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The Exponentiated Generalized Power Generalized Weibull Distribution: Properties and Applications

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Abstract: This paper introduces a new flexible extension of power generalized Weibull distribution which contains many life-time distributions as sub-models. The hazard rate function of the proposed distribution is useful and suitable for monotone and non-monotone hazard behaviors that are more likely to be observed in real-life situations. Statistical properties of the new model are studied including; quantile, moment generating, reliability, hazard, and reverse hazard functions. Further, the moments, incomplete moments, mean deviations, Bonferroni and Lorenz curves, order statistics densities are derived. The maximum likelihood estimation method is used to estimate the distribution parameters. The effectiveness and usefulness of the new distribution are accomplished through four different real-life applications.

Keywords: power generalized Weibull distribution, maximum likelihood estimation, moment, hazard rate function, order statistics, Bonferroni and Lorenz curves

1 Introduction

Bagdonavicius and Nikulin [1] were originally proposed the three-parameter power generalized Weibull (PGW) distribution as a generalization of the well-known Weibull distribution by introducing an additional shape parameter. The cumulative density function (cdf) of the PGW distribution is

$$G(x) = 1 - exp1 - (1 + \lambda x^{\alpha}\ )^{\theta}\ , \ \ \alpha, \lambda, \theta > 0, \quad x > 0 \tag{1}$$

and the corresponding probability density function (pdf) is

$$g\left(x\right) = \ \lambda\alpha\theta x^{\alpha-1}\left(1+\lambda x^{\alpha}\right)^{\theta-1} exp\left\{1-\left(1+\lambda x^{\alpha}\right)^{\theta}\right\}, \quad \alpha, \lambda, \theta > 0, \quad x > 0 \tag{2}$$

where λ is a scale parameter, and α, θ are two shape parameters. The standard Weibull distribution is a special case of (1) when $\theta = 1$. This distribution can be also considered as an extension of exponential distributions. Nikulin and Haghighi[2] pointed out that the hazard rate function of the PGW distribution has nice and flexible properties and can be constant, monotone and non-monotone shaped. They also illustrated that this distribution is often used for constructing accelerated failures times models. Some statistical properties of the *PGW* distribution have studied been by Nikulin and Haghighi[3]. Cordeiro et al. 2013 defined a new class of distributions using additional two shape parameters to the baseline distribution. The cdf and the pdf of this class respectively are

$$F\left(x;\ b,\ \gamma\right) = \ \left[1-\left(1-G\left(x\right)\right)^{b}\right]^{\gamma}, \tag{3}$$

and

$$f\left(x;\ b,\ \gamma\right) = \ b\,\gamma\,g\left(x\right)\left(1-G\left(x\right)\right)^{b-1}\left[1-\left(1-G\left(x\right)\right)^{b}\right]^{\gamma-1}, \tag{4}$$

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where $G(x)$ represents the cdf of any continuous distribution and $b > 0$ and $\gamma > 0$ are the extra two shape parameters. Recently, based on idea of Cordeiro, Ortega [4], many authors have introduced new distributions. For example, Andrade, Rodrigues [5] introduced exponentiated generalized Gumbel distribution. Rodrigues, Percontini [] introduced the exponentiated generalized Lindley distribution and studied some of its properties and reliability characterization. El-Bassiouny, El-Damcese [7] introduced exponentiated generalized Weibull-Gompertz distribution. MirMostafaee, Alizadeh [8] introduced exponentiated generalized power Lindley distribution. They used the maximum likelihood and diagonally weighted least squares methods to estimation parameters of the new distribution. Nasiru, Mwita [9] introduced the exponentiated generalized exponential Dagum distribution and derived its statistical properties. By the same manner, in this paper, we study the so-called exponentiated generalized power generalized Weibull (EGPGW) distribution.

The motivations for conducting this study can be summarized as follows:

(i)The proposed distribution extends variety known distributions previously described in the literature.

(ii)The new distribution has constant, increasing, decreasing and upside-down bathtub hazard rate function.

(iii) Generate a new distribution with elastic properties and greater ability to describe phenomena in several fields.

(iv) Flexibility of the new distribution due to adding the extra two shape parameters to the baseline distribution.

One of them controls the tail behavior and the other adds entropy to the center of density function.

The main aim of this paper is to introduce a new extension of generalization power generalized Weibull distribution and study its statistical properties. The rest of this paper is organized as follow. In section 2, the exponentiated generalized power generalized Weibull distribution is introduced. In section 3, statistical properties of the new model such quantile function, moment generating function, reliability function, hazard function and the reverse hazard function are explored. Also, the moments, incomplete moments, mean deviations and Bonferroni and Lorenz curves and the order statistics densities are derived. In section 4, the reliability functions are derived. In section 5, the pdf of order statistics of the EGPGW model is introduced. Section 6 investigates the maximum likelihood estimation. In section 7, four real data sets are conducted to illustrate the usefulness and applicability of the proposed model. The concluding comments are given in section 8.

