute{\mathrm{w}}(\mathrm{t})$ then*

$$\mathscr{N}_{\mathrm{V}}^{\alpha}(\mathrm{w}\circ\acute{\mathrm{w}})(\mathrm{t})=\mathrm{w}'(\acute{\mathrm{w}}(\mathrm{t}))\mathscr{N}_{\mathrm{V}}^{\alpha}\acute{\mathrm{w}}(\mathrm{t}).$$

*Remark.*From the above definition, it is not difficult to extend the order of the derivative for $0\leq n-1<\alpha\leq n$ by putting

$$\mathscr{N}_{\mathrm{V}}^{\alpha}\mathrm{w}(\mathrm{t})=\lim_{\varepsilon\to 0}\frac{\mathrm{w}^{(n-1)}(\mathrm{t}+\varepsilon\mathrm{V}(\mathrm{t},\alpha))-\mathrm{w}^{(n-1)}(\mathrm{t})}{\varepsilon}. \quad (6)$$

If $\mathrm{w}^{(n)}$ exists on some interval $I\subseteq\mathbb{R}$, then we have

$$\mathscr{N}_{\mathrm{V}}^{\alpha}\mathrm{w}(\mathrm{t})=\mathrm{V}(\mathrm{t},\alpha)\mathrm{w}^{(n)}(\mathrm{t}),\qquad 0\leq n-1<\alpha\leq n.$$

We can defined the following associate integral (see [15]). Throughout the work we will consider that the integral operator kernel V defined below is an absolutely continuous function. Let I be an interval $I\subseteq\mathbb{R}$, $\tau_1,\mathrm{t}\in I$ and $\alpha\in\mathbb{R}$. The integral operator $\mathscr{J}_{\mathrm{V},\tau_1}^{\alpha}$ is defined for every locally integrable function w on I and an absolutly continuous V on $I\times(0,1]$ as

$$\mathscr{I}^{\alpha}_{V,\tau_1}(w)(t)=\int_{\tau_1}^{t}\frac{w(\eta)}{V(\eta,\alpha)}d\eta=\int_{\tau_1}^{t}w(\eta)d_V\eta,\qquad t>\tau_1. \tag{7}$$

Obviously

$$\mathscr{I}^{\alpha}_{V,t}(w)(\tau_1)=\int_{t}^{\tau_1}\frac{w(\eta)}{V(\eta,\alpha)}d\eta=-\mathscr{I}^{\alpha}_{V,\tau_1}(w)(t),$$

$$\mathscr{I}^{\alpha}_{V,\tau_1}(w)(\tau_2)=\int_{\tau_1}^{\tau_2}\frac{w(\eta)}{V(\eta,\alpha)}d\eta=\mathscr{I}^{\alpha}_{V,\tau_1}(w)(t)+\mathscr{I}^{\alpha}_{V,t}(w)(\tau_2).$$

Throughout the work we use the functions Γ (see [16]) and Γ_k (see [17]):

$$\Gamma(t)=\int_0^{\infty}\eta^{t-1}e^{-\eta}d\eta,\qquad \Re(t)>0,$$

$$\Gamma_k(t)=\int_0^{\infty}\eta^{t-1}e^{-\frac{\eta^k}{k}}d\eta,\qquad \Re(t)>0,k>0.$$

It is clear that if $k\to 1$ we have $\Gamma_k(t)\to\Gamma(t)$, $\Gamma_k(t+k)=t\Gamma_k(t)$ and

$$\Gamma_k(zt)=(k)^{\frac{t}{k}-1}\Gamma\left(\frac{t}{k}\right).$$

Unlike the rest of the known local derivatives, the order of our operator is determined by n fractional indices and n positive functions, which can be rearranged to retrieve the integer derivative. The n indexes and the n functions give the generalized derivative greater freedom and a more complex dynamics than the derivatives of a single index and single kernel, in addition, being a very general derivative, it may be the case that it is conformable for some indices and non-conformable for others, it is a local operator as in the case of integer order. Thus, our operator is similar to the Gateaux Derivative (see [18]), but it is definitely not the same. We present a differential operator in which the order of the derivative now depends on multi-fractional indices, preserving almost all the properties of integer-order derivatives, as is often the case with generalized local derivatives. At the end of the work, we present potential physical applications of the defined operator and some methodological remarks.

