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# Convergence of Fuzzy Conformable Laplace Transform

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**Abstract:** The fuzzy conformable Laplace transform is defined in this study, and then some of its well-known features are studied. Furthermore, existence theorems for two convergences of the fuzzy conformable Laplace transforms are presented. As a result, as a new application of fuzzy Laplace transforms, we examine the solutions in state-space description of fuzzy linear continuous-time systems under generalized conformable differentiability.

**Keywords:** Fuzzy conformable Laplace transform, generalized conformable derivatives, Fuzzy fractional differential, fuzzy valued function.

## 1 Introduction

Several definitions of fractional derivative have been proposed by various researchers over the years. Riemann-Liouville [2] and Caputo fractional derivative [3] are the most prevalent fractional derivatives. We refer to [4] for more information on the features of Riemann, Caputo, and other related fractional definitions.

Although many mathematicians utilize the most common fractional derivatives, such as Riemann-Liouville (Rl) and Caputo, there are many studies in the literature that show that these fractional derivatives have various flaws, as shown in the following [5,6]. In RL, the derivative of a constant is not equal to zero.

– In RL initial conditions must be given in RL sense.
– For the Caputo derivative, to be differentiable the function must be differentiable in classical sense.
– In both RL and Caputo derivatives; chain rule is not satisfied.
– In both RL and Caputo derivatives; derivatives for integer order does not coincides with the derivatives in classical sense.

All of these concerns fueled the search for a new derivative. R. Khalil et al. developed a new derivative called conformable derivative in 2014, which meets numerous fractional derivative features that cannot be achieved by existing fractional derivatives [5,6].

This derivative is quite similar to the definition of derivative in the classical limit form, and it's very simple to work with. As a result, it was immediately accepted and the focus of numerous research studies [6,7,9]. We use fuzzy conformable Laplace transformations in this paper.

This is how the paper is organized. Section 2 explains the fundamental ideas. The general formula for the fuzzy conformable Laplace transforms for a fuzzy valued function $f$ is discovered in Section 3, as well as the general inverse fuzzy Laplace transform, an existence theorem, and some key features. The generic formula for the convergence of fuzzy conformable Laplace transform is given in Section 4. Consequently, Under generalized conformable differentiability, solutions of the state-space description of fuzzy conformable linear continuous-time systems are examined in Section 5. Finally, conclusions are given in Section 6.

## 2 Preliminaries

Let us denote by $\mathbb{R}_{\mathscr{F}} = \{u : \mathbb{R} \to [0,1]\}$ the class of fuzzy subsets of the real axis satisfying the following properties:

(i) u is normal i.e, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,

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(ii) u is fuzzy convex i.e for $x,y \in \mathbb{R}$ and $0<\lambda \le 1$,

$$u(\lambda x+(1-\lambda)y) \ge \min[u(x),u(y)]$$

(iii) u is upper semicontinuous,
(iv)$[u]^0 = cl\{x\in\mathbb{R}|u(x)>0\}$ is compact.

Then $\mathbb{R}_{\mathscr{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R}\subset\mathbb{R}_{\mathscr{F}}$. For $0<\alpha\le 1$ denote $[u]^\alpha=\{x\in\mathbb{R}|u(x)\ge\alpha\}$, then from $(i)$ to $(iv)$ it follows that the $\alpha$-level sets $[u]^\alpha\in P_K(\mathbb{R})$ for all $0\le\alpha\le 1$ is a closed bounded interval which is denoted by $[u]^\alpha=[u_1^\alpha,u_2^\alpha]$. By $P_K(\mathbb{R})$ we denote the family of all nonempty compact convex subsets of $\mathbb{R}$, and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual.

**Theorem 1.1.** [10] If $u\in\mathbb{R}_{\mathscr{F}}$, then

(i)$[u]^\alpha\in P_K(\mathbb{R})$ for all $0\le\alpha\le 1$
(ii)$[u]^{\alpha_2}\subset[u]^{\alpha_1}$ for all $0\le\alpha_1\le\alpha_2\le 1$
(iii)$\{\alpha_k\}\subset[0,1]$ is a nondecreasing sequence which converges to $\alpha$ then

$$[u]^\alpha=\bigcap_{k\ge 1}[u]^{\alpha_k}$$

Conversely, if $A_\alpha=\{[u_1^\alpha,u_2^\alpha];\alpha\in(0,1]\}$ is a family of closed real intervals verifying $(i)$ and $(ii)$, then $\{A_\alpha\}$ defined a fuzzy number $u\in\mathbb{R}_{\mathscr{F}}$ such that
$[u]^\alpha=A_\alpha$ for $0<\alpha\le 1$ and $[u]^0=\overline{\bigcup_{0<\alpha\le 1}A_\alpha}\subset A_0$.

**Lemma 1.1.** [11] Let $u,v:\mathbb{R}\to[0,1]$ be the fuzzy sets. Then $u=v$ if and only if $[u]^\alpha=[v]^\alpha$ for all $\alpha\in[0,1]$.

**Definition 1.1.**[12] A fuzzy number $u$ in parametric form is a pair $(u_1^\alpha,u_2^\alpha)$ of functions $u_1^\alpha,u_2^\alpha$, $\alpha\in[0,1]$, which satisfy the following requirements:

1.$u_1^\alpha$ is a bounded increasing left continuous function in $(0,1]$, and right continuous at 0,
2.$u_2^\alpha$ is a bounded decreasing left continuous function in $(0,1]$, and right continuous at 0,
3.$u_1^\alpha\le u_2^\alpha$, $0\le\alpha\le 1$.

