Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160624

Estimation of Sameera Distribution Parameters with Applications to Real Data

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Received: 22 Jul. 2022, Revised: 22 Sep. 2022, Accepted: 12 Oct. 2022
Published online: 1 Nov. 2022

Abstract: In this paper, a new two-parameter continuous distribution called Sameera distribution is proposed. Some statistical properties of this distribution are derived such as: moment-generating function, moments, and related measures, reliability analysis and associated functions. Also, the distribution of order statistics and the quantile function are presented. The Shannon, Rényi, and Tsallis entropies are derived. The methods of maximum likelihood estimation, ordinary and weighted least squares, Anderson-Darling, Cramer-Von Mises, and maximum product spacing are used to estimate the distribution parameters. A simulation study is performed to investigate the performance of these methods. Real data applications show that the proposed distribution can provide a better fit than several competitive distributions.

Keywords: Mixing distribution, Sameera distribution, moments, reliability analysis, Rényi entropy, methods of estimation, moment-generating function

1 Introduction

In statistics, modeling lifetime data is an important issue in many fields including biomedical sciences, economics, finance, engineering. A lot of continuous distributions have introduced for modeling such data. Many ways are recently used to propose new models such as the mixture of two or more distributions. These distributions are used in many fields of life such as: medicine, environment, biostatistics, and many others. Several distributions have been proposed from mixing distributions, for example, [1] suggested Darna distribution as a mixture of $Exp\left(\frac{\beta}{\alpha}\right)$ and $\Gamma\left(3,\frac{\beta}{\alpha}\right)$ with mixing proportion $\frac{2\alpha^2}{2\alpha^2+\beta^2}$. [2] employed the concept of mixture distributions using the $Exp(\beta)$ and $\Gamma(\alpha-1,\beta)$, with mixture proportions $\frac{1}{\alpha\beta+1}$ and $\frac{\alpha\beta}{\alpha\beta+1}$, to suggest a new two parameters distribution called Alzoubi distribution. [3] employed the same concept to suggest Benrabia distribution as a mixture of exponential and gamma distributions. Alzoubi, et al. [4] proposed a new distribution called Loai distribution as a mixture of $Lindley(\theta)$ and $gamma(3,\theta)$. The properties of this distribution are studied. Different methods of parameter estimation are employed. Gharaibeh distribution is proposed by [5] as a four components mixture of $exp(\beta)$, $\Gamma(2,\beta)$, $\Gamma(4,\beta)$ and $\Gamma(6,\beta)$ with mixing proportions $\frac{\beta^6}{\beta^6+\beta^4+\beta^2+1}$, $\frac{\beta^4}{\beta^6+\beta^4+\beta^2+1}$, $\frac{\beta^2}{\beta^6+\beta^4+\beta^2+1}$ and $\frac{1}{\beta^6+\beta^4+\beta^2+1}$; respectively. [6] suggested a new lifetime distribution using mixture of distributions technique, called Karam distribution. On the other hand, rank transmutation map suggested by [7] is another technique used to propose new distributions. For example, [8] used this map to generate the transmuted Mukherjee-Islam distribution. This map is also used to make a generalization of the new Weibull-Pareto distribution [10]. Many other transmuted distributions have been suggested using this map, including transmuted Janardan distribution [9], transmuted gamma-Gompertz distribution [11], transmuted Ishita distribution [12], transmuted Aradhana distribution [13] and many others, for example [15], [16], [17], [18], [19], [20], [21].

The idea of combining kernel functions with a distribution is used to generate new distributions. For example, [22] compounded the biweight kernel function with the exponential distribution to propose the biweight

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exponential distribution by combining the biweight kernel function and the exponential distribution. [23] used Epanechnikov kernel function and the exponential distribution to generate the Epanechnikov exponential distribution.

A random variable X is said to have a mixture of k distributions $f_1(x),\cdots,f_k(x)$, if its probability density function (pdf) $g(x)=\sum_{i=1}^{k}a_i f_i(x)$ with $0\le a_i\le 1$ is the mixing weight, such that $\sum_{i=1}^{k}a_i=1$.

In this article, we adopt the idea of mixing distributions of exponential with parameter β and gamma with parameters α and β to suggest a new two parameters distribution called Sameera distribution with mixing proportions $a_1=\frac{\alpha^2\beta}{1+\alpha^2\beta}$ and $a_2=\frac{1}{1+\alpha^2\beta}$, denoted as $X\sim SamD(\alpha,\beta)$. Also, we want to prove that the suggested distribution is more flexible than the base distribution based on some real lifetime data.

This paper is organized as follows, in Section 2, we define the probability density and the cumulative distribution function of Sameera distribution. In Section 3, we consider some statistical properties including the moment generating function, the moments, and some related measures. In Section 4, the reliability analysis functions are derived. In Section 5, we describe the density of order statistics and the quantile function. Sections 6 and 7 derive the Bonferroni and Lorenz curves and the entropies; respectively. Section 8 implements the mean absolute deviation about the mean and median. In Section 9 we used different methods of estimation to estimate the model parameters. Section 10 introduces the stress-strength reliability. In Section 11, we provide a simulation study for the methods of estimation. Section 12 presents applications to real lifetime data sets. Finally, in Section 13 we sum up the article.

2 Sameera Distribution

In this section, we define the probability density function (pdf) and the cumulative distribution function (cdf) of the proposed distribution with graphical illustration for both of them.

Definition 1. *A random variable X is said to have Sameera distribution if its pdf is defined as:*

$$\psi(x|\alpha,\beta)=\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta}+\frac{x^{\alpha-1}\beta^{\alpha}}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x}$$
$$x>0,\quad \alpha>0,\beta>0 \qquad (1)$$

Note: For $\alpha=1$, SamD reduces to the exponential distribution with parameter β.

The cumulative distribution function of Sameera distribution is given by

$$\Psi(x|\alpha,\beta)=\frac{1}{1+\alpha^2\beta}\left[\alpha^2\beta(1-e^{-\beta x})+\gamma(\alpha,\beta x)\right],$$
$$x>0,\quad \alpha>0,\beta>0, \qquad (2)$$

where $\gamma(\alpha,x)=\int_0^x t^{\alpha-1}e^{-t}dt$, is the lower incomplete gamma function. The graphs of the pdf and cdf of Sameera distribution are presented in Figures 1 and 2; respectively.

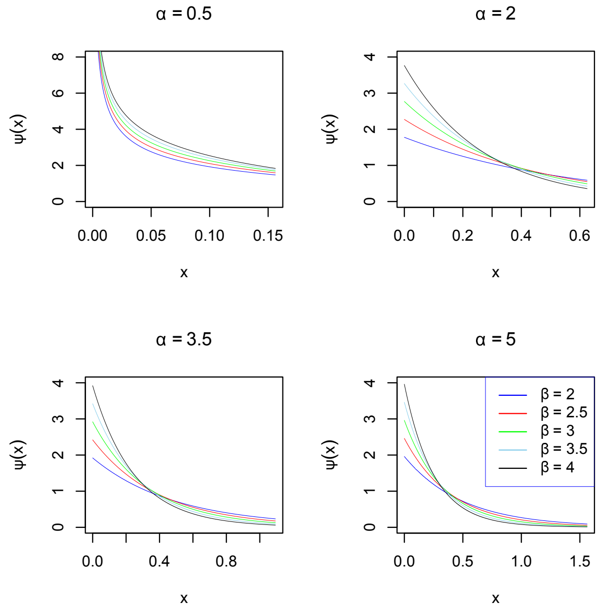

Fig. 1: The pdf of SamD when $\alpha=0.5$, 2, 3.5 and 5 for different values of β

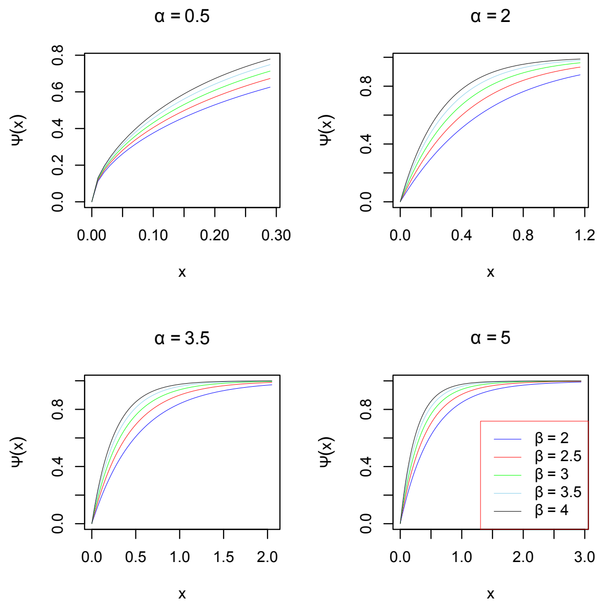

Fig. 2: The cdf of SamD when $\alpha=0.5$, 2, 3.5 and 5 for different values of β

3 Moments and Moment Generating Function

Moment generating function and the r^{th} moment are derived in this section. Accordingly, the mean, variance, kurtosis, skewness and coefficient of variation are calculated.