2 Main Results

Now we are in a position to define the differential operator, the central object of this work.

Definition 1. *Let* $w:[0,+\infty)\to\mathbb{R}$, $\alpha_i\in(0,1]$ *for* $i=1,2,\cdots,n$ *and* $V(t,\alpha)$ *be some absolutly continuous function on* $I\times(0,1]$. *Then, the* $\mathscr{N}$ *generalized derivative multi-index of* $\mathscr{N}$ *of order* $\alpha_1+\alpha_2+\cdots+\alpha_n$ *is defined by*

$$\mathscr{N}_V^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}w(t)=\lim_{\varepsilon\to 0}\frac{w(t+\varepsilon V(t,\alpha_1,\alpha_2,\ldots,\alpha_n))-w(t)}{\varepsilon}, \tag{8}$$

for $t>0$.

For the kernel function, let's consider the following characteristics:

–$V(t,\alpha_1,\alpha_2,\ldots,\alpha_n)\neq 0,\forall t\in\mathbb{R}^+$;
–$V(t,\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n)\neq V(t,\alpha_1,\ldots,\alpha_j,\ldots,\alpha_n)$, $\forall i\neq j$, *with* $i,j=1,2,\ldots,n$

Here we will use some cases of F defined using the $\mathfrak{E}_{\tau_1,\tau_2}(.)$, *the classic definition of Mittag–Leffler function with* $\Re(\tau_1)$, $\Re(\tau_2)>0$. *Also we consider* $\mathfrak{E}_{\tau_1,\tau_2}(.)_k$ *is the k-th term of* $\mathfrak{E}_{\tau_1,\tau_2}(.)$. *If* w *is* $\mathscr{N}$ *generalized differentiable for* $0<\alpha_1,\alpha_2,\ldots,\alpha_n\leq 1$, *and*

$$\lim_{t\to 0^+}\mathscr{N}_V^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}w(t),$$

exists, then define

$$\mathscr{N}_V^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}w(0)=\lim_{t\to 0^+}\mathscr{N}_V^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}w(t).$$

Remark. In the event that it is fulfilled

$$\lim_{(\alpha_1,\alpha_2,\ldots,\alpha_n)\to(1,1,\ldots,1)}V(t,\alpha_1,\alpha_2,\ldots,\alpha_n)=1, \tag{9}$$

then we will say that it is a conformable $\mathscr{N}$ generalized derivative multi-index, otherwise, we will say that it is non-conformable.

Remark. We consider $n=1$, if $V(t,\alpha)=t^{1-\alpha}$ then from Definition 1 we obtain the conformable derivative of Khalil *et al.* Other kernel choices give us different known local derivatives (see [19,20] for example).

Remark. If $n=2$, we obtain the "$\alpha_1\alpha_2$ derivative" and with

$$\varepsilon V(t,\alpha_1,\alpha_2)=w(t)^{\alpha_2}\mathfrak{E}_{1,1}[\varepsilon k^{1-\alpha_1}(t)],$$

then we have the "α_1,α_2hkl conformable derivative" of w of order $\alpha_1+\alpha_2$ (see [21]).

Remark. The important thing about the Definition 1 and that it shows its generality and scope, is the fact that we can include generalized derivatives of a new type (not reported in the literature), "mixed", since they can be conformable for indices and non-conformables. Of course, the result will be a non-conformable derivative since it does not satisfy (9).