A crisp number $k$ is simply represented by $u_1^\alpha=u_2^\alpha=k$.
For arbitrary $u=(u_1^\alpha,u_2^\alpha)$, $v=(v_1^\alpha,v_2^\alpha)$ and $\lambda>0$ we define addition and scalar multiplication by $\lambda$ see [16, 11]:

$$[u+v]^\alpha=[u_1^\alpha+v_1^\alpha,u_2^\alpha+v_2^\alpha]$$

$$[\lambda u]^\alpha=\lambda[u]^\alpha=\begin{cases}[\lambda u_1^\alpha,\lambda u_2^\alpha] \; if\; \lambda\ge 0\\ [\lambda u_2^\alpha,\lambda u_1^\alpha] \; if\; \lambda<0,\end{cases}$$

It is well known that the conformable derivative ($q$-derivative) for fuzzy mapping was initially introduced in [14] and it is based on the H-difference of sets s follows:

**Definition 1.2.** Let $u,v\in\mathbb{R}_{\mathscr{F}}$. If there exists $w\in\mathbb{R}_{\mathscr{F}}$ such as $u=v+w$ then $w$ is called the $H$-difference of $u,v$ and it is denoted $u\ominus v$.

Define $d:\mathbb{R}_{\mathscr{F}}\times\mathbb{R}_{\mathscr{F}}\to\mathbb{R}_+\cup\{0\}$ by the equation

$$d(u,v)=\sup_{\alpha\in[0,1]} d_H([u]^\alpha,[v]^\alpha),\;\; for\; all\; u,v\in\mathbb{R}_{\mathscr{F}}$$

where $d_H$ is the Hausdorff metric .

$$d_H([u]^\alpha,[v]^\alpha)=\max\{|u_1^\alpha-v_1^\alpha|,|u_2^\alpha-v_2^\alpha|\}$$

where $u=(u_1^\alpha,u_2^\alpha)$, $v=(v_1^\alpha,v_2^\alpha)\subset\mathbb{R}$ is utilized in Bede and Gal [16]. Then, it is easy to see that $d$ is a metric in $\mathbb{R}_{\mathscr{F}}$ and has the following properties [12]

$(i)d(u+w,v+w)=d(u,v),\quad\forall u,v,w\in\mathbb{R}_{\mathscr{F}}$,
$(ii)d(ku,kv)=|k|d(u,v)$, $\forall k\in\mathbb{R}$, $u,v\in\mathbb{R}_{\mathscr{F}}$,
$(iii)d(u+v,w+e)\le d(u,w)+d(v,e)$, $\forall u,v,w,e\in\mathbb{R}_{\mathscr{F}}$
$(iv)(d,\mathbb{R}_{\mathscr{F}})$ is a complete metric space.

**Definition 1.3.**[13] Let $f:\mathbb{R}\to\mathbb{R}_{\mathscr{F}}$ be a fuzzy-valued function. If for arbitrary fixed $t_0\in\mathbb{R}$ and $\varepsilon>0$ a $\delta>0$ such that

$$|t-t_0|<\delta\Longrightarrow d(f(t),f(t_0))<\varepsilon$$

$f$ is said to be continuous.

**Theorem 1.2.**[8] For $t_0\in\mathbb{R}$, the fuzzy fractional differential equation

$$u^{(q)}=F(t,u),u(t_0)=u_0\in\mathbb{R}_{\mathscr{F}} \tag{1}$$

where $F:\mathbb{R}\times\mathbb{R}_{\mathscr{F}}\longrightarrow\mathbb{R}_{\mathscr{F}}$ is supposed to be continuous, is equivalent to one of the integral equations for all $q\in(0,1]$ :

$$u(t)=u_0+\int_{t_0}^{t}x^{q-1}F(x,u(x))dx,\;\forall t\in[t_0,t_1]$$

or

$$u_0=u(t)+(-1)\int_{t_0}^{t}x^{q-1}F(x,u(x))dx,\;\forall t\in[t_0,t_1]$$

on some interval $(t_0,t_1)\subset\mathbb{R}$, depending on the strongly differentiability considered, (i) or (ii), respectively. Here the equivalence between two equations means that any solution of an equation is a solution too for the other one.

## *2.1 The fuzzy conformable fractional differentiability*

**Definition 2.1.** [14] Let $F:(0,a)\to\mathbb{R}_{\mathscr{F}}$ be a fuzzy function. $q^{\text{th}}$ order " fuzzy conformable fractional derivative" of $F$ is defined by

$$T_q(F)(t)=\lim_{\varepsilon\to 0^+}\frac{F(t+\varepsilon t^{1-q})\ominus F(t)}{\varepsilon}=\lim_{\varepsilon\to 0^+}\frac{F(t)\ominus F(t-\varepsilon t^{1-q})}{\varepsilon}.$$

for all $t > 0$, $q \in (0,1)$. Let $F^{(q)}(t)$ stands for $T_q(F)(t)$. Hence

$$F^{(q)}(t) = \lim_{\varepsilon \to 0^+} \frac{F(t+\varepsilon t^{1-q}) \ominus F(t)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{F(t) \ominus F(t-\varepsilon t^{1-q})}{\varepsilon}.$$

If $F$ is $q$-differentiable in some $(0,a)$, and $\lim_{t\to 0^+} F^{(q)}(t)$ exists, then

$$F^{(q)}(0) = \lim_{t\to 0^+} F^{(q)}(t)$$

and the limits (in the metric $d$)

**Remark 2.1.** From the definition, it directly follows that if $F$ is $q$-differentiable then the multi valued mapping $F_\alpha$ is $q$-differentiable for all $\alpha \in [0,1]$ and

$$T_q F_\alpha = [F^{(q)}(t)]^\alpha \quad (2)$$

Here $T_q F_\alpha$ is denoted the conformable fractional derivative of $F_\alpha$ of order $q$.

