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Transient Analysis of Chemical Queue with Catastrophes and Server Repair

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Abstract: In this paper we deduced explicitly expressions for the transient state distribution for a queueing problem having "chemical" rules with arbitrary number of customers present initially in the system, in addition to having the possibility of catastrophe and hence system repair. Our calculations based on using generating function and Laplace transformation techniques, the obtained solution of the probabilities distribution enables us to recover the well known formula and other cases such as the transient solution of the standard $M/M/1/\infty$ queue with $\lambda = \mu$. Finally, the theory is underpinned by numerical results.

Keywords: Stochastic processes, Random walk, Catastrophe, Repair process, Chemical queue

1 Introduction

In recent years, the attention has been focused on queueing models, see Böhm and Homik [1], which produced from some chemical reactions so-called chemical queueing models where a molecule is modelled as an infinitely long chain of atoms joined by the links of equal lengths. The links are subject to random shocks and causes the atoms to move and the molecule to diffuse, for more details, see Stochmayer et al.[2] which seem to be the first to study this type of process in a chemical-physics context. Chemical queueing processes have been also discussed by Conolly et al.[3], who based their analysis on the direct approach. Tarabia and Colleagues [4,5] have given a thorough analysis by means of a uniformlization procedure. They formulated the system of partial difference equations satisfied by the transition functions of the embedded discrete-time Markov chain, cleverly guessed its solutions, and then provided a proof by induction.
In our work we will concentrate on the chemical queue model that includes the effect of the catastrophes and repair. The model can be described as a single server queue with infinite capacity. If the catastrophe occurs at the busy server randomly, this leading to annihilate all customers in the system immediately and the server gets inactivated, i.e. the server is subject to failure and has to repair it.
In this paper, we attempt to obtain the transient probabilities analytically in a closed form by dividing the generating function into two parts the first part is for the odd states and the second part is for the even states. From the obtained transient solution many special cases such as Conolly el al.[3] , Tarabia et al. [4] and Krishna Kumar et al. [6] are recovered in closed forms. Numerical calculations are carried out to illustrate the behaviour of the transient probabilities and their corresponding means.

2 Model Description

Consider a chemical queueing model with catastrophes and server failure and repairs. The arrival process of customers is Poisson process with mean arrival rateλ during times that the server is working. Assume that the customers are served on first-come, first-served discipline with the service time following an exponential distribution with mean λ.
The capacity of the system is infinite. When the system is idle or busy, catastrophes occur at the service station according to Poisson process of rateγ. Whenever a catastrophe occurs at the busy server all customers in the system are destroyed immediately and the server gets inactivated, i.e. the server is subject to failure and has to repair it. The repair times of

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failed server are i.i.d, according to an exponential distribution with parameter η.
After repairing the server, it becomes ready to serve new customers. Let $Q(t)$ be the probability that the server is under repair at the instant t with $Q(0)=0$. It is easy to see that the given model can be described by Markov process $\{X(t), t \geq 0\}$ where $X(t)$ denotes the number of customers in the system at time t. Let

$$P_n(t) = pr(X(t) = n|X(0) = 0)$$

From the above assumptions the resulting stochastic behavior is described by a set of what are essentially Chapman-Kolmogorov forward equations which can be given as:

$$\acute{Q}(t) = -(\eta+\gamma)Q(t)+\gamma, \tag{1}$$

$$\acute{P}_0(t) = -(\gamma+\lambda)P_0(t)+\lambda P_1(t)+\eta Q(t), \tag{2}$$

$$\acute{P}_{2n}(t) = -(\gamma+\mu+\lambda)P_{2n}(t)+\lambda P_{2n+1}(t)+\mu P_{2n-1}(t), \tag{3}$$

$$\acute{P}_{2n+1}(t) = -(\gamma+\mu+\lambda)P_{2n+1}(t)+\lambda P_{2n}(t)+\mu P_{2n+2}(t). \tag{4}$$

Notice that the normalization condition can be written as following:

$$Q(t)+\sum_{i=0}^{\infty} P_i(t) = 1, \tag{5}$$

from Eq.(1), we can easily obtain the repair function as

$$Q(t) = \frac{\gamma}{\eta+\gamma}[1-\exp(-(\eta+\gamma)t)]. \tag{6}$$

Let us define the generating function as:

$$P(s,t) = Q(t)+\sum_{n=0}^{\infty} P_n(t)s^n$$

which can be rewritten as:

$$P(s,t) = Q(t)+G(s,t)+H(s,t) \tag{7}$$

where

$$G(s,t) = \sum_{n=0}^{\infty} P_{2n}(t)s^{2n} \tag{8}$$

$$H(s,t) = \sum_{n=0}^{\infty} P_{2n+1}(t)s^{2n+1} \tag{9}$$

Using the *p. g. f.* given in Eq. (7), the system of Eqs. (2)-(4) can be turned to the following two differential equations

$$\acute{G}(s,t)+(\gamma+\mu+\lambda)G(s,t) = (\frac{\lambda}{s}+\mu s)H(s,t)+\mu P_0(t)+\eta Q(t) \tag{10}$$

$$\acute{H}(s,t)+(\gamma+\mu+\lambda)H(s,t) = (\lambda s+\frac{\mu}{s})G(s,t)-\frac{\mu}{s}P_0(t) \tag{11}$$