Theorem 2. *Let* w_1 *and* w_2 *be* $\mathscr{N}$ *generalized derivative multi-index at a point* $t>0$ *and* $\alpha_i\in(0,1]$, $\forall i=1,\ldots,n$. *Then*

a) $\mathscr{N}_V^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\xi_1w_1+\xi_2w_2)(t)=\xi_1\mathscr{N}_V^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(w_1)(t)+\xi_2\mathscr{N}_V^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(w_2)(t)$;

b) $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{t}^r) = \mathrm{V}(\mathrm{t},\alpha_1,\alpha_2,\ldots,\alpha_n)r\mathrm{t}^{r-1}$, $r\in\mathbb{R}$;

c) $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(r) = 0$, $r\in\mathbb{R}$;

d) $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w}_1\mathrm{w}_2)(\mathrm{t}) = \mathrm{w}_1\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w}_2)(\mathrm{t}) + \mathrm{w}_2\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w}_1)(\mathrm{t})$;

e) $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\frac{\mathrm{w}_1}{\mathrm{w}_2})(\mathrm{t}) = \frac{\mathrm{w}_2\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w}_2)(\mathrm{t}) - \mathrm{w}_1\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w}_2)(\mathrm{t})}{\mathrm{w}_2^2(\mathrm{t})}$;

f) If, in addition, w_1 *is differentiable then* $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w}_1) = \mathrm{V}(\mathrm{t},\alpha_1,\alpha_2,\ldots,\alpha_n)\mathrm{w}_1'(\mathrm{t})$.

One of the basic results in the applications of generalized derivatives are the following results. The proof of the next theorem is obtained very easily, following the classical way.

Theorem 3(Chain Rule). *Let* w_2 *be a* $\mathscr{N}$ *generalized derivative multi-index function at a point* $\mathrm{t}>0$, w_1 $\mathscr{N}$ *generalized derivative multi-index function at point* $\mathrm{w}_2(\mathrm{t})>0$ *and* $\alpha_i\in(0,1]$, $\forall i=1,\ldots,n$, *then we have*

$$\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w}_1\circ\mathrm{w}_2)(\mathrm{t}) = \frac{d\mathrm{w}_1(\mathrm{w}_2(\mathrm{t}))}{d\mathrm{t}}\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}_2(\mathrm{t}).$$

Theorem 4(Rolle's Theorem). *Let* $\mathrm{w}\in C([\tau_1,\tau_2],\mathbb{R})$, *with* $\tau_1>0$, *be a given function that satisfies*

i) w *is* $\mathscr{N}$ *generalized derivative multi-index function on* (τ_1,τ_2) *for* $\alpha_i\in(0,1]$, $\forall i=1,\ldots,n$;
ii) $\mathrm{w}(\tau_1)=\mathrm{w}(\tau_2)$.

Then, there exists $\mathfrak{s}\in(\tau_1,\tau_2)$ *such that* $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s})=0$.

Proof. We prove this using contradiction. From assumptions, since w is continuous in $[\tau_1,\tau_2]$, and $\mathrm{w}(\tau_1)=\mathrm{w}(\tau_2)$, there is $\mathfrak{s}\in(\tau_1,\tau_2)$, at least one, which is a point of local extreme. By other hand, how w is $\mathscr{N}$ generalized derivative multi-index function in (τ_1,τ_2) for $\alpha_i\in(0,1]$, $\forall i=1,\ldots,n$ we have

$$\begin{aligned}
&\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s})\\
&= \mathscr{N}_1^{\alpha_1}\mathrm{w}(\mathfrak{s}^+)\\
&= \lim_{h\to 0^+}\frac{\mathrm{w}(\mathfrak{s}+h\mathrm{V}(\mathfrak{s},\alpha_1,\alpha_2,\ldots,\alpha_n))-\mathrm{w}(\mathfrak{s})}{h}\\
&= \mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s}^-)\\
&= \lim_{h\to 0^-}\frac{\mathrm{w}(\mathfrak{s}+h\mathrm{V}(\mathfrak{s},\alpha_1,\alpha_2,\ldots,\alpha_n))-\mathrm{w}(\mathfrak{s})}{h},
\end{aligned}$$

but $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s}^+)$ and $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s}^-)$ have opposite signs. Hence $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s})=0$. If $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s}^+)$ and $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s}^-)$ they have the same sign then as $\mathrm{w}(\tau_1)=\mathrm{w}(\tau_2)$, we have that w is constant and the result is trivially followed. This concludes the proof.