**Example 2.1.** We consider the fuzzy mapping $F : \mathbb{R} \to \mathbb{R}_{\mathscr{F}}$ defined by $F(t) = \lambda t^q$, where $\lambda$ is a fuzzy number defined by $[\lambda]^\alpha = [1+\alpha, 3-\alpha]$. Then

–if $t \geq 0$, $F(t) = [(1+\alpha)t^q, (3-\alpha)t^q]$
–if $t < 0$, $F(t) = [(3-\alpha)t^q, (1+\alpha)t^q]$

However $F$ is $q$-differentiable on $\mathbb{R}$ and $[F(t)]^\alpha = \lambda$.

**Theorem 2.1.**[14] Let $F : (0,a) \to \mathbb{R}_{\mathscr{F}}$ be fuzzy function, where $F_\alpha(t) = \left[f_1^\alpha(t), f_2^\alpha(t)\right]$, $\alpha \in [0,1]$.

(i)If $F$ is $q_{(1)}$-differentiable, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are $q$-differentiable and

$$\left[F^{(q_{(1)})}(t)\right]^\alpha = \left[(f_1^\alpha)^{(q)}(t), (f_2^\alpha)^{(q)}(t)\right].$$

(ii)If $F$ is $q_{(2)}$-differentiable, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are $q$-differentiable and

$$[F^{(q_{(2)})}(t)]^\alpha = \left[(f_2^\alpha)^{(q)}(t), (f_1^\alpha)^{(q)}(t)\right].$$

**Theorem 2.2.** [14] Let $q \in (0,1]$

(i)If $F$ is $(1)$-differentiable and $F$ is $q_{(1)}$-differentiable then

$$T_{q_{(1)}} F(t) = t^{1-q} D_1^1 F(t)$$

(ii)If $F$ is $(2)$-differentiable and $F$ is $q_{(2)}$-differentiable then

$$T_{q_{(2)}} F(t) = t^{1-q} D_2^1 F(t)$$

Note that the definition of $(n)$-differentiable or $\left(D_n^1\right)$ for $n \in 1,2$ see [16, 20, 17].

## 3 Fuzzy conformable Laplace transform

**Definition 3.1.**[7] The conformable fractional exponential function is defined for every $t \geq 0$ by:

$$E_q(p,t) = e^{p\frac{t^q}{q}}, \quad (3)$$

where $p \in \mathbb{R}$ and $0 < q \leq 1$.

Suppose that $f$ is a fuzzy-valued function and $p$ is a real parameter. We define the fuzzy conformable Laplace transform of $f$ as following:

**Definition 3.2.** Let $0 < q \leq 1$ and the fuzzy conformable Laplace transform of fuzzy-valued function $f$ is defined as follows:

$$\widehat{F}_q(p) = \mathbf{L}_q(f(t)) = \int_0^\infty E_q(-p,t) f(t) \mathrm{d}_q t = \int_0^\infty E_q(-p,t) f(t) t^{q-1} \mathrm{d}t \quad (4)$$

$$\widehat{F}_q(p) = \mathbf{L}_q(f(t)) = \lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f(t) \mathrm{d}_q t = \lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f(t) t^{q-1} \mathrm{d}t \quad (5)$$

i.e.,

$$\widehat{F}_q(p) = \left[\lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f_1^\alpha(t) \mathrm{d}_q t, \lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f_2^\alpha(t) \mathrm{d}_q t\right] = \left[\lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f_1^\alpha(t) t^{q-1} \mathrm{d}t, \lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f_2^\alpha(t) t^{q-1} \mathrm{d}t\right] \quad (6)$$

whenever the limits exist.

Denote by $\mathscr{L}_q[g(t)]$ the classical conformable Laplace transform of order $q$ starting from zero of crisp function $g(t)$. Then the lower and upper fuzzy conformable Laplace transform of the fuzzy-valued function are denoted based on the lower and upper of this function $f$ as follows:

$$\widehat{F}_q^\alpha(p) = \mathbf{L}_q(f^\alpha(t)) = [\mathscr{L}_q f_1^\alpha(t), \mathscr{L}_q f_2^\alpha(t))]$$

where $q \in (0,1]$, and

$$\mathscr{L}_q f_1^\alpha(t) = \int_0^\infty E_q(-p,t) f_1^\alpha(t) \mathrm{d}_q t = \lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f_1^\alpha(t) \mathrm{d}_q t, = \lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f_1^\alpha(t) t^{q-1} \mathrm{d}t,$$

$$\mathscr{L}_q f_2^\alpha(t) = \int_0^\infty E_q(-p,t) f_2^\alpha(t) \mathrm{d}_q t = \lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f_2^\alpha(t) \mathrm{d}_q t = \lim_{\tau\to\infty} \int_0^\tau E_q(-p,t) f_2^\alpha(t) t^{q-1} \mathrm{d}t$$

In order to solve fuzzy conformable differential equations, it is necessary to know the fuzzy conformable Laplace transform of the derivative of $f$,