Theorem 1. *The moment generating function of Sameera distribution can be expressed as follows*

$$M_X(t) = \frac{1}{1+\alpha^2\beta}\left(\frac{\alpha^2\beta^2}{\beta-t} + \left(\frac{\beta}{\beta-t}\right)^{\alpha}\right), \quad t<\beta. \quad (3)$$

Proof.

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) = \int_0^\infty e^{tx}\psi(x)dx \\ &= \int_0^\infty \left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{t-\beta x}dx \\ &= \frac{1}{1+\alpha^2\beta}\left(\frac{\alpha^2\beta^2}{\beta-t} + \left(\frac{\beta}{\beta-t}\right)^{\alpha}\right), \quad t<\beta. \end{aligned}$$

Theorem 2. *The r^{th} moment of Sameera distribution can be expressed as follows*

$$E(X^r) = \frac{1}{1+\alpha^2\beta}\left[\alpha^2\frac{\Gamma(r+1)}{\beta^{r-1}} + \frac{\Gamma(r+\alpha)}{\beta^r\Gamma(\alpha)}\right] \quad (4)$$

Proof. Let X have a $SamD(\alpha,\beta)$, then the r^{th} moment is

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r\psi(x)dx \\ &= \int_0^\infty x^r\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x}dx \\ &= \int_0^\infty \left(\frac{x^r\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{r+\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x}dx \\ &= \frac{1}{1+\alpha^2\beta}\Big[\int_0^\infty \alpha^2\beta^2x^re^{-\beta x}dx \\ &\quad + \int_0^\infty \frac{1}{\Gamma(\alpha)}x^{r+\alpha-1}\beta^\alpha e^{-\beta x}dx\Big] \\ &= \frac{1}{1+\alpha^2\beta}\left[\alpha^2\frac{\Gamma(r+1)}{\beta^{r-1}} + \frac{\Gamma(r+\alpha)}{\beta^r\Gamma(\alpha)}\right] \end{aligned}$$

The first four moments can be found by substituting r = 1, 2, 3 and 4 in (4). Thus

$$\begin{aligned} \mu = E(X) &= \frac{\alpha}{1+\alpha^2\beta}\left[\alpha + \frac{1}{\beta}\right] \\ E(X^2) &= \frac{\alpha}{1+\alpha^2\beta}\left[\frac{2\alpha^2}{\beta} + \frac{(1+\alpha)}{\beta^2}\right] \\ E(X^3) &= \frac{\alpha}{1+\alpha^2\beta}\left[\frac{6\alpha^2}{\beta^2} + \frac{(1+\alpha)(2+\alpha)}{\beta^3}\right] \\ E(X^4) &= \frac{\alpha}{1+\alpha^2\beta}\left[\frac{24\alpha^2}{\beta^3} + \frac{(1+\alpha)(2+\alpha)(3+\alpha)}{\beta^4}\right] \end{aligned}$$

The variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis of the random variable $X \sim SamD(\alpha,\beta)$ are given; respectively, by:

$$\begin{aligned} Var(X) = \sigma^2 &= E(X^2) - (E(X))^2 \\ &= \frac{2\alpha^5\beta^2 - \alpha^4\beta^2 + \alpha^4\beta + \alpha^3\beta + \alpha}{\beta^2(\alpha^2\beta+1)^2} \\ C.V &= \frac{\sigma}{\mu} = \frac{\sqrt{2\alpha^5\beta^2 - \alpha^4\beta^2 + \alpha^4\beta + \alpha^3\beta + \alpha}}{\beta\alpha^2+\alpha} \\ sk(X) &= \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3} \\ &= \frac{\begin{pmatrix} \alpha^7\beta^2 + 2\alpha^6\beta^3 - 6\alpha^6\beta^2 + 11\alpha^5\beta^2 \\ -\alpha^5\beta + 7\alpha^3\beta + 2\alpha \end{pmatrix}}{(2\alpha^5\beta^2 - \alpha^4\beta^2 + \alpha^4\beta + \alpha^3\beta + \alpha)^{\frac{3}{2}}} \\ ku(X) &= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2E(X^2) - 3\mu^4}{\sigma^4} \\ &= \frac{\begin{pmatrix} \alpha^{10}\beta^3 + 60\alpha^9\beta^4 + 10\alpha^9\beta^3 \\ -3\alpha^8\beta^4 + 77\alpha^8\beta^3 + 7\alpha^8\beta^2 \\ +140\alpha^7\beta^3 + 62\alpha^7\beta^2 + 137\alpha^6\beta^2 \\ +17\alpha^6\beta + 136\alpha^5\beta^2 + 64\alpha^5\beta \\ +97\alpha^4\beta + 8\alpha^4 + 50\alpha^3\beta \\ +24\alpha^3 + 19\alpha^2 + 6\alpha \end{pmatrix}}{\beta^4\left(\alpha^2\beta+1\right)^4} \end{aligned}$$

Figures 3 - 7 show the three dimensions plots of the mean, standard deviation, skewness, excess kurtosis and the coefficient of variation of Sameera distribution for different values of α and β. The figures show that the distribution is skewed right with heavier tail than the normal distribution as all values of excess kurtosis = kurtosis − 3 [24] are positive. Also it shows that all measure values decrease as the values of α and β increase.

4 Reliability analysis

If T is a random variable that follows Sameera distribution, then the survival or reliability function, hazard, cumulative hazard function, the reversed hazard rate and odd functions corresponding to (1) and (2) are respectively, defined by

$$\begin{aligned} R(t) &= 1 - \Psi(t) \\ &= 1 - \frac{1}{1+\alpha^2\beta}\left[\alpha^2\beta(1-e^{-\beta x}) + \gamma(\alpha,x)\right] \end{aligned}$$

$$\begin{aligned} h(t) &= \frac{\psi(t)}{1-\Psi(t)} \\ &= \frac{\left(\alpha^2\beta^2 + \frac{x^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}\right)e^{-\beta x}}{1+\alpha^2\beta - \left[\alpha^2\beta(1-e^{-\beta x}) + \gamma(\alpha,x)\right]} \\ H(t) &= -ln(1-\Psi(t)) \\ &= -ln\left[1 - \frac{1}{1+\alpha^2\beta}\left[\alpha^2\beta(1-e^{-\beta x}) + \gamma(\alpha,x)\right]\right] \end{aligned}$$

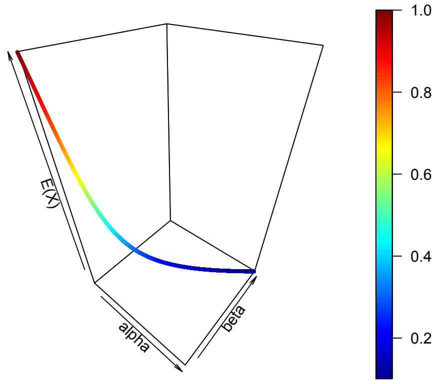

Fig. 3: The three dimension plot of the mean of SamD for different values of α and β.

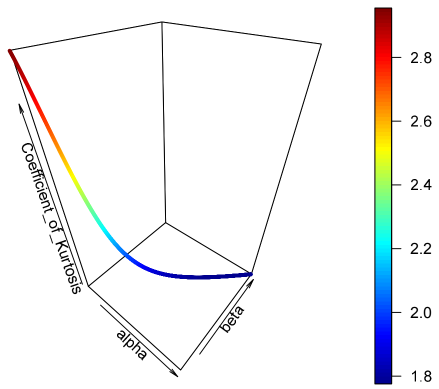

Fig. 6: The three dimension plot of the coefficient of kurtosis of SamD for different values of α and β.