Theorem 5(Mean Value Theorem). *Let* $\mathrm{w}\in C([\tau_1,\tau_2],\mathbb{R})$, *with* $\tau_1>0$, *be a function satisfies* w *is* $\mathscr{N}$ *generalized derivative multi-index function on* (τ_1,τ_2), $\alpha_i\in(0,1]$, $\forall i=1,\ldots,n$. *Then, exists* $\mathfrak{s}\in(\tau_1,\tau_2)$ *such that*

$$\begin{aligned}
&\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s})\\
&= \left[\frac{\alpha_1,\alpha_2,\ldots,\alpha_n(\tau_2)-\alpha_1,\alpha_2,\ldots,\alpha_n(\tau_1)}{\tau_2-\tau_1}\right]\mathrm{V}(\mathfrak{s},\alpha_1,\alpha_2,\ldots,\alpha_n).
\end{aligned}$$

Proof. Consider the function

$$\acute{\mathrm{w}}(\mathrm{t}) = \mathrm{w}(\mathrm{t})-\mathrm{w}(\tau_1)-\left[\frac{\mathrm{w}(\tau_2)-\mathrm{w}(\tau_1)}{\tau_2-\tau_1}\right](\mathrm{t}-\tau_1).$$

The auxiliary function w_2 satisfies all the conditions of Rolle's Theorem and, therefore, exists $\tau_1<\mathfrak{s}<\tau_2$ such that $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\acute{\mathrm{w}}(\mathfrak{s})=0$. Then, we have

$$\begin{aligned}
&\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\acute{\mathrm{w}}(\mathrm{t})\\
&= \mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w}(\mathrm{t})-\mathrm{w}(\tau_1))\\
&\quad -\frac{\mathrm{w}(\tau_2)-\mathrm{w}(\tau_1)}{\tau_2-\tau_1}\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{t}-\tau_1),
\end{aligned}$$

and from here it follows that

$$\begin{aligned}
&\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\acute{\mathrm{w}}(\mathfrak{s})\\
&= \mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathfrak{s})\\
&\quad -\frac{\mathrm{w}(\tau_2)-\mathrm{w}(\tau_1)}{\tau_2-\tau_1}\mathrm{V}(c,\alpha_1,\alpha_2,\ldots,\alpha_n)=0,
\end{aligned}$$

from where

$$\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}[\mathrm{w}(\mathfrak{s})] = \frac{\mathrm{w}(\tau_2)-\mathrm{w}(\tau_1)}{\tau_2-\tau_1}\mathrm{V}(\mathfrak{s},\alpha_1,\alpha_2,\ldots,\alpha_n).$$

This concludes the proof.

Theorem 6. *Let* $\mathrm{w}\in C([\tau_1,\tau_2],\mathbb{R})$, *with* $\tau_1>0$, *be a given function that satisfies* w *is* $\mathscr{N}$ *generalized derivative multi-index function on* (τ_1,τ_2), $\alpha_i\in(0,1]$, $\forall i=1,\ldots,n$. *If* $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathrm{t})=0$ *for all* $\mathrm{t}\in(\tau_1,\tau_2)$, *then* w *is a constant on* $[\tau_1,\tau_2]$.

Proof. It is sufficient to apply the theorem of the mean value to the function w over any non-degenerate interval contained in $[\tau_1,\tau_2]$.