**Definition 3.3.**[7] Let $0 < q \leq 1$ and $f^{(q)}(t)$ be a conformable fractional integral fuzzy-value function, and $f(t)$ is the primitive of $f^{(q)}(t)$ on $[0,\infty)$. Then

$(i)$if $f$ is $q_{(1)}$-differentiable:

$$\mathbf{L}_q\left[f^{(q)}(t)\right] = p\mathbf{L}_q\left[f(t)\right] \ominus f(0) \tag{7}$$

$(ii)$if $f$ is $q_{(2)}$-differentiable:

$$\mathbf{L}_q\left[f^{(q)}(t)\right] = (-f(0)) \ominus ((-p)\mathbf{L}_q\left[f(t)\right]) \tag{8}$$

In the fuzzy Laplace theory, we need to use absolute value of fuzzy-valued function $f$. To this end, we define two kinds of absolute values as following:

**Definition 3.4.**[15] Let us consider fuzzy-valued function $f$ defined in the parametric form $f^{\alpha}(t) = [f_1^{\alpha}(t), f_2^{\alpha}(t)]$.

–We say that $f$ is (1)-absolute value function, if

$$\forall \alpha \in [I_1, I_2] \subseteq [0,1]: \quad |f^{\alpha}(t)| = [|f_1^{\alpha}(t)|, |f_2^{\alpha}(t)|]$$

–We say $f$ is (2)-absolute value function, if

$$\forall \alpha \in [I_1, I_2] \subseteq [0,1]: \quad |f^{\alpha}(t)| = [|f_2^{\alpha}(t)|, |f_1^{\alpha}(t)|]$$

provided that $|f^{\alpha}(t)|$ define the $\alpha$-cuts of a fuzzy-valued function.

**Theorem 3.1.**[15] Let us consider fuzzy-valued function $f$ as $f^{\alpha}(t) = [f_1^{\alpha}(t), f_2^{\alpha}(t)]$, where $f_1^{\alpha}(t)$ and $f_2^{\alpha}(t)$ are lower and upper functions of $f$ for all $\alpha \in [0,1]$, respectively. Then:

1.if $f_1^{\alpha}(t) \geq 0$ for all $\alpha$, then $f$ is (1)-absolute fuzzy-valued function.
2.if $f_2^{\alpha}(t) \leq 0$ for all $\alpha$, then $f$ is (2)-absolute fuzzy-valued function.

**Theorem 3.2.**[15] The absolute value of fuzzy-valued function $f$ is always a positive fuzzy-valued function.

## 4 Convergence of Fuzzy conformable Laplace transform

Now, we need to distinguish two special modes of convergence of the fuzzy conformable Laplace integral.

**Definition 4.1.** Let $q \in (0,1]$, the integral (4) is said to be absolutely convergent if:

$$\lim_{\tau\to\infty} \int_0^{\tau} \left|E_q(-p,t) f(t)\right| \mathrm{d}_q t = \lim_{\tau\to\infty} \int_0^{\tau} \left|E_q(-p,t) f(t) t^{q-1}\right| \mathrm{d}t \tag{9}$$

exists, i.e.,

$$\lim_{\tau\to\infty} \int_0^{\tau} E_q(-p,t) \left|f_1^{\alpha}(t)\right| \mathrm{d}_q t \;,\; \lim_{\tau\to\infty} \int_0^{\tau} E_q(-p,t) \left|f_2^{\alpha}(t)\right| \mathrm{d}_q t \tag{10}$$

exists.

If $\mathbf{L}_q(f(t))$ does converge absolutely and if $f(t)$ be (1)-absolute, then

$$\left|\int_{\tau}^{\tau'} E_q(-p,t) f(t) \mathrm{d}_q t\right| = \left[\left|\int_{\tau}^{\tau'} E_q(-p,t) f_1^{\alpha}(t) \mathrm{d}_q t\right| , \left|\int_{\tau}^{\tau'} E_q(-p,t) f_2^{\alpha}(t) \mathrm{d}_q t\right|\right] \tag{11}$$

$$\left|\int_{\tau}^{\tau'} E_q(-p,t) f(t) \mathrm{d}_q t\right| \preceq \left[\int_{\tau}^{\tau'} E_q(-p,t) \left|f_1^{\alpha}(t)\right| \mathrm{d}_q t , \int_{\tau}^{\tau'} E_q(-p,t) \left|f_2^{\alpha}(t)\right| \mathrm{d}_q t\right] \tag{12}$$

$$\left|\int_{\tau}^{\tau'} E_q(-p,t) f(t) \mathrm{d}_q t\right| = \int_{\tau}^{\tau'} E_q(-p,t) \left|f(t)\right| \mathrm{d}_q t \longrightarrow \widetilde{0} \tag{13}$$

as $\tau \longrightarrow \infty$, for all $\tau' > \tau$. This, implies that $\mathbf{L}_q(f(t))$ also converges. Similar case holds when $f$ is (2)-absolute.

**Remark 4.1.** Notice that $\preceq$ is an ordering defined as follows: Let us consider two arbitrary fuzzy number $u$ and $v$. We say that $u$ is smaller or equal than $v$ and denote by $u \preceq v$ if and only if $u_1^{\alpha} \leq v_1^{\alpha}$ and $u_2^{\alpha} \leq v_2^{\alpha}$, $\forall\, \alpha \in [0,1]$.