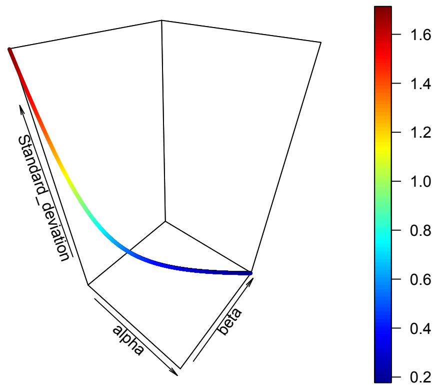

Fig. 4: The three dimension plot of the standard deviation of SamD for different values of α and β.

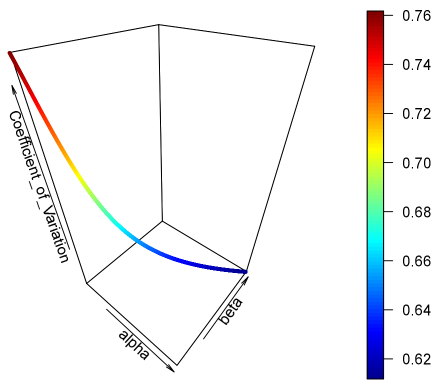

Fig. 7: The three dimension plot of the coefficient of variation of SamD for different values of α and β.

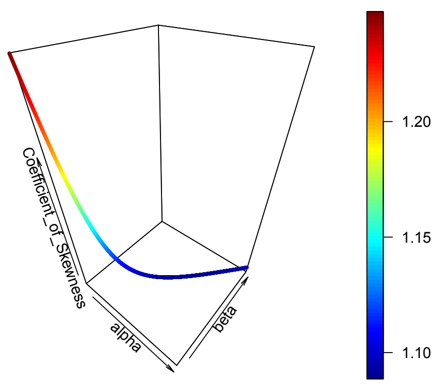

Fig. 5: The three dimension plot of the coefficient of skewness of SamD for different values of α and β.

$$rh(t) = \frac{\psi(t)}{\Psi(t)} = \frac{\left(\alpha^2\beta^2 + \frac{x^{\alpha-1}\beta^{\alpha}}{\Gamma(\alpha)}\right)e^{-\beta x}}{\left[\alpha^2\beta(1-e^{-\beta x}) + \gamma(\alpha,x)\right]},$$

$$O(t) = \frac{\Psi(t)}{1-\Psi(t)} = \frac{\left[\alpha^2\beta(1-e^{-\beta x}) + \gamma(\alpha,x)\right]}{1+\alpha^2\beta - \left[\alpha^2\beta(1-e^{-\beta x}) + \gamma(\alpha,x)\right]}$$

Figures 8 - 11 show that the reliability, hazard rate, reversed hazard rate and the cumulative hazard rate functions of SamD. They show that the hazard rate and reversed hazard rate functions are decreasing functions as the value of x is increasing. While the cumulative hazard function is increasing.

5 Order Statistics and Quantile Function

In this section, we will derive the distribution of first, n^{th} and j^{th} order statistics and the quantile function of Sameera distribution.

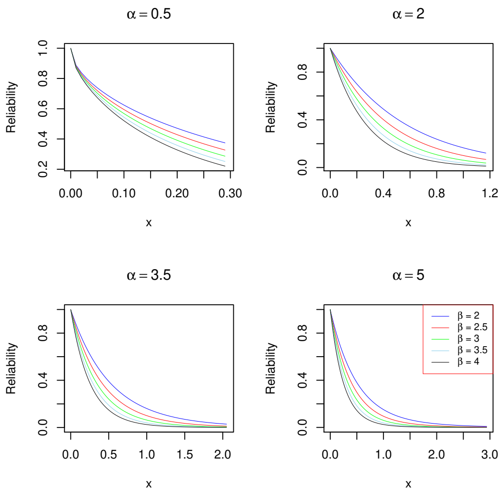

Fig. 8: The reliability function of SamD for $\alpha = 0.5, 2, 3.5, 5$.

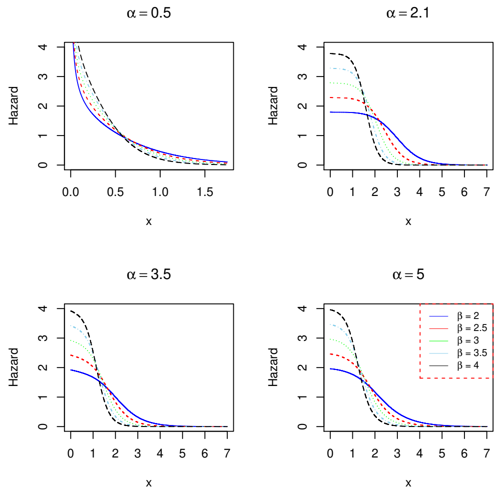

Fig. 9: The hazard rate function of SamD for $\alpha = 0.5, 2, 3.5, 5$.

5.1 Order statistics

Let $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ be the order statistics of the random sample $X_1, X_2, \cdots, X_n$ selected from Sameera distribution. Then the pdf of the j^{th} order statistics $X_{(j)}$ is defined as

$$g_{(j)}(x) = j\binom{n}{j}[\Psi(x)]^{j-1}[1-\Psi(x)]^{n-j}\psi(x) \qquad (5)$$

By replacing (1) and (2) in (5) and using binomial theorem, we get

$$\psi_{(j)}(x) = j\binom{n}{j}\left[\frac{1}{1+\alpha^2\beta}\right]^n\left[\alpha^2\beta(1-e^{-\beta x})+\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right]^{j-1} \times \left[1+\alpha^2\beta-\left[1-\alpha^2\beta(1-e^{-\beta x})+\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right]^{n-j}\right] \times \left(\alpha^2\beta^2+\frac{x^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}\right)e^{-\beta x}$$

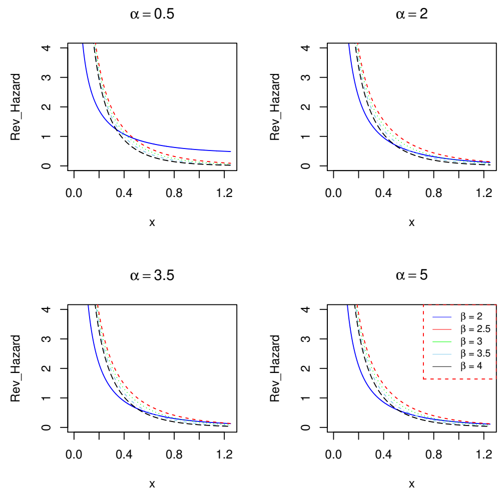

Fig. 10: The reversed hazard rate function of SamD for $\alpha = 0.5, 2, 3.5, 5$.

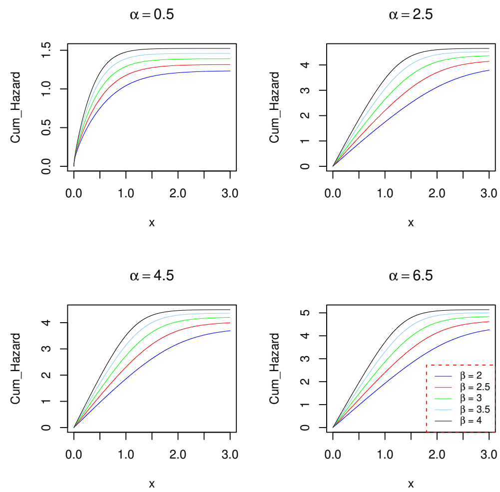

Fig. 11: The cumulative hazard rate function of SamD for $\alpha = 0.5, 2, 3.5, 5$.

The distribution of the minimum order statistic $X_{(1)} = min(X_1, X_2, \cdots, X_n)$ and the largest order statistic $X_{(n)} = max(X_1, X_2, \cdots, X_n)$ can be computed by replacing j in the previous equation by 1 and n; respectively. So, we get

$$\psi_{(1)}(x) = n\left[\frac{1}{1+\alpha^2\beta}\right]^n \left(\alpha^2\beta^2 + \frac{x^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}\right) e^{-\beta x} \times \left[1+\alpha^2\beta - \left[\alpha^2\beta(1-e^{-\beta x}) + \gamma(\alpha,x)\right]\right]^{n-1}$$

$$\psi_{(n)}(x) = n\left[\frac{1}{1+\alpha^2\beta}\right]^n \left(\alpha^2\beta^2 + \frac{x^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}\right) \times \left[\alpha^2\beta(1-e^{-\beta x}) + \gamma(\alpha,x)\right]^{n-1} e^{-\beta x}$$

5.2 Quantile function

The quantile function of a probability distribution with cdf, $\Psi(x)$, is defined by $q = \Psi^{-1}(x_q)$, where $0 < q < 1$. Then, the quantile function of Sameera distribution is given by

$$Q_p = \frac{1}{\beta}\gamma^{-1}\left(\alpha, \frac{\Gamma(\alpha)}{\alpha^2\beta}\left[p(1+\alpha^2\beta) - \frac{\alpha^2\beta}{G^{-1}(x)}\right]\right), \quad (6)$$

where $\gamma^{-1}(.,.)$ is the inverse of the lower incomplete gamma function.