As a consequence of the previous theorem we have.

Corollary 1. *Let* $a>0$ *and* $\mathrm{V}_1,\mathrm{V}_2:[\tau_1,\tau_2]\to\mathbb{R}$ *be functions such that for all* $\alpha\in(0,1)$,

$$\mathscr{N}_{\mathrm{V}_1}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{V}_1(\mathrm{t}) = \mathscr{N}_{\mathrm{V}_1}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{V}_2(\mathrm{t}), \qquad \forall\mathrm{t}\in(\tau_1,\tau_2).$$

Then there exist a constant $\mathfrak{s}$ *such that* $\mathrm{V}_1(\mathrm{t})=\mathrm{V}_2(\mathrm{t})+\mathfrak{s}$.

*Remark.*Something that we must point out is that this generalized operator does not satisfy the Index Law, valid in the fractional case. That is to say,

$$\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\left(\mathscr{N}_{\mathrm{V}}^{\langle\beta_1,\beta_2,\ldots,\beta_n\rangle}\right)\neq\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1+\beta_1,\alpha_2+\beta_2,\ldots,\alpha_n+\beta_n\rangle},$$

unless the kernel V is an additive function, with respect to the second variable.

Let $C^1[\tau_1,\tau_2]$ be the set of functions f with first ordinary derivative continuous on $[\tau_1,\tau_2]$, we consider the following norms on $C^1[\tau_1,\tau_2]$,

$$\|\mathrm{V}\|_C=\max_{[\tau_1,\tau_2]}|\mathrm{w}(\mathrm{w})|,$$

$$\|\mathrm{V}\|_{C^1}=\left\{\max_{[\tau_1,\tau_2]}|\mathrm{w}(\mathrm{t})|+\max_{[\tau_1,\tau_2]}\left|\mathrm{w}'(\mathrm{t})\right|\right\}.$$

Theorem 7.*For a function* $\mathrm{w}\in C^1[\tau_1,\tau_2]$ *and* $\mathrm{t}\in[\tau_1,\tau_2]$, *we have*

$$\left|\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathrm{t})\right|\leq K(\alpha_1,\alpha_2,\ldots,\alpha_n)\|\mathrm{V}\|_C\max_{\mathrm{t}\in[\tau_1,\tau_2]}|\mathrm{w}(\mathrm{t})|. \tag{10}$$

*Remark.*The constant $K(\alpha_1,\alpha_2,\ldots,\alpha_n)$ of the theorem can depend on other parameters, as in the case of the Katugampola operator ($n=1$), where ρ will appear. It is easily obtained from the previous definitions.

Theorem 8.*The derivative generalized multi-index* $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathrm{t})$ *is a bounded operators from* $C^1[\tau_1,\tau_2]$ *to* $C[\tau_1,\tau_2]$ *with*

$$\left|\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathrm{t})\right|\leq K\|\mathrm{V}\|_C\|\mathrm{w}\|_{C^1}. \tag{11}$$

where the constant K, may be depend of derivative frame considered.

*Proof.*Given $\mathrm{t}\in[\tau_1,\tau_2]$ and $\mathrm{w}\in C^1[\tau_1,\tau_2]$, using simple properties of norm and previous theorem, the result follows.

*Remark.*From previou results we obtain that the derivative $\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathrm{t})$ is well defined.

Taking into account the previous ideas, we can define the generalized multi-index partial derivatives as follows.