Now, we consider another type of convergence as follows:

**Definition 4.2.** Let $q \in (0,1]$ and the integral (4) is said to converge uniformly for arbitrary $p$ in domain $\Omega$, if for any $\varepsilon > 0$, there exists some number $\tau_0$ such that if $\tau \geq \tau_0$, then

$$\left|\int_{\tau}^{\infty} E_q(-p,t) f(t) \mathrm{d}t\right| \preceq \varepsilon \cdot \widetilde{1}$$

Now, we investigate the continuity requirement for some fuzzy-valued function which will be possesses the fuzzy conformable Laplace transform.

**Definition 4.3.**[1] A fuzzy-valued function $f$ has a jump discontinuity at the point $t_0$ if both limits

$$\lim_{t\to t_0^-} f(t) = \left[\lim_{t\to t_0^-} f_1^{\alpha}(t), \lim_{t\to t_0^-} f_2^{\alpha}(t)\right] = \left[f_1^{\alpha}\left(t_0^-\right), f_2^{\alpha}\left(t_0^-\right)\right],$$

$$\lim_{t\longrightarrow t_0^+} f(t) = \left[\lim_{t\to t_0^+} f_1^{\alpha}(t), \lim_{t\to t_0^+} f_2^{\alpha}(t)\right] = \left[f_1^{\alpha}\left(t_0^+\right), f_2^{\alpha}\left(t_0^+\right)\right]$$

exist and

$$f\left(t_0^-\right) \neq f\left(t_0^+\right)$$

Here, $t \longrightarrow t_0^-$ and $t \longrightarrow t_0^+$ mean that $t \longrightarrow t_0$ from the left and the right, respectively.

**Definition 4.4.** A fuzzy-valued function $f$ is piecewise continuous on the interval $[0,\infty)$ if

1. $\lim_{t \longrightarrow 0^+} f(t) = f(0^+)$ exists.
2. $f$ is continuous on every finite interval $(0,b)$ expect possibly at a finite number of points $\tau_1, \tau_2, \ldots, \tau_n$ in $(0,b)$ at which $f$ has jump discontinuity.

Also, considered fuzzy-valued function $f$ is also bounded which means that

$$|f(t)| \preceq M_i \cdot \widetilde{1}, \quad \tau_i < t < \tau_{i+1}, \quad i = 1,2,3,\ldots,n-1$$

for finite positive constants $M_i$.
In order to integrate piecewise continuous fuzzy-valued functions form 0 to $b$, one simply integrates $f$ over each of the subintervals and takes the sum of these integrals, that is,

$$\int_0^b f(t) \mathrm{d}_q t = \int_0^{\tau_1} f(t) \mathrm{d}_q t + \int_{\tau_1}^{\tau_2} f(t) \mathrm{d}_q t + \cdots + \int_{\tau_n}^{b} f(t) \mathrm{d}_q t$$

$$\int_0^b f(t) t^{q-1} \mathrm{d}t = \int_0^{\tau_1} f(t) t^{q-1} \mathrm{d}t + \int_{\tau_1}^{\tau_2} f(t) t^{q-1} \mathrm{d}t + \cdots + \int_{\tau_n}^{b} f(t) t^{q-1} \mathrm{d}t$$

where $q \in (0,1]$.

Now, we consider the definition of exponential order of fuzzy-valued functions which is necessary for future purpose.

**Definition 4.5.** Let $q \in (0,1]$ and a fuzzy-valued function $f$ has exponential order $s$ if there exist constants $M > 0$ and $s$ such that for some

$$t_0 \geq 0, \quad |f(t)| \preceq M\, E_q(s,t) \cdot \widetilde{1},\ t \geq t_0.$$

Consequently, we show that a large class of fuzzy-valued functions can possesses the fuzzy conformable Laplace transform.

**Theorem 4.1.** If fuzzy-valued function $f$ be bounded piecewise continuous on $[0,\infty)$ and of exponential order $s$, then the fuzzy Laplace transform $\widehat{F}(p) = \mathbf{L}_q(f(t)), \forall q \in (0,1]$ exists for $p > s$ and converges absolutely.

**Proof.** Let $q \in (0,1]$ and since $f$ is assumed to have exponential order property, we get

$$|f(t)| \preceq M_1 E_q(s,t) \cdot \widetilde{1},$$

for some real $s$ and $t \geq t_0$. Also, $f$ is piecewise continuous on $[0,t_0]$ is bounded, we obtain:

$$|f(t)| \preceq M_2 \cdot \widetilde{1}, \quad 0 < t < t_0$$

Since $E_q(s,t)$ has a positive minimum on $[0,t_0]$, a constant $M$ can be chosen sufficiently large as $M = \max\{M_1, M_2\}$ so that $|f(t)| \preceq M E_q(s,t) \cdot \widetilde{1}$, $t > 0$. Therefore,

$$\int_0^{\tau} \left| E_q(-p,t) f(t) \right| \mathrm{d}_q t \preceq M \cdot \widetilde{1} \int_0^{\tau} \mathrm{e}^{-(x-s)\frac{t^q}{q}} \mathrm{d}_q t = \left( \frac{M}{x-s} - \frac{M \mathrm{e}^{-(x-s)\frac{\tau^q}{q}}}{x-s} \right) \cdot \widetilde{1}$$

By passing $\tau \longrightarrow \infty$ and since $p = x > s$, we get

$$\int_0^{\infty} \left| E_q(-p,t) f(t) \right| \preceq \frac{M}{x-s} \cdot \widetilde{1}$$

Thus, the fuzzy conformable Laplace integral converges absolutely and hence converges for $p > s$.