Taking Laplace transform for Eqs.(10)- (11), we get

$$\begin{aligned}(z+\gamma+\mu+\lambda)G^*(s,z)-(\frac{\lambda}{s}+\mu s)H^*(s,z) &= \mu P_0^*(z)+1+\eta Q^*(z)\\ (\lambda s+\frac{\mu}{s})G^*(s,z)-(z+\gamma+\mu+\lambda)H^*(s,z) &= \frac{\mu}{s}P_0^*(z)\end{aligned} \tag{12}$$

which can be solved to obtained $G^*(s,z)$ and $H^*(s,z)$ as following:

$$G^*(s,Z) = \frac{(-\mu Z + \frac{\mu}{s})P_0^*(z) - Z(\eta Q^*(Z)+1)}{-Z^2 + (\frac{\lambda}{s} + \mu s)(\lambda s + \frac{\mu}{s})}, \tag{13}$$

$$H^*(s,Z) = \frac{(\frac{\mu}{s}Z - \mu(\lambda s + \frac{\mu}{s}))P_0^*(z) - (\lambda s + \frac{\mu}{s})(\eta Q^*(Z)+1))}{-Z^2 + (\frac{\lambda}{s} + \mu s)(\lambda s + \frac{\mu}{s})}, \tag{14}$$

where
$Z = (z+\gamma+\mu+\lambda), \mu = \frac{a-b}{2}, \lambda = \frac{a+b}{2}, A^2 = Z^2 - a^2, B^2 = Z^2 - b^2$ and
$Q^*(Z) = \frac{\gamma}{z(z+\eta+\gamma)}$.

Clearly, Eq. (13) and Eq. (14)have the same denominator with the roots given as:

$$\alpha^2, \beta^2 = \frac{(A \pm B)^2}{a^2 - b^2}, \tag{15}$$

where β^2 have the minus sign and α^2 takes the Positive sign, with $\beta^2\alpha^2 = 1$ and $\beta^2 + \alpha^2 = \frac{Z^2 - (\lambda^2+\mu^2)}{\lambda\mu}$.
Since numerator of $G^*(s,Z)$ equal zero at the roots of the denominator then

$$P_0^*(Z) = \frac{Z + \lambda\beta + \frac{\mu}{\beta}}{\mu[\frac{1}{\beta}(Z + \frac{\lambda}{\beta} + \mu\beta) - (Z + \lambda\beta + \frac{\mu}{\beta})]}(\eta Q^*(Z)+1),$$

$$P_0^*(Z) = \frac{1}{\mu[\frac{1}{\beta}\frac{Z + \frac{\lambda}{\beta} + \mu\beta}{Z + \lambda\beta + \frac{\mu}{\beta}} - 1]}(\eta Q^*(Z)+1),$$

after some calculation

$$P_0^*(Z) = \frac{\lambda\beta^2 + \mu}{\mu[Z - (\lambda\beta^2 + \mu)]}(\eta Q^*(Z)+1)$$

where,

$$\lambda\beta^2 + \mu = Z + \frac{1}{2\mu}(\sqrt{Z-a}\sqrt{Z+b})(\sqrt{Z-a}\sqrt{z+b} - \sqrt{Z+a}\sqrt{Z-b}) \tag{16}$$

Then $P_0^*(Z)$ can be simplified as following:

$$P_0^*(Z) = \frac{\sqrt{(Z+a)(Z-b)} - \sqrt{(Z-a)(Z+b)}}{(a-b)\sqrt{(Z-a)(Z+b)}}(\eta Q^*(Z)+1) \tag{17}$$

which can be written for simplicity as

$$P_0^*(Z) = R^*(Z)(\eta Q^*(Z)+1), \tag{18}$$

where $R^*(Z) = \frac{\sqrt{(Z+a)(Z-b)} - \sqrt{(Z-a)(Z+b)}}{(a-b)\sqrt{(Z-a)(Z+b)}}$.
Using the partial fraction technique to Eqs. (13)- (14), the coefficients of s^{2n} and s^{2n+1} can be obtained as

$$P_{2n}^*(Z) = \beta^{2n}P_0^*(Z), \tag{19}$$

$$P_{2n+1}^*(Z) = \frac{Z^2 + ab - AB}{Z(a+b)}\beta^{2n}P_0^*(Z). \tag{20}$$

Taking the inverse of Laplace to find the transient probabilities:

$$\ell^{-1}P_0^*(Z) = \ell^{-1}R^*(Z)(\eta Q^*(Z)+1) = \ell^{-1}R^*(Z)\eta Q^*(z) + \ell^{-1}R^*(Z), \tag{21}$$

using the convolution theory, we get

$$P_0(t) = \eta \int_0^t R(Z)Q(t-Z)dZ + R(t) \quad (22)$$

where

$$R(t) = \frac{e^{-at}}{a-b}(aF(at) - bF(-bt) - ab\int_0^t F(as)F(-b(t-s))ds), \quad (23)$$

and

$$F(x) = I_0(x) + I_1(x), a = \lambda + \mu, b = \lambda - \mu.$$

3 Special Cases

In this section we introduced some special cases which emerging from the our general solution:
case 1: If $\eta = 0$, this turned our system to the system of chemical queue without catastrophe,

$$P_0(t) = R(t) = \frac{e^{-at}}{a-b}(aF(at) - bF(-bt) - ab\int_0^t F(as)F(-b(t-s))ds), \quad (24)$$

which agree with Conolly et al. [3]
Case 2 : If $\lambda = \mu$, in Eq.(15) gives $b = 0$our formula transformed to the known randomised random walk forms