Definition 2.*Given a real valued function* $\mathrm{w}:\mathbb{R}^n\to\mathbb{R}$ *and* $\overrightarrow{a}=(a_1,\ldots,a_n)\in\mathbb{R}^n$ *a point whose ith component is positive. Then the N-derivative generalized multi-index partial of* w *of order* $\alpha_1,\alpha_2,\cdots,\alpha_n$ *in the point* $\overrightarrow{a}=(a_1,\cdots,a_n)$ *is defined by*

$$\mathscr{N}_{\mathrm{V}_i,\mathrm{t}_i}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\overrightarrow{a})=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big[\mathrm{w}\Big(a_1,\ldots,a_i+\varepsilon\mathrm{V}_i(a_i,\alpha_1,\alpha_2,\ldots,\alpha_n),\ldots,a_n\Big)-\mathrm{w}(a_1,\ldots,a_i,\ldots,a_n)\Big], \tag{12}$$

if it exists, is denoted $\mathscr{N}_{\mathrm{V}_i,\mathrm{t}_i}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\overrightarrow{a})$, *and called the ith* $\mathscr{N}$ *generalized partial derivative multi-index of* w *of the order* $\alpha_1,\alpha_2,\ldots,\alpha_n$ *at* $\overrightarrow{a}$.

*Remark.*If a real valued function f with n variables has all generalized partial derivatives of the order $\alpha_1,\alpha_2,\ldots,\alpha_n$ at $\overrightarrow{a}$, each $a_i>0$, then the generalized $\alpha_1,\alpha_2,\ldots,\alpha_n$-gradient multi-index of w of the order $\alpha_1,\alpha_2,\ldots,\alpha_n$ at $\overrightarrow{a}$ is

$$\nabla_{\mathscr{N}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\overrightarrow{a})=\left(\mathscr{N}_{\mathrm{t}_1}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}f(\overrightarrow{a}),\ldots,\mathscr{N}_{\mathrm{t}_n}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\overrightarrow{a})\right). \tag{13}$$

Now, we give the definition of a general fractional integral. Throughout the work we will consider that the integral operator kernel V defined below is an absolutely continuous function.

Definition 3.*Let* $I\subseteq\mathbb{R}$, $\tau_1,\mathrm{t}\in I$ *and* $\alpha_i\in(0,1]$, $\forall i=1,\ldots,n$. *The integral operator* $J_{\mathrm{V},\tau_1}^{\langle\beta_1,\beta_2,\ldots,\beta_n\rangle}$, *is defined for every locally integrable function* w *on I as*

$$\begin{aligned}\left[\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\mathrm{w}(\mathrm{t})\right]^{-1}&=J_{\mathrm{V},\tau_1}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w})(\mathrm{t})\\&=\int_{\tau_1}^{\mathrm{t}}\frac{\mathrm{w}(\eta)}{\mathrm{V}(\eta,\alpha_1,\alpha_2,\ldots,\alpha_n)}\mathrm{d}\eta\\&=\int_{\tau_1}^{\mathrm{t}}\mathrm{w}(\eta)\,\mathrm{d}d_F\eta,\qquad \mathrm{t}>\tau_1.\end{aligned} \tag{14}$$

*Remark.*It is easy to see that the case of the J_{V}^{α} operator defined above contains ($n=1$), as particular cases, the integral operators obtained from conformable and non-conformable local derivatives and even, for appropriate kernels, it may contain known fractional integral operators.

We can define the function space $L_{\alpha}^{r}[\tau_1,\tau_2]$ as the set of functions over $[\tau_1,\tau_2]$ such that

$$\left(J_{\mathrm{V},\tau_1}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}[\mathrm{w}(\mathrm{t})]^r(\tau_2)\right)<+\infty.$$

The following two results establish the relationship between the generalized operators defined above.

Theorem 9.*Let* $I\subseteq\mathbb{R}$, $\tau_1\in I$, $\alpha_i\in(0,1]$, $\forall i=1,\ldots,n$ *and* w *a* $\mathscr{N}$ *generalized derivative multi-index function on I such that* w' *is a locally integrable function on I. Then, we have for all* $\mathrm{t}\in I$

$$J_{\mathrm{V},\tau_1}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}\left(\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle}(\mathrm{w})\right)(\mathrm{t})=\mathrm{w}(\mathrm{t})-\mathrm{w}(\tau_1).$$