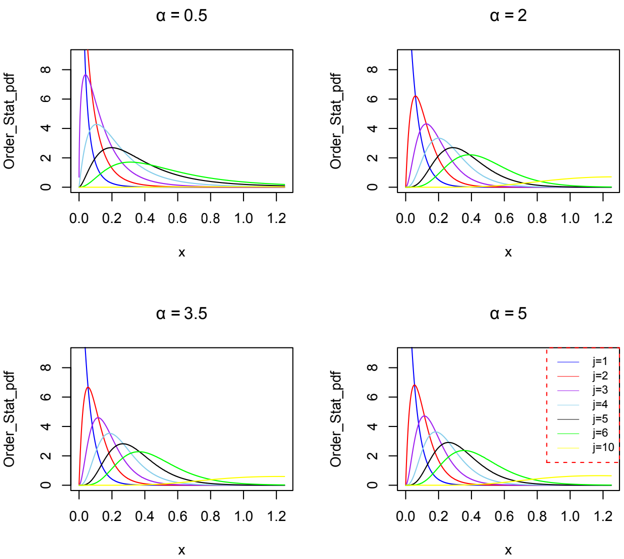

Fig. 12: The probability density functions of the first to sixth and tenth order statistics for a sample of size 10 of Sameera distribution when $\beta = 2$.

Figure 13 shows the quantile plot for different values of q. The selected values are $q = 0.05$, 0.25, 0.5, 0.75. 0.95 and 0.99. The figure shows that the line represent each value of q intersects the x-axis (horizontal line) in one point only, which means that the quantile function has exactly one solution regardless the value of q. This solution can no be determined in a close form, so one can use numerical methods to find this solution.

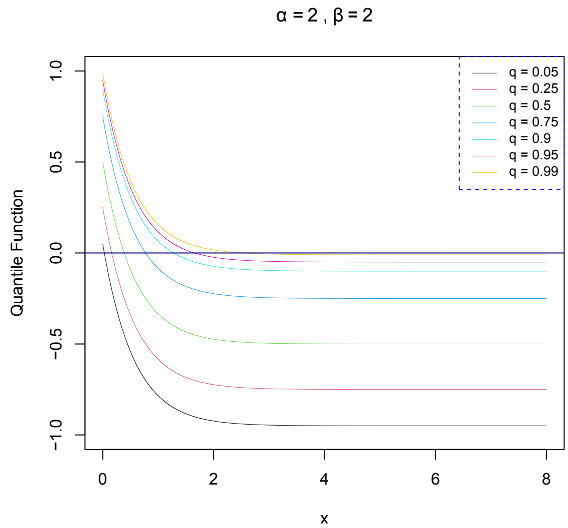

Fig. 13: The quantile function of SamD when $\alpha = \beta = 2$

6 Bonferroni and Lorenz Curves

The Bonferroni [26] and Lorenz [27] curves of a random variable X that follows Sameera distribution are defined, respectively, as

$$B(p) = \frac{1}{p\mu}\int_0^q x\psi(x)dx,$$

$$L(p) = \frac{1}{\mu}\int_0^q x\psi(x)dx,$$

where $q = \Psi^{-1}(p);\ p \in [0,1]$ *and* $\mu = E(X)$.

Hence the Bonferroni and Lorenz curves of Sameera distribution are, respectively, given by:

$$B(p) = \frac{\beta}{p\alpha(1+\alpha\beta)}\left[\alpha^2(1-(1+\beta q))e^{-\beta q} + \frac{\gamma(\alpha+1,\beta q)}{\beta}\right]$$

$$L(p) = \frac{\beta}{\alpha(1+\alpha\beta)}\left[\alpha^2(1-(1+\beta q))e^{-\beta q} + \frac{\gamma(\alpha+1,\beta q)}{\beta}\right]$$

7 Entropy

The Shannon entropy [28] is introduced to measure the uncertainty about an event accompanying with a discrete random variable. It can, also be defined for a continuous random variable X as:

$$H(X) = \int_0^\infty \psi(x)log(\psi(x))dx, \quad (7)$$

where X is a non-negative random variable with probability density function $\psi(x)$. Thus for Sameera distribution, the Shannon entropy is

$$H(X) = \int_0^\infty \left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right) e^{-\beta x} \times log\left(\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right) e^{-\beta x}\right) dx,$$

[29] defined Tsallis entropy based on a discrete probabilities p_i subject to the condition $\sum_{i=0}^{\infty} p_i = 1$ as

$$S_\delta = \frac{k}{\delta - 1}\left(1 - \sum_{i=0}^{\infty} p_i\right), \quad (8)$$

For continuous random variables with probability density function $\psi(x)$. Tsallis entropy is defined as:

$$S_\delta = \frac{k}{\delta - 1}\left(1 - \int_0^\infty (\psi(x))^\delta dx\right), \quad (9)$$

where $\delta \in \mathbb{R}$, $\delta \neq 1$. Thus for Sameera distribution with pdf defined in (1), Tsallis entropy is defined to be

$$\begin{aligned} S_\delta &= \frac{k}{\delta-1}\left(1 - \int_0^\infty \left(\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right) e^{-\beta x}\right)^\delta dx\right) \\ &= \frac{k}{\delta-1}\left(1 - \int_0^\infty \sum_{j=0}^{\delta}\binom{\delta}{j}\left(\frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)^j \left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta}\right)^{\delta-j} e^{-\beta\delta x} dx\right) \\ &= \frac{k}{\delta-1}\left(1 - \sum_{j=0}^{\delta}\binom{\delta}{j}\left(\frac{\alpha^{2(\delta-j)}\beta^{2(\delta-j)+j\alpha}}{(1+\alpha^2\beta)^\delta(\Gamma(\alpha))^j}\right) \times \int_0^\infty x^{j(\alpha-1)} e^{-\beta\delta x} dx\right) \\ &= \frac{k}{\delta-1}\left(1 - \sum_{j=0}^{\delta}\binom{\delta}{j}\left(\frac{\alpha^{2(\delta-j)}\beta^{2(\delta-j)+j\alpha}}{(1+\alpha^2\beta)^\delta(\Gamma(\alpha))^j}\right) \times \frac{\Gamma(j(\alpha-1)+1)}{\beta^{(j(\alpha-1)+1)}}\right) \\ &= \frac{k}{\delta-1}\left(1 - \sum_{j=0}^{\delta}\binom{\delta}{j}\left(\frac{\alpha^{2(\delta-j)}\beta^{2\delta-j-1}\Gamma(j(\alpha-1)+1)}{(1+\alpha^2\beta)^\delta(\Gamma(\alpha))^j}\right)\right) \end{aligned}$$

The limit of Tsallis entropy as δ approaches 1, results in Shannon entropy [29].

The Re'nyi entropy [30] of order δ for a continuous random variable $X \sim SamD(\alpha, \beta)$ can be derived similar to Tsallis entropy. It is defined as:

$$\begin{aligned} R_\delta &= \left(\frac{1}{1-\delta}\right) log\left[\int_0^\infty \psi^\delta(x)dx\right] \\ &= \left(\frac{1}{1-\delta}\right) log\left[\sum_{j=0}^{\delta}\binom{\delta}{j} \times \left(\frac{\alpha^{2(\delta-j)}\beta^{2\delta-j-1}\Gamma(j(\alpha-1)+1)}{(1+\alpha^2\beta)^\delta(\Gamma(\alpha))^j}\right)\right] \end{aligned}$$

Table 2 shows the numerical results of Shannon, Tsallis and Re'nyi entropies for different values of α of 1 to 1.14 with step of 0.02 and values of β of 1-1.10 with step 0.02. For Tsallis and Re'nyi entropies the value of $\delta = 5$ is used. These values are calculated using the R software [31].