Now we will investigate the basic properties of the fuzzy Laplace transforms.

**Theorem 4.2.**[7](Linearity) Let $f(t)$, $g(t)$ be continuous fuzzy-valued functions, $q \in (0,1]$ and $c_1, c_2$ two real constants, then

$$\mathbf{L}_q\left[c_1 f(t) + c_2 g(t)\right] = c_1 \mathbf{L}_q\left[f(t)\right] + c_2 \mathbf{L}_q\left[g(t)\right]. \tag{14}$$

Uniform convergence We have already seen that for fuzzy-valued function $f$ which is piecewise continuous on $[0,\infty)$ and of exponential order, the fuzzy conformable Laplace integral (4) converges absolutely, that is,

$$\int_0^{\infty} \left| E_q(-p,t) f(t) \right| \mathrm{d}_q t$$

converges. Moreover, for fuzzy-valued function $f$, the fuzzy conformable Laplace integral (4) uniformly converges. Since, if $|f(t)| \preceq M \cdot \widetilde{1} \cdot E_q(s,t)$, $t \geq t_0$, then we have

$$\left| \int_0^{\infty} E_q(-p,t) f(t) \mathrm{d}_q t \right| \preceq \int_{t_0}^{\infty} E_q(-x,t) |f(t)| \mathrm{d}_q t \preceq M \cdot \widetilde{1} \int_{t_0}^{\infty} \mathrm{e}^{-(x-s)\frac{t^q}{q}} \mathrm{d}_q t = \left( \frac{M \mathrm{e}^{-(x-s)\frac{t_0^q}{q}}}{x-s} \right) \cdot \widetilde{1} \tag{15}$$

provided $x = p > s$. Considering $x \geq x_0 > s$, gives an upper bound for the last expression:

$$\frac{M \mathrm{e}^{-(x-s)\frac{t_0^q}{q}}}{x-s} \leq \frac{M \mathrm{e}^{-(x_0-s)\frac{t_0^q}{q}}}{x_0-s} \tag{16}$$

By choosing $t_0$ sufficiently large, we can make the term on the right-hand side of (16) arbitrarily small; that is, for a given $\varepsilon > 0$, there exists a positive value $T > 0$ such that $\left| \int_{t_0}^{\infty} E_q(-p,t) f(t) \mathrm{d}_q t \right| \preceq \varepsilon$ whenever $t_0 \geq T$ for all values of $p$ with $p \geq x_0 > s$. This is precisely the condition required for the uniform convergence of the conformable Laplace integral in the region $p \geq x_0 > s$.

**Theorem 4.3.** If fuzzy-valued function $f$ is piecewise continuous on $[0,\infty)$ and has exponential order $s$, then

$$\widehat{F}_q(p) = \mathbf{L}_q(f(t)) \longrightarrow 0, \; as \; p \longrightarrow \infty$$

where $q \in (0,1]$.

**Proof.** Let $q \in (0,1]$, using previous results, we have

$$\left| \int_0^\infty E_q(-p,t) f(t) d_q t \right| \preceq \frac{M}{x-s} \cdot \widetilde{1}, \quad p = x > s$$

then, $x \longrightarrow \infty$ gives the result.

In order to apply the fuzzy conformable Laplace transform to physical problems, it is necessary to invoke the inverse transform.

If $\mathbf{L}_q(f(t)) = \widehat{F}(p)$, then the inverse of fuzzy conformable Laplace transform is denoted by $\mathbf{L}_q^{-1}(\widehat{F}(p)) = f(t), t \geq 0$ which maps the fuzzy conformable Laplace transform of a fuzzy-valued function $f$ back to original fuzzy-valued function $f$. Note that $\mathbf{L}_q^{-1}$ is linear, that is, $\mathbf{L}^{-1}(\lambda \widehat{F}_q(p) + \gamma \widehat{G}_q(p)) = \lambda f(t) + \gamma g(t)$ if

$$\mathbf{L}_q(f(t)) = \widehat{F}_q(p), \quad \mathbf{L}_q(g(t)) = \widehat{G}_q(p)$$

and $\lambda, \gamma$ are arbitrary positive or negative constants. This follows from the linearity of $\mathbf{L}_q$ and holds in the domain to $\widehat{F}_q$ and $\widehat{G}_q$.

Now, we present useful result for determining the fuzzy conformable Laplace transforms and their inverses.

**Theorem 4.4.**[7] Let $0 < q \leq 1$ and $f(t)$ is continuous fuzzy-value function and $\mathbf{L}_q[f(t)] = F(p)$, then

$$\mathbf{L}_q[E_q(a,t) f(t)] = \widehat{F}_q(p-a)$$

where $E_q(a,t)$ is real value function and $P - a > 0$.

## 5 Application

Let us consider the following system:

$$\begin{cases} x^{(q)} = f(t,x,u) \\ y = g(t,x,u) \end{cases} \tag{17}$$

where $q \in (0,1]$, $x \in \mathbb{R}^n_{\mathscr{F}}, y \in \mathbb{R}^p_{\mathscr{F}}, u \in \mathbb{R}^m_{\mathscr{F}}, f : R \times \mathbb{R}^n_{\mathscr{F}} \times \mathbb{R}^m_{\mathscr{F}} \longrightarrow \mathbb{R}^n_{\mathscr{F}}$ and $g : R \times \mathbb{R}^n_{\mathscr{F}} \times \mathbb{R}^m_{\mathscr{F}} \longrightarrow \mathbb{R}^p_{\mathscr{F}}$. Here, $t$ denotes time and $u, y$ denote fuzzy system input and output, respectively. Note that $x^{(q)} = \frac{\partial^q x}{\partial^q t} = t^{1-q} \frac{\partial x}{\partial t}$ is computed under generalized conformable differentiability.