*Proof.*Since w' is a locally integrable function on I, we have

$$J_{\mathrm{V},\tau_1}^{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle}\left(\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle}(\mathrm{w})\right)(\mathrm{t})$$
$$=\int_{\tau_1}^{\mathrm{t}}\frac{\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle}(\mathrm{w})(\tau_1)}{\mathrm{V}(\eta,\alpha_1,\alpha_2,\dots,\alpha_n)}d\eta$$
$$=\int_{\tau_1}^{\mathrm{t}}\mathrm{w}'(\eta)\,d\eta$$
$$=\mathrm{w}(\mathrm{t})-\mathrm{w}(\tau_1),$$

which is the desired result.

Theorem 10.*Let $I\subseteq\mathbb{R}$, $\tau_1\in I$ and $\alpha_i\in(0,1]$, $\forall i=1,\dots,n$. Then we have*

$$\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle}\left(J_{\mathrm{V},\tau_1}^{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle}(\mathrm{w})(\mathrm{t})\right)=\mathrm{w}(\mathrm{t}),$$

for every continuous function w *on I and $\tau_1,\mathrm{t}\in I$.*

*Proof.*Let w be a continuous function w on I. Proposition 9 gives for every $\tau_1,\mathrm{t}\in I$,

$$\left(J_{\mathrm{V},\tau_1}^{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle}(\mathrm{w})(\mathrm{t})\right)'=\left(\int_{\tau_1}^{\mathrm{t}}\frac{\mathrm{w}(\eta)}{\mathrm{V}(\eta,\alpha_1,\alpha_2,\dots,\alpha_n)}d\eta\right)'$$
$$=\frac{\mathrm{w}(\mathrm{t})}{\mathrm{V}(\mathrm{t},\alpha_1,\alpha_2,\dots,\alpha_n}.$$

So, we have

$$\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle}\left(J_{\mathrm{V},\tau_1}^{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle}(\mathrm{w})(\mathrm{t})\right)$$
$$=\mathrm{V}(\mathrm{t},\alpha_1,\alpha_2,\dots,\alpha_n)\left(J_{\mathrm{V},\tau_1}^{\alpha}(\mathrm{w})(\mathrm{t})\right)'$$
$$=\mathrm{V}(\mathrm{t},\alpha_1,\alpha_2,\dots,\alpha_n)\frac{\mathrm{w}(\mathrm{t})}{\mathrm{V}(\mathrm{t},\alpha_1,\alpha_2,\dots,\alpha_n)}$$
$$=\mathrm{w}(\mathrm{t}).$$

Sometimes, the kernel of the integral operator may not be the same as the derivative operator, from the theoretical point of view it does not affect, in fact what it does is complicate expressions and some elementary properties.

3 Applications to differential equations

The fractional derivative present here is local by nature, hence any comparison with classical fractional derivatives is erroneous, because we are considering mathematical objects of different kinds. Our operator is local and the derivatives of Caputo, Riemann-Liouville, etc. are global (in [22] a comparison of this type is presented).

Next, to simplify the calculations, we will use our generalized derivative with two indices in this section. We know that in the classical fractional case

$$D^{\alpha}\left(\sin\left(\tau_2\mathrm{t}\right)\right)=\tau_2^{\alpha}\sin\left(\tau_2\mathrm{t}+\frac{\pi}{4}\right),$$

but what's up if $\alpha=\frac{1}{2n}$, $n\in\mathbb{N}$, and $\tau_2=-1$? However using our definition, and Theorem 2-(f), we have no problem

$$\mathscr{N}_{\mathrm{V}}^{\langle\frac{1}{2},\frac{1}{2}\rangle}(\sin(-\mathrm{t}))=-e^{\frac{\mathrm{t}}{2}+\mathrm{t}^{-\frac{1}{2}}}\cos\mathrm{t},$$

with $n=1$ and

$$\mathrm{V}(\mathrm{t},\alpha_1,\alpha_2)=e^{(1-\alpha_1)\mathrm{t}}e^{\mathrm{t}^{-\alpha_2}}.$$