8 Mean Deviations about Mean and Median

One of the good measure of variability from the mean of the data is the mean deviation about mean or median [32]. Hence, for Sameera distribution they are defined respectively, as

$$\begin{aligned} MD_{mean} &= E|X-\mu| = \int_0^\infty |x-\mu|\psi(x)dx \\ &= \int_0^\mu (\mu-x)\psi(x)dx + \int_\mu^\infty (x-\mu)\psi(x)dx \\ &= 2\int_0^\mu (\mu-x)\psi(x)dx \\ &= 2\mu\Psi(\mu) - 2\int_0^\mu x\psi(x)dx \end{aligned}$$

Hence,

$$MD_{mean} = \frac{2}{1+\alpha^2\beta}\left\{\left[\frac{(\alpha\beta+1)}{1+\alpha^2\beta}\left(\frac{\alpha^2\beta(\alpha+1)}{1+\alpha^2\beta} \times \left(1-e^{-\beta\mu}\right) + \gamma(\alpha,\beta\mu)\right) - \left[\alpha^2\left(1-(\beta\mu+1)e^{-\beta\mu}\right) + \frac{\gamma(\alpha+1,\beta\mu)}{\beta}\right]\right]\right\},$$

where $\gamma(a,x)$ is the incomplete gamma function and $\mu = \frac{\alpha}{1+\alpha^2\beta}\left[\alpha + \frac{1}{\beta}\right]$. The mean absolute deviation about the median (M) is defined similar to the mean absolute deviation about mean as:

$$MD_{med} = \frac{2}{1+\alpha^2\beta}\left\{\left[\frac{(\alpha\beta+1)}{1+\alpha^2\beta}\left(\frac{\alpha^2\beta(\alpha+1)}{1+\alpha^2\beta} \times \left(1-e^{-\beta M}\right) + \gamma(\alpha,\beta M)\right) - \left[\alpha^2\left(1-(\beta M+1)e^{-\beta M}\right) + \frac{\gamma(\alpha+1,\beta M)}{\beta}\right]\right]\right\},$$

where M is he median of Sameera distribution and it is the solution of the equation $\Psi(M) = 0.5$.

9 Methods of Estimation

9.1 Maximum likelihood method

The likelihood function, $L(x,\alpha,\beta)$, for a random sample $X_1,X_2,\dots X_n$ selected from Sameera distribution, is defined by

$$\begin{aligned}L(x,\alpha,\beta) &= \prod_{j=1}^{n}\psi(x_j,\alpha,\beta)\\ &= \prod_{j=1}^{n}\left[\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta}+\frac{x_i^{\alpha-1}\beta^{\alpha}}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x_i}\right]\\ &= \left(\frac{1}{1+\alpha^2\beta}\right)^n\prod_{i=1}^{n}\left[\left(\alpha^2\beta^2+\frac{x_i^{\alpha-1}\beta^{\alpha}}{\Gamma(\alpha)}\right)e^{-\beta x_i}\right]\\ &= \left(\frac{1}{1+\alpha^2\beta}\right)^n\prod_{i=1}^{n}\left[\left(\alpha^2\beta^2+\frac{x_i^{\alpha-1}\beta^{\alpha}}{\Gamma(\alpha)}\right)\right]e^{-\beta\sum_{i=1}^{n}x_i}\end{aligned}$$

then, the natural logarithm will give

$$\begin{aligned}\xi &= \ln(L(x,\alpha,\beta))\\ &= -n\ln\left(1+\alpha^2\beta\right)+\left[\sum_{i=1}^{n}\ln\left(\alpha^2\beta^2+\frac{(x_i\beta)^{\alpha}}{x_i\Gamma(\alpha)}\right)\right]\\ &\quad -\beta\sum_{i=1}^{n}x_i \end{aligned}\qquad (10)$$

Deriving with respect to α and β, we get

$$\left\{\begin{aligned}\frac{\partial\xi}{\partial\alpha} &= \left(\frac{-2n\alpha\beta}{1+\alpha^2\beta}\right)\\ &\quad +\sum_{j=1}^{n}\left[\frac{\left(\begin{array}{c}\alpha\beta^2(x_i\Gamma(\alpha))^2-x_i\Gamma'(\alpha)\\ +(\beta x_i)^{\alpha}\left(x_i\Gamma(\alpha)\ln(\beta x_i)\right)\end{array}\right)}{(x_i\Gamma(\alpha))[(x_i\Gamma(\alpha))+(\beta x_i)^{\alpha}]}\right]\\ \frac{\partial\xi}{\partial\beta} &= \left(\frac{-n\alpha^2}{1+\alpha^2\beta}\right)-\sum_{j=1}^{n}x_i\\ &\quad +\sum_{j=1}^{n}\left[\frac{2\alpha^2\beta x_i\Gamma(\alpha)+\alpha^2x_i(\beta x_i)^{\alpha-1}}{\alpha^2\beta^2x_i\Gamma(\alpha)+(\beta x_i)^{\alpha}}\right]\end{aligned}\right\}\qquad (11)$$

The MLE $(\hat\alpha,\hat\beta)$ of (α,β) can be obtained by solving the system of equation $\left\{\frac{\partial\xi}{\partial\alpha}=0,\frac{\partial\ell}{\partial\beta}=0\right\}$.

The system of equations in (11) has no explicit analytical solution, hence, it can be solved numerically using any iterartive numerical method.

9.2 Ordinary and weighted least square methods

This subsection shows other methods for estimation of the model parameters, which are the ordinary least squares (OLSE) and the weighted least squares (WLSE) estimators. They are suggested by [35].

Let $X_{(1)},X_{(2)},\dots,X_{(n)}$ be the order statistics of the random sample $X_1,X_2,\dots,X_n$ which has a cdf defined by (2). The OLS of α and β can be obtained by minimizing

$$\sum_{i=1}^{n}\left[\Psi(x_{(i)};\alpha,\beta)-\frac{i}{n+1}\right]^2 \qquad (12)$$

with respect to α and β. In our case, the $\hat\alpha_{OLSE}$ and $\hat\beta_{OLSE}$ are obtained by minimizing

$$\sum_{i=1}^{n}\left[\frac{1}{1+\alpha^2\beta}\left[\alpha^2\beta(1-e^{-\beta x_{(i)}})+\frac{\gamma(\alpha,\beta x_{(i)})}{\Gamma(\alpha)}\right]-\frac{i}{n+1}\right]^2,$$

with respect to the two parameters α and β.

The weighted least squares estimators (WLSE) of α and β can be obtained by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\Psi(x_{(i)};\alpha,\beta)-\frac{i}{n+1}\right]^2, \qquad (13)$$

So, in SamD case, the $\hat\alpha_{WLSE}$ and $\hat\beta_{WLSE}$ can be obtained by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{1}{1+\alpha^2\beta}\left[\alpha^2\beta(1-e^{-\beta x_{(i)}})+\frac{\gamma(\alpha,\beta x_{(i)})}{\Gamma(\alpha)}\right]-\frac{i}{n+1}\right]^2,$$

with respect to α and β; respectively. Some results of these estimators are shown in the simulation study in Section 10.

9.3 Method of maximum product of spacings

An alternative estimation method to the maximum likelihood (ML) method, [33], [34] proposed the method of maximum product spacing (MPS). This method relies on maximizing the geometric mean of the spacings of the data with respect to the parameters. The MPS method provides consistent and asymptotically efficient estimators whether MLE exists or not. For a random sample $X_1,X_2,\dots,X_n$ of size n and $X_{(1)},X_{(2)},\dots,X_{(n)}$ be the order statistics of the random sample. The uniform spacings is defined as:

$$\Xi_i(\alpha,\beta) = \Psi(x_{(i)}|\alpha,\beta)-\Psi(x_{(i-1)}|\alpha,\beta),\ i=1,\dots,n+1,$$

where $\Psi(x_{(0)}|\alpha,\beta)=0$ and $\Psi(x_{(n+1)}|\alpha,\beta)=1$. It is clear that $\sum_{i=1}^{n+1}\Xi(\alpha,\beta)=1$.