4 Numerical Results

In this section computational results are carried out to illustrate the above derived theoretical part. Taking the values $\lambda = 2, \mu = 1, \gamma = 0.3$ the effect of repair time rate η on the idle probability $P_0(t)$ behavior is shown in Fig. 1.
From Fig. 1. we can notice that the value of $P_0(t)$ increases as η increases.

Fig. 2. shows the values of $P_0(t)$ tends to zero as tends infinity and the system show has no steady state when $\eta = 0$. Also, the values of $P_0(t)$ increase with increasing the values of γ with fixed η as expected.

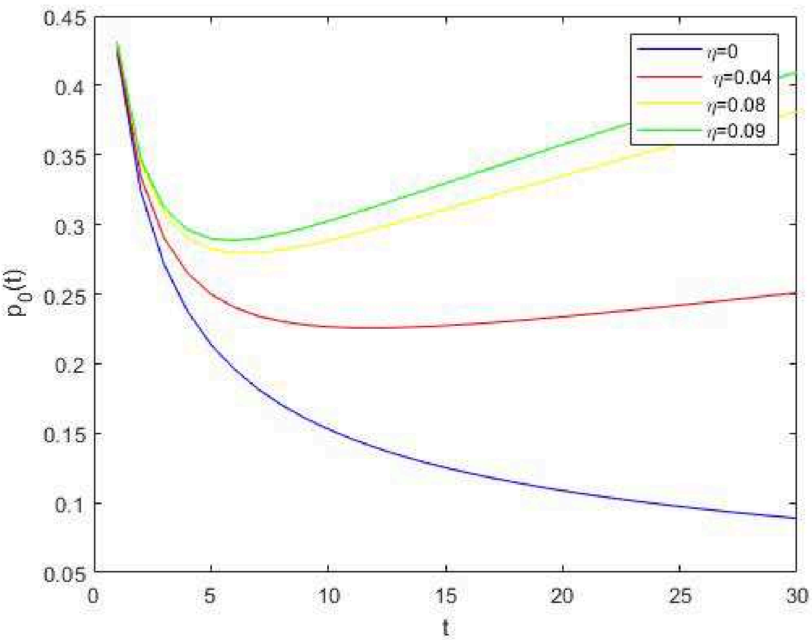

Fig. 1: The effect of various values of η on the value of $P_0(t)$ with $\lambda = 2$ and $\mu = 1$

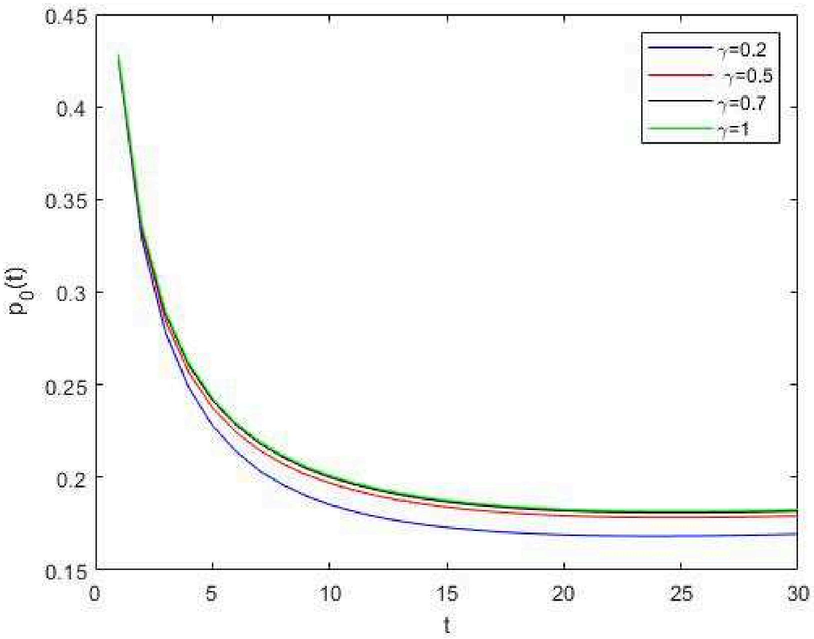

Fig. 2: The effect of various values of γ on the value of $P_0(t)$ with $\lambda = 2$ and $\mu = 1$

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Conflicts of Interest

There are no conflicts of interest declared by the authors for the publication of this paper..

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