In addition, there are functions like

$$\mathrm{w}(\mathrm{t})=e^{(1-\alpha_1)\mathrm{t}}e^{\mathrm{t}^{-\alpha_2}},$$

whose fractional derivative in the classical sense is very difficult to calculate, if not impossible, while using our definition is very easy, so we have

$$\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2\rangle}\left(e^{-\left((1-\alpha_1)\mathrm{t}+\mathrm{t}^{-\alpha_2}\right)}\right)=-\left((1-\alpha_1)-\alpha_2\mathrm{t}^{-\alpha_2-1}\right).$$

Consider the very simple differential equation

$$D^{\alpha}\mathrm{w}+\mathrm{w}=\mathrm{t}^{-\left(\alpha_1+\frac{\alpha_2}{\alpha_1}\right)}e^{\left(\mathrm{t}^{-\alpha_2}-\mathrm{t}\right)}.$$

This is a differential equation, whose independent term is a biparametric function, very common in different applications. If we try to solve it using classical derivatives D^{α} as the Caputo or Riemann-Liouville definition, then must use either the Laplace transform or the fractional power series technique. By other hand, if D^{α} is our definition, we rewrite it as a generalized derivative with two indices, like this

$$\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2\rangle}\mathrm{w}+\mathrm{w}=\mathrm{t}^{-\left(\alpha_1+\frac{\alpha_2}{\alpha_1}\right)}e^{\left(\mathrm{t}^{-\alpha_2}-\mathrm{t}\right)},$$

easily we obtain that

$$\mathrm{w}(\mathrm{t})=-\frac{1}{\alpha_1}\left[\mathrm{t}^{-\left(\alpha_1+\frac{\alpha_1}{\alpha_2}\right)}e^{\mathrm{t}^{-\alpha_2}-\mathrm{t}}\right]e^{-\frac{1}{\alpha_1}\Gamma\left(-\frac{\alpha_2}{\alpha_1},\frac{1}{\mathrm{t}^{\alpha_1}}\right)},$$

is a particular solution. We would like to add an additional application of our fractional derivative to solve ordinary differential equations. Thus, consider the following linear first-order differential equation:

$$\mathrm{w}'+\left((\alpha_1-1)+\alpha_2\mathrm{t}^{-\alpha_2-1}\right)\mathrm{w}=e^{(1-\alpha_1)\mathrm{t}+\mathrm{t}^{-\alpha_2}}\mathrm{t}^{r},\qquad r\in\mathbb{R}. \tag{15}$$

It is clear that this equation can be written this way:

$$\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2\rangle}\left[\mathrm{w}(\mathrm{t})e^{(1-\alpha_1)\mathrm{t}+\mathrm{t}^{-\alpha_2}}\right]=\mathscr{N}_{\mathrm{V}}^{\langle\alpha_1,\alpha_2\rangle}\left[\mathrm{t}^{r}\right].$$

According to Corollary 1, we easily obtain that:

$$\mathrm{w}(\mathrm{t})e^{(1-\alpha_1)\mathrm{t}+\mathrm{t}^{-\alpha_2}}=\mathrm{t}^{r}+C.$$

From where the general solution sought can be obtained without problems.

4 Conclusion

The $\mathbb{F}\mathrm{D}q - \mathbb{D}\mathbb{P}$ has been investigated in this work in detail. The investigation of this particular equation provides us with a powerful tool in modeling most scientific phenomena without the need to remove most parameters which have an essential role in the physical interpretation of the studied phenomena. $\mathbb{F}\mathrm{D}q - \mathbb{D}\mathbb{P}$ has been studied on a time scale under some B.Cs. An application that describe the motion of a particle in the plane has been provided to support our results' validity and applicability in fields of physics and engineering.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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