The MPS estimators of the distribution parameters α and β denoted by $\hat\alpha_{MPS}$ and $\hat\beta_{MPS}$ can be obtained by maximizing the geometric mean of the spacings, that is,

$$K(\alpha,\beta|x) = \left(\prod_{i=1}^{n+1}\Xi_i(\alpha,\beta)\right)^{\frac{1}{n+1}} \qquad (14)$$

Now, the natural logarithm of (14) gives

Table 1: The Shannon, Rényi and Tsallis entropies for different values of α, β and $\delta = 5$

β	α	Shannon	Rényi	Tsallis
1.00	1.00	1.76141	-0.00074	0.24926
1.00	1.02	1.76382	-0.00073	0.24927
1.00	1.04	1.76615	-0.00072	0.24928
1.00	1.06	1.76841	-0.00071	0.24929
1.00	1.08	1.77061	-0.00070	0.24930
1.00	1.10	1.77275	-0.00069	0.24931
1.00	1.12	1.77482	-0.00068	0.24932
1.00	1.14	1.77684	-0.00068	0.24932
1.02	1.00	1.74108	-0.00081	0.24919
1.02	1.02	1.74350	-0.00079	0.24921
1.02	1.04	1.74585	-0.00078	0.24922
1.02	1.06	1.74812	-0.00077	0.24923
1.02	1.08	1.75034	-0.00076	0.24924
1.02	1.10	1.75249	-0.00075	0.24925
1.02	1.12	1.75458	-0.00074	0.24926
1.02	1.14	1.75661	-0.00073	0.24927
1.04	1.00	1.72113	-0.00087	0.24913
1.04	1.02	1.72357	-0.00086	0.24914
1.04	1.04	1.72593	-0.00085	0.24915
1.04	1.06	1.72822	-0.00084	0.24916
1.04	1.08	1.73045	-0.00083	0.24917
1.04	1.10	1.73262	-0.00082	0.24918
1.04	1.12	1.73472	-0.00080	0.24920
1.04	1.14	1.73676	-0.00080	0.24920
1.06	1.00	1.70156	-0.00095	0.24905
1.06	1.02	1.70401	-0.00093	0.24907
1.06	1.04	1.70639	-0.00092	0.24908
1.06	1.06	1.70870	-0.00091	0.24909
1.06	1.08	1.71094	-0.00089	0.24911
1.06	1.10	1.71312	-0.00088	0.24912
1.06	1.12	1.71523	-0.00087	0.24913
1.06	1.14	1.71729	-0.00086	0.24914
1.08	1.00	1.68235	-0.00103	0.24897
1.08	1.02	1.68481	-0.00101	0.24899
1.08	1.04	1.68721	-0.00099	0.24901
1.08	1.06	1.68953	-0.00098	0.24902
1.08	1.08	1.69178	-0.00097	0.24903
1.08	1.10	1.69397	-0.00095	0.24905
1.08	1.12	1.69610	-0.00094	0.24906
1.08	1.14	1.69817	-0.00093	0.24907
1.08	1.16	1.70018	-0.00092	0.24908
1.10	1.00	1.66349	-0.00111	0.24889
1.10	1.02	1.66596	-0.00109	0.24891
1.10	1.04	1.66837	-0.00107	0.24893
1.10	1.06	1.67071	-0.00106	0.24894
1.10	1.08	1.67298	-0.00104	0.24896
1.10	1.10	1.67518	-0.00103	0.24897
1.10	1.12	1.67732	-0.00102	0.24898
1.10	1.14	1.67940	-0.00100	0.24900

Table 2: Numerical results for Shannon, Rényi and Tsallis entropies for Sameera distribution using different values of α and β with δ=5.

$$\begin{aligned} NL(\alpha,\beta|x) &= \frac{1}{n+1}\sum_{i=1}^{n+1} ln(\Xi_i(\alpha,\beta)) \\ &= \frac{1}{n+1}\Bigg\{\sum_{i=1}^{n+1} -ln(1+\alpha^2\beta) \\ &\quad +ln\left[\begin{array}{c} \alpha^2\beta(1-e^{-\beta x_{(i)}})+\frac{\gamma(\alpha,\beta x_{(i)})}{\Gamma(\alpha)} \\ -\alpha^2\beta(1-e^{-\beta x_{(i-1)}})+\frac{\gamma(\alpha,\beta x_{(i-1)})}{\Gamma(\alpha)} \end{array}\right]\Bigg\} \quad (15) \end{aligned}$$

The MPS estimators $\hat{\alpha}_{MPS}$ and $\hat{\beta}_{MPS}$ can be obtained by solving the following nonlinear system of equations with respect to the parameters α and β.

$$\frac{\partial NL(\alpha,\beta|x)}{\partial\alpha} = 0$$
$$\frac{\partial NL(\alpha,\beta|x)}{\partial\beta} = 0,$$

9.4 Methods of minimum distances

[36] introduced the method of minimum distances to obtain strong consistent estimators. It is defined by considering a random sample of size n, say $X_1,\cdots,X_n$ with cdf $\Psi(x|\alpha,\beta)$. Assuming that $\Psi_n(x)$ is the empirical distribution function based on the sample $\mathbf{x} = (x_1,\cdots,x_n)$. If $(\hat{\alpha},\hat{\beta})$ is the vector of estimators of (α,β), then $\Psi(x|\hat{\alpha},\hat{\beta})$ is an estimator of $\Psi(x|\alpha,\beta)$. Assuming $(\hat{\alpha},\hat{\beta})$ exist, such that

$$d[\Psi(x|\hat{\alpha},\hat{\beta}),\Psi_n(x)] = inf\{d[\Psi(x|\alpha,\beta),\Psi_n(x)]\},$$

where $d[.,.]$ is the distance between $\Psi(x|\hat{\alpha},\hat{\beta})$ and $\Psi_n(x)$, then $(\hat{\alpha},\hat{\beta})$ is called the minimum-distance estimator of (α,β) [37].

9.4.1 Cramer-Von-Mises method

Cramer-Von-Mises and Anderson–Darling methods of estimation are the most famous methods of minimizing the test statistics between the theoretical and empirical cdfs.

Cramer-Von-Mises method [38], [39] usually denoted as W^2, is a method used in one-sample applications to compare between the theoretical cumulative distribution function $\Psi^*(x)$ of a random variable and a given empirical distribution $\Psi_n(x)$ using the goodness of fit. It is also used as a part of the minimum distance method of estimation. It is defined as

$$\tau^2 = \int_{-\infty}^{\infty}[\Psi_n(x)-\Psi^*(x)]^2\,d\Psi^*(x)$$

For a random sample of size n with observed values $x_1,\cdots,x_n$ sorted in an ascending order, the Cramer-Von Mises test statistic value is [40].

$$W^2 = n\tau^2 = \sum_{i=0}^{n}\left[\Psi(x_{(i)},\alpha,\beta)-\frac{2i-1}{2n}\right]^2+\frac{1}{12n}$$

Thus for a random sample of size n from Sameera distribution with observed with observed values $x_1,\cdots,x_n$ sorted in an ascending order. The Cramer-Von Mises test statistic value is

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{1}{1+\alpha^2\beta}\left[\alpha^2\beta(1-e^{-\beta x_{(i)}}) + \frac{\gamma(\alpha,\beta x_{(i)})}{\Gamma(\alpha)}\right] - \frac{2i-1}{2n}\right]^2$$

The Cramer-Von Mises estimators $(\hat{\alpha},\hat{\beta})$ of (α,β) can be obtained by minimizing W^2.

9.4.2 Method of Anderson-Darling

The method of Anderson-Darling for estimating the distribution parameters was introduced by [41]. It is defined as

$$\begin{aligned}A(\alpha,\beta) &= -n - \frac{1}{n}\sum_{i=0}^{n}(2i-1)\Big[log[\Psi(x_{(i)};\alpha,\beta)] + \log\overline{\Psi}(x_{(n+1-i)};\alpha,\beta)\Big] \\ &= -n - \sum_{i=0}^{n}\left(\frac{2i-1}{n}\right) \qquad (16)\\ &\times\left(\begin{array}{c} log\left[\frac{1}{1+\alpha^2\beta}\left[\alpha^2\beta(1-e^{-\beta x_{(i)}}) + \frac{\gamma(\alpha,\beta x_{(i)})}{\Gamma(\alpha)}\right]\right] + \log\left[\frac{\alpha^2\beta}{1+\alpha^2\beta}\right. \\ \left[\alpha^2\beta(1-e^{-\beta x_{(n+1-i)}}) + \frac{\gamma(\alpha,\beta x_{(n+1-i)})}{\Gamma(\alpha)}\right]\Big]\end{array}\right), \qquad (17)\end{aligned}$$

where $\overline{\Psi} = 1-\Psi$. The estimators $\hat{\alpha}_{AD}$ and $\hat{\beta}_{AD}$ can be obtained by minimizing (16).