Also, first equation in Eq. (17) is called the fuzzy state conformable equation, the second is called the fuzzy output conformable equation and both equations, together, are called the state-space description of fuzzy continuous-conformable time system. In linear case, Eq. (17) will be as follows:

$$\begin{cases} x^{(q)} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{18}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}, D \in R^{p\times m}$ and $u : R \longrightarrow \mathbb{R}^m_{\mathscr{F}}$ is assumed to be continuous fuzzy-valued function. Note that, in Eq. (18), $x$ denotes the fuzzy state vector, $u$ denotes the fuzzy system input, and $y$ denotes the fuzzy system output.

Also, for given fuzzy initial conditions $x(t_0) = x_0 \in \mathbb{R}^n_{\mathscr{F}}$, $t_0 \in R$ and for a given fuzzy input $u$, the equivalent integral form of first equation in Eq. (18) is stated under case $(q_1)$-differentiability by

$$\begin{aligned} \phi(t,t_0,x_0) &= \Phi(t,t_0)x_0 \\ &+ \int_{t_0}^t t^{q-1} \Phi(t,s) Bu(s) \mathrm{d}s, \quad t \in R \end{aligned} \tag{19}$$

and also is expressed under case $(q_2)$-differentiability by

$$\begin{aligned} \phi(t,t_0,x_0) &= \Phi(t,t_0)x_0 \\ &\ominus (-1). \int_{t_0}^t t^{q-1} \Phi(t,s) Bu(s) \mathrm{d}s, \quad t \in R \end{aligned} \tag{20}$$

where $\Phi$ denotes the fuzzy state transition matrix of A. Note that our calculations are concentrated on the case $(q_1)$-differentiability and the type of $(q_2)$-differentiability is completely similar. To this end, $A, B, C$ and $D$ are assumed positive.

Moreover, the fuzzy total system response is given by

$$y(t) = C\Phi(t,t_0)x_0 + C\int_{t_0}^t t^{q-1} \Phi(t,s) Bu(s) \mathrm{d}s + Du(t), \quad t \in R \tag{21}$$

Note that Eq. (21) can be viewed as sum of two components, the fuzzy zero-input response:

$$\Psi(t,t_0,x_0,0) = C\Phi(t,t_0)x_0 \tag{22}$$

and the fuzzy zero-state response:

$$\rho(t,t_0,0,u) = C\int_{t_0}^t t^{q-1} \Phi(t,s) Bu(s) \mathrm{d}s + Du(t) \tag{23}$$

Let us consider $x_0 = 0$ in Eq. (21); then using the Dirac delta ($\delta$), the fuzzy impulse response of system (18) is given as follows:

$$\begin{aligned} y(t) &= \int_{t_0}^t [C\Phi(t,\tau)B + D\delta(t-\tau)] u(\tau) \mathrm{d}_q\tau \\ &= \int_{t_0}^t [C\Phi(t,\tau)B + D\delta(t-\tau)] u(\tau) t^{q-1} \mathrm{d}\tau \\ &= \int_{t_0}^t H(t,\tau) u(\tau) \mathrm{d}_q\tau \\ &= \int_{t_0}^t H(t,\tau) u(\tau) t^{q-1} \mathrm{d}\tau \end{aligned} \tag{24}$$

where $H(t,\tau)$ denotes the impulse response matrix of system (18) given by

$$H(t,\tau)=\begin{cases} C\Phi(t,\tau)B+D\delta(t-\tau), & t\geq\tau \\ 0, & t<\tau \end{cases} \tag{25}$$

Since, $\Phi(t,t_0)=\mathrm{e}^{A(t-t_0)}$, the solution of first equation in (18) is given by

$$\phi(t,t_0,x_0)=\mathrm{e}^{A\left(\frac{t^q}{q}-\frac{t_0^q}{q}\right)}x_0+\int_{t_0}^{t}\mathrm{e}^{A\left(\frac{t^q}{q}-\frac{s^q}{q}\right)}Bu(s)\mathrm{d}_q s. \tag{26}$$

Hence, the fuzzy total response of system (18) is given by

$$y(t)=C\mathrm{e}^{A\left(\frac{t^q}{q}-\frac{t_0^q}{q}\right)}x_0+C\int_{t_0}^{t}\mathrm{e}^{A\left(\frac{t^q}{q}-\frac{s^q}{q}\right)}Bu(s)\mathrm{d}_q s+Du(t) \tag{27}$$

Similarly, the fuzzy zero-state response of system (18) is given by

$$\begin{aligned} y(t) &= \int_{t_0}^{t}\left[C\mathrm{e}^{A\left(\frac{t^q}{q}-\frac{\tau^q}{q}\right)}B+D\delta(t-\tau)\right]u(\tau)\mathrm{d}\tau \quad (28)\\ &= \int_{t_0}^{t}H(t,\tau)u(\tau)\mathrm{d}\tau \end{aligned}$$