10 Stress-Strength Reliability

Suppose that X and Y are two independent random variables from Sameera distribution, where X represents the strength of the system and Y is the stress applied to this system [42]. The component failed to work at the moment that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X > Y$. The stress- strength model is defined as $p(Y < X)$, [43].

$$p(Y<X) = \int_0^\infty\int_0^x\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right) \times\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{y^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x}e^{-\beta y}dydx$$

Using the power series [44] for $e^{-\beta y}$, we can write the above integration as:

$$\begin{aligned}p(Y<X) &= \int_0^\infty\int_0^x\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x} \times\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{y^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)\sum_{i=0}^{n}\frac{(-1)^i\beta^i y^i}{i!}dydx \\
&= \int_0^\infty\int_0^x\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x} \times\sum_{i=0}^{n}\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{y^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)\frac{(-1)^i\beta^i y^i}{i!}dydx \\
&= \int_0^\infty\int_0^x\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x} \times\sum_{i=0}^{n}\left(\frac{(-1)^i\alpha^2\beta^{i+2}y^i}{i!(1+\alpha^2\beta)} + \frac{(-1)^i\beta^{i+\alpha}y^{\alpha+i-1}}{i!(1+\alpha^2\beta)\Gamma(\alpha)}\right)dydx \\
&= \sum_{i=0}^{n}\int_0^\infty\left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right) \times\left(\frac{(-1)^i\alpha^2\beta^{i+2}x^{i+1}}{(i+1)!(1+\alpha^2\beta)} + \frac{(-1)^i\beta^{i+\alpha}x^{\alpha+i}}{i!(\alpha+i)(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x}dx \\
&= \sum_{i=0}^{n}\int_0^\infty\left(\begin{array}{c}\frac{(-1)^i\alpha^4\beta^{i+4}x^{i+1}}{(i+1)!(1+\alpha^2\beta)^2} + \frac{(-1)^i\alpha^2\beta^{i+\alpha+2}x^{\alpha+i}}{i!(\alpha+i)(1+\alpha^2\beta)^2\Gamma(\alpha)} + \\ \frac{(-1)^i\alpha^2\beta^{i+\alpha+2}x^{\alpha+i}}{(i+1)!(1+\alpha^2\beta)^2\Gamma(\alpha)} + \frac{(-1)^i\beta^{i+2\alpha}x^{2\alpha+i-1}}{i!(\alpha+i)(1+\alpha^2\beta)^2(\Gamma(\alpha))^2}\end{array}\right)\times e^{-\beta x}dx \\
&= \sum_{i=0}^{n}\left(\begin{array}{c}\frac{(-1)^i\alpha^4\beta^{i+4}\Gamma(i+2)}{(i+1)!\beta^{i+2}(1+\alpha^2\beta)^2} \\ + \frac{(-1)^i\alpha^2\beta^{i+\alpha+2}\Gamma(\alpha+i+1)}{i!(\alpha+i)\beta^{\alpha+i+1}(1+\alpha^2\beta)^2\Gamma(\alpha)} \\ + \frac{(-1)^i\alpha^2\beta^{i+\alpha+2}\Gamma(\alpha+i+1)}{(i+1)!\beta^{\alpha+i+1}(1+\alpha^2\beta)^2\Gamma(\alpha)} \\ + \frac{(-1)^i\beta^{i+2\alpha}\Gamma(2\alpha+i)}{i!(\alpha+i)\beta^{2\alpha+i}(1+\alpha^2\beta)^2(\Gamma(\alpha))^2}\end{array}\right) \\
&= \sum_{i=0}^{n}\left(\begin{array}{c}\frac{(-1)^i\alpha^4\beta^2}{(1+\alpha^2\beta)^2} + \frac{(-1)^i\alpha^2\beta\Gamma(\alpha+i)}{i!(1+\alpha^2\beta)^2\Gamma(\alpha)} \\ + \frac{(-1)^i\alpha^2\beta\Gamma(\alpha+i+1)}{(i+1)!(1+\alpha^2\beta)^2\Gamma(\alpha)} + \frac{(-1)^i\Gamma(2\alpha+i)}{i!(\alpha+i)(1+\alpha^2\beta)^2(\Gamma(\alpha))^2}\end{array}\right) \\
&= \sum_{i=0}^{n}\frac{(-1)^i}{(1+\alpha^2\beta)^2}\left(\begin{array}{c}\alpha^4\beta^2 + \frac{\alpha^2\beta\Gamma(\alpha+i)}{i!\Gamma(\alpha)} \\ + \frac{\alpha^2\beta\Gamma(\alpha+i+1)}{(i+1)!\Gamma(\alpha)} + \frac{\Gamma(2\alpha+i)}{i!(\alpha+i)(\Gamma(\alpha))^2}\end{array}\right)\end{aligned}$$

11 Simulation study

In this section, a simulation study is performed to test the accuracy of the estimators of the Sameera distribution parameters with the help of R software [31]. For this purpose, $N = 1000$ samples are generated, each of size 15, 100, 150, 200, 300, and 500 for values of $\alpha = \beta = 0.5$ using (10).
For each sample, the estimators of the parameter space $\phi = (\alpha,\beta)$ using MLE, OLS, WLS, MPS, CVM, and AD methods of estimation with their mean square error (MSE) and the bias are obtained. Then, the average bias (AB) and the average mean square error (AMSE) are calculated as follows:

$$AB(\hat{\phi}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi),\ AMSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi)^2$$

The results of this simulation are summarized in Tables 3 and 4.

Table 3: Parameter Estimates and their average biases and average mean squares errors, when $\alpha = 0.5$.

n	Method	$\hat{\alpha}$	$AB(\hat{\alpha})$	$AMSE(\hat{\alpha})$
15	MLE	0.5899	0.0899	0.0641
	OLS	0.5041	0.0041	0.0067
	WLS	0.5303	0.0303	0.0556
	AD	0.5361	0.0361	0.0449
	CVM	0.6114	0.1114	0.1036
	MPS	0.7421	0.2421	0.1633
100	MLE	0.5112	0.0112	0.0035
	OLS	0.5114	0.0114	0.0013
	WLS	0.5078	0.0078	0.0040
	AD	0.5046	0.0046	0.0037
	CVM	0.5088	0.0088	0.0042
	MPS	0.5319	0.0319	0.0046
150	MLE	0.5049	0.0049	0.0021
	OLS	0.5102	0.0102	0.0009
	WLS	0.5024	0.0024	0.0024
	AD	0.5039	0.0039	0.0023
	CVM	0.5065	0.0065	0.0030
	MPS	0.5210	0.0210	0.0027
200	MLE	0.5047	0.0047	0.0016
	OLS	0.5103	0.0103	0.0008
	WLS	0.5018	0.0018	0.0018
	AD	0.5037	0.0037	0.0017
	CVM	0.5061	0.0061	0.0022
	MPS	0.5172	0.0172	0.0019
300	MLE	0.5042	0.0042	0.0009
	OLS	0.5111	0.0111	0.0006
	WLS	0.5024	0.0024	0.0010
	AD	0.5014	0.0014	0.0011
	CVM	0.5023	0.0023	0.0014
	MPS	0.5116	0.0116	0.0012
500	MLE	0.5015	0.0015	0.0006
	OLS	0.5110	0.0110	0.0005
	WLS	0.5004	0.0004	0.0007
	AD	0.5006	0.0006	0.0007
	CVM	0.5012	0.0012	0.0008
	MPS	0.5068	0.0068	0.0007

Tables 3 and 4 show the values of the average bias, and the average of mean squares errors for $\hat{\alpha}$ and $\hat{\beta}$. They show that these values decrease with increasing sample sizes, thus the estimates behave in a standard manner for different values of α and β. Also, it indicates that the MLEs are asymptotically unbiased and consistent. Table 5 summarizes the preferences of the methods of estimation. It shows that for small sample sizes the OLS method is the best based on AB and AMSE. Based on AMSE the OLS is the best method regardless the sample size. Based on AB it alternates between OLS, WLS, and AD methods.

12 Real Data Applications

In this section, we show the flexibility of the proposed distribution by considering two real-life time data sets and comparing its goodness of fit with some existing distributions.

The first data set is reported in [45] and represents a failure time of 50 items. The failure times are given in Table 6.

The second data set consists of 46 observations reported on active repair times (hours) for an airborne communication transceiver analyzed by [46] and given in Table 7.