**Example 5.1.** Let us consider $A=\begin{bmatrix}a_1 & a_2\end{bmatrix}^{\mathrm{T}}, B=[a_1]^{\mathrm{T}}$, $C=[a_1], D=0$ and consider the fuzzy initial condition in the parametric form $x^\alpha(0)=(x_1^\alpha(0),x_2^\alpha(0))^{\mathrm{T}}$, $u$ is unit step and $t\geq 0$, where $a_1=\begin{bmatrix}0 & 1\end{bmatrix}, a_2=\begin{bmatrix}0 & 0\end{bmatrix}, x_1^\alpha(0)=\tilde{0}$ and $x_2^\alpha(0)=[-2+\alpha,-\alpha]$ for all $\alpha\in[0,1]$. Then, the solution of system (18) is obtained for all $0\leq\alpha\leq 1$ as follows:

$$\begin{cases} x_1^\alpha(t)=\frac{t^2}{2}\cdot\widetilde{1}-[\alpha,2-\alpha]\times t \\ x_2^\alpha(t)=t\times\widetilde{1}-[\alpha,2-\alpha] \end{cases} \tag{29}$$

Moreover, the fuzzy total system response $y(t)=Cx(t)$ is derived as follows:

$$\begin{aligned} y^\alpha(t,t_0,x_0,u) &= \Psi^\alpha(t,t_0,x_0,0)+\rho^\alpha(t,t_0,0,u) \quad (30)\\ &= [t-2+\alpha,t-\alpha] \end{aligned}$$

Let us consider Eq. (28); then the fuzzy impulse response matrix $H$ of system is given by

$$H(t-\tau)=\begin{cases} C\mathrm{e}^{A(t-\tau)}B+D\delta(t-\tau), & t\geq\tau \\ 0, & t<\tau \end{cases} \tag{31}$$

We take the conformable Laplace transform of (28) where $q\in(0;1]$; then we obtain $\widehat{y}_q(p)=\widehat{H}_q(p)\widehat{u}_q(p)$ where $\widehat{y}_q(p)=L_q\{y(t)\}$ and so on for $H(t)$ and $u(t)$. Note that $\widehat{H}(s)$ is called the transfer function matrix of system (18). Now, we try to obtain $\widehat{H}_q$ in a direct manner by first taking the conformable Laplace transform of system (18) to obtain

$$p\widehat{x}_q(p)\ominus x(0) = A\widehat{x}_q(p)+B\widehat{u}_q(p)\widehat{y}_q(p) \tag{32}$$

$$\widehat{y}_q(p) = C\widehat{x}_q(p)+D\widehat{u}_q(p). \tag{33}$$

So, we get:

$(\widehat{x}_1)_q^\alpha(p)=(pI-A)^{-1}x^\alpha(0)+(pI-A)^{-1}B(\widehat{u}_1)_q^\alpha(p);$

$(\widehat{x}_2)_q^\alpha(p)=(pI-A)^{-1}x^\alpha(0)+(pI-A)^{-1}B(\widehat{u}_2)_q^\alpha(p);$

for $\alpha\in[0,1]$. Then, the lower and upper function of $\widehat{y}_q(p)$ are determined as follows:

$$\begin{aligned} (\widehat{y}_1)_q^\alpha(p) &= C(pI-A)^{-1}x^\alpha(0) \quad (34)\\ &+ C(pI-A)^{-1}B(\widehat{u}_1)_q^\alpha(p)+D(\widehat{u}_1)_q^\alpha(p) \\ (\widehat{y}_2)_q^\alpha(p) &= C(pI-A)^{-1}x(0;\alpha) \quad (35)\\ &+ C(pI-A)^{-1}B(\widehat{u}_2)_q^\alpha(p) \quad (36) \end{aligned}$$

$+D(\widehat{u}_2)_q^\alpha(p)$

and then, using the inverse of fuzzy Laplace transform we can easily compute the lower and upper function of $y(t)$ in the parametric form as following:

$$y_1^\alpha(t)=L_q^{-1}(\widehat{y}_1)_q^\alpha(p) \tag{37}$$

$$\begin{aligned} y_1^\alpha(t) &= C\mathrm{e}^{A\frac{t^q}{q}}x_1^\alpha(0) \quad (38)\\ &+ C\int_{t_0}^{t}t^{q-1}\mathrm{e}^{A(\frac{t^q}{q}-\frac{s^q}{q})}Bu_1^\alpha(p)\mathrm{d}_q p+Du_1^\alpha(p) \end{aligned}$$

$$y_2^\alpha(t)=L_q^{-1}(\widehat{y}_2)_q^\alpha(p) \tag{39}$$

$$\begin{aligned} y_2^\alpha(t) &= C\mathrm{e}^{A\frac{t^q}{q}}x_2^\alpha(0) \quad (40)\\ &+ C\int_{t_0}^{t}t^{q-1}\mathrm{e}^{A(\frac{t^q}{q}-\frac{s^q}{q})}Bu_2^\alpha(p)\mathrm{d}_q p+Du_2^\alpha(p) \end{aligned}$$

If $x(0)=\widetilde{0}$, then the fuzzy zero-state response is given by

$$\widehat{y}_q(p)=\left[C(pI-A)^{-1}B+D\right]\widehat{u}_q(p)$$

$$\widehat{y}_q(p)=\widehat{H}_q(p)\widehat{u}(s)$$

where $\widehat{H}_q(p)=C(pI-A)^{-1}B+D$, denotes the transfer function of system (18).

## 6 Conclusion

We obtained two convergence results for fuzzy conformable Laplace transforms in this research, and we used the fuzzy Laplace transforms to define the state-space of a fuzzy continuous-time system under generalized conformable differentiability. We proposed several efficient results for fuzzy Laplace transforms to achieve this goal. It was the first application that this article looked into.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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