The goodness of fit of the proposed distribution is compared with the following distributions:

–Gamma distribution [47]
$f(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)},\ x > 0,\ \alpha > 0,\ \beta > 0.$
–Weibull distribution [48]
$f(x) = \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-(x/\beta)^{\alpha}},\ x > 0,\ \alpha > 0,\ \beta > 0$
–Lindley distribution [49]: $f(x) = \frac{\alpha^2(1+x)e^{-\alpha x}}{1+\alpha},\ x > 0,\ \alpha > 0$
–Shanker distribution [50]: $f(x) = \frac{\alpha^2(\alpha+x)}{\alpha^2+1} e^{-\alpha x},\ x > 0, \alpha > 0$
–Exponential distribution [51]: $f(x) = \alpha e^{-\alpha x},\ x > 0,\ \alpha > 0$

Table 4: Parameter Estimates and their average biases and average mean squares errors, when $\beta = 0.5$.

n	Method	$\hat{\beta}$	$AB(\hat{\beta})$	$AMSE(\hat{\beta})$
15	MLE	0.6615	0.1615	0.1441
	OLS	0.5167	0.0167	0.0070
	WLS	0.5512	0.0512	0.1236
	AD	0.5748	0.0748	0.1136
	CVM	0.6769	0.1769	0.2361
	MPS	0.8908	0.3908	0.3507
100	MLE	0.5224	0.0224	0.0096
	OLS	0.5210	0.0210	0.0016
	WLS	0.5139	0.0139	0.0107
	AD	0.5121	0.0121	0.0102
	CVM	0.5212	0.0212	0.0151
	MPS	0.5646	0.0646	0.0153
150	MLE	0.5121	0.0121	0.0063
	OLS	0.5212	0.0212	0.0014
	WLS	0.5061	0.0061	0.0074
	AD	0.5117	0.0117	0.0073
	CVM	0.5139	0.0139	0.0100
	MPS	0.5417	0.0417	0.0086
200	MLE	0.5104	0.0104	0.0045
	OLS	0.5219	0.0219	0.0012
	WLS	0.5048	0.0048	0.0053
	AD	0.5063	0.0063	0.0051
	CVM	0.5112	0.0112	0.0069
	MPS	0.5339	0.0339	0.0059
300	MLE	0.5058	0.0058	0.0028
	OLS	0.5200	0.0200	0.0009
	WLS	0.5029	0.0029	0.0033
	AD	0.5038	0.0038	0.0036
	CVM	0.5068	0.0068	0.0047
	MPS	0.5242	0.0242	0.0038
500	MLE	0.5023	0.0023	0.0017
	OLS	0.5216	0.0216	0.0008
	WLS	0.5002	0.0002	0.0019
	AD	0.5009	0.0009	0.0020
	CVM	0.5043	0.0043	0.0027
	MPS	0.5152	0.0152	0.0021

Table 5: Best methods based on simulation results

n	$AB(\hat{\alpha})$	$AMSE(\hat{\alpha})$	$AB(\hat{\beta})$	$AMSE(\hat{\beta})$
15	OLS	OLS	OLS	OLS
100	AD	OLS	AD	OLS
150	WLS	OLS	WLS	OLS
200	WLS	OLS	WLS	OLS
300	AD	OLS	WLS	OLS
500	WLS	OLS	WLS	OLS

H

Table 6: Data 1: Failure times

0.12	0.43	0.92	1.14	1.24	1.61	1.93
2.38	4.51	5.09	6.79	7.64	8.45	11.90
11.94	13.01	13.25	14.32	17.47	18.10	18.66
19.23	24.39	25.01	26.41	26.80	27.75	29.69
29.84	31.65	32.64	35.00	40.70	42.34	43.05
43.40	44.36	45.40	48.14	49.10	49.44	51.17
58.62	60.29	72.13	72.22	72.25	72.29	85.20
89.52						

Table 7: Data 2: Repair times (hours)for an airborne communication transceiver

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7
0.7	0.7	0.8	0.8	1.0	1.0	0.80	1.0	1.1
1.3	1.5	1.5	1.5	1.5	2.0	2.0	2.2	2.5
2.7	3.0	3.0	3.3	3.3	4.0	4.0	4.5	4.7
5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3	22.0
24.55								

For comparison, we consider the following goodness of fit criteria: -2lnL, Akaike Information Criterion (AIC) [52], Corrected Akaike Information Criterion (CAIC) [53], Bayesian Information Criterion (BIC) [54], Kolmogorov-Smirnov Statistic (KS-Statistic) and its p-value [55], where

$$AIC = -2lnL + 2k, \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$
$$BIC = -2lnL + kln(n), \qquad KS = \sup_x |F_n(x) - F_0(x)|,$$

where L is the likelihood function, k is the number of parameters, n is the sample size and $F_n(x)$ is the empirical distribution function.

Table 8 shows that the Sameera distribution has the smallest values of -2lnL, AIC, CAIC, BIC, and KS statistics with the highest p-values compared with other fitted distributions. This indicates that the proposed distribution is more adequate in fitting both data sets than other distributions.

For both data sets, the MLEs of the parameters of the fitted distributions along with their corresponding confidence intervals are computed and the results are summarized in Table 9.

13 Conclusion

This article suggested a new two parameter continuous distribution called Sameera distribution. Its statistical properties are discussed thoroughly including: the moments and their related measures, moment-generating function, reliability analysis functions, mean deviation about the mean and median, Bonferroni and Lorenz curves, Shannon, Rényi and Tsallis

Table 8: $-2lnL$, AIC, AICC, BIC, KS statistic and the p-values of the fitted distributions.

Data	Distribution	$-2logL$	AIC	CAIC
1	Sameera	435.4424	439.4423	439.6976
	Gamma	440.5202	444.5202	444.7755
	Weibull	440.698	444.698	444.9534
	Lindley	453.9486	455.9486	456.032
	Shanker	463.371	465.3709	465.4543
	Exponential	440.7134	442.7134	442.7968
2	Sameera	204.0448	208.0447	208.3238
	Gamma	209.8618	213.2cm619	214.141
	Weibul	208.9394	212.9394	213.2185
	Lindley	219.9694	221.9694	222.0603
	Shanker	223.5056	225.5056	225.5965
	Exponential	210.0124	212.0124	212.1033
Data	**Distribution**	**BIC**	**KS**	**p-value**
1	Sameera	443.2664	0.0769	0.9067
	Gamma	448.3442	0.1224	0.4092
	Weibull	448.5221	0.1111	0.5309
	Lindley	457.8607	0.1426	0.2377
	Shanker	467.283	0.153	0.1736
	Exponential	444.6254	0.1138	0.5008
2	Sameera	211.702	0.117	0.5548
	Gamma	217.5192	0.1454	0.2855
	Weibul	216.5967	0.1204	0.5174
	Lindley	223.7981	0.2339	0.0131
	Shanker	227.3342	0.2471	0.0073
	Exponential	213.2cm411	0.1597	0.1915

Table 9: The MLEs of the parameters of the fitted distributions and their corresponding confidence intervals using data 1 and 2

Data	Distribution	Par.	MLE	SE	95%CI	
					LB	UB
1	Sameera	α	3.5675	0.5101	2.5678	4.5673
		β	0.0771	0.0080	0.0614	0.0928
	Gamma	α	0.9260	0.1618	0.6089	1.2432
		β	0.0307	0.0070	0.0170	0.0444
	Weibull	α	1.0150	0.1211	0.7777	1.2523
		β	30.3490	4.4168	21.6920	39.0059
	Lindley	β	0.0643	0.0064	0.0517	0.0769
	Shanker	β	0.0666	0.0066	0.0536	0.0796
	Exponential	β	0.0331	0.0047	0.0239	0.0422
2	Sameera	α	8.2550	2.2255	3.2cm931	12.6170
		β	0.3652	0.0507	0.2659	0.4645
	Gamma	α	0.9322	0.1701	0.5988	1.2655
		β	0.2585	0.0615	0.1380	0.3791
	Weibull	α	0.8986	0.0958	0.7109	1.0863
		β	3.3919	0.5910	2.2335	4.5503
	Lindley	β	0.4663	0.0499	0.3685	0.5641
	Shanker	β	0.5109	0.0493	0.4144	0.6074
	Exponential	β	0.2774	0.0409	0.1972	0.3570

entropies, order statistics, quantile function, and the stress-strength reliability. Estimates of the distribution parameters are attained using MLE, OLS, WLS, MPS, CV, and AD methods. A simulation study using these methods is conducted as well. It revealed that the estimators are approximately unbiased and consistent. Sameera distribution is used for fitting two real data sets. The results showed that Sameera outperformed competence distributions.

Acknowledgements

We are very grateful to experts for their appropriate and constructive suggestions to improve this template.

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