2 The Exponentiated Generalized Power Generalized Weibull Distribution

By inserting the pdf and cdf of PGW distribution Eqs (1) and (2) as baseline function in to Eqs (3) and (4), we get the cdf and pdf of the EGPGW distribution, as follow :

$$F(x) = \left[1 - exp\left\{b\left[1 - (1 + \lambda x^{\alpha})^{\theta}\right]\right\}\right]^{\gamma}\ ,\ \alpha, \lambda, \theta,\ b, \gamma > 0\,, x > 0 \tag{5}$$

and

$$\begin{aligned} f(x) &= \alpha\lambda\theta b\gamma\, x^{\alpha-1}(1 + \lambda x^{\alpha})^{\theta-1} exp\left\{b\left[1 - (1 + \lambda x^{\alpha})^{\theta}\right]\right\} \\ &\times \left[1 - exp\left\{b\left[1 - (1 + \lambda x^{\alpha})^{\theta}\right]\right\}\right]^{\gamma-1}, \alpha, \lambda, \theta,\ b, \gamma > 0\,, x > 0 \end{aligned} \tag{6}$$

The PGW distribution is a special case of (5) when, $\gamma = b = 1$. The pdf and cdf for various values of the parameters $\alpha, \lambda, \theta,\ b$ and γ are plotting in Fig. 1, 2 respectively.

2.1 Special distributions of the EGPGW distribution

The EGPGW distribution having number of sub-models are; exponentiated power generalized Weibull (EPGW), generalized power generalized Weibull (GPGW), power generalized Weibull distribution (PGW), Weibull (W), exponentiated Weibull (EW), Nadarajah-Haghighi (NH), exponentiated Nadarajah-Haghighi distribution (ENH), exponentiated generalized Nadarajah-Haghighi distribution (EGNH), Rayleigh (R), exponential (E), exponentiated exponential (EE), distributions. Sub-models of EGPGW distribution are listed in Table 1 for selected values of the parameters.

Table 1 **Sub-models of the EGPGW distribution**

Model	λ	α	θ	b	γ	Author(s)
PGW	-	-	-	1	1	Nikulin and Haghighi [2]
EPGW	-	-	-	1	-	Pena-Ramirez, Guerra [10]
GPGW	-	-	-	-	1	Selim[11]
W	-	-	1	1	1	Weibull[12]
EW	-	-	1	1	-	Mudholkar, Srivastava [13]
NH	-	1	-	1	1	Nadarajah and Haghighi [14]
ENH	-	1	-	1	-	Lemonte [15]
EGNH	-	1	-	-	-	VedoVatto, Nascimento [16]
R	-	2	1	1	1	Rayleigh [17]
E	-	1	1	1	1	Exponential distribution
E	1	1	1	-	1	Exponential distribution
EE	-	1	1	1	-	Gupta and Kundu [18]

3 The Statistical Properties of EGPGW Distribution

In this section, we derive some of the distributional properties of EGPGW distribution including, the quantile function, the moments, moment generating function, skewness, kurtosis and random variables generation function, incomplete moments, mean deviations, and mean deviations.

3.1 Quantile function

The quantity function is a function that has a wide number of applications. It is used to generate random variables and to obtain statistical measures such as location measures. The definition of the q-th quantile is the real solution of the following equation

$$F\left(x_q\right) = q,\ \ 0 \leq q \leq 1$$

Thus, the quantile function Q(q) corresponding of the EGPGW distribution is

$$Q\left(q\right) = \lambda^{-\frac{1}{\alpha}} \left\{ \left[1 - \frac{ln\left(1 - q^{\frac{1}{\gamma}}\right)}{b} \right]^{\frac{1}{\theta}} - 1 \right\}^{\frac{1}{\alpha}} \tag{7}$$

The median $M(X)$ of EGPGW distribution can be obtained from the previous function, by setting $q = 0.5$, as follows

$$M\left(x\right) = \lambda^{-\frac{1}{\alpha}} \left\{ \left[1 - \frac{ln\left(1 - 0.5^{\frac{1}{\gamma}}\right)}{b} \right]^{\frac{1}{\theta}} - 1 \right\}^{\frac{1}{\alpha}} \tag{8}$$

3.2 Skewness and kurtosis

The skewness and kurtosis measures are used in statistical analyses to characterize a distribution or a data set. The Bowley's skewness measure based on quartiles (Kenney and Keeping [[19]]) is given by

$$Sk = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{9}$$

and the Moors' kurtosis measure based on octiles (Moors [20]) is given by

$$Ku = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{10}$$

The skewness and kurtosis measures based on quantiles like Bowley's skewness and Moors' kurtosis have a number of advantages compared to the classical measures of skewness and kurtosis, e.g. they are less sensitive to outliers and they exist for the distributions even without defined the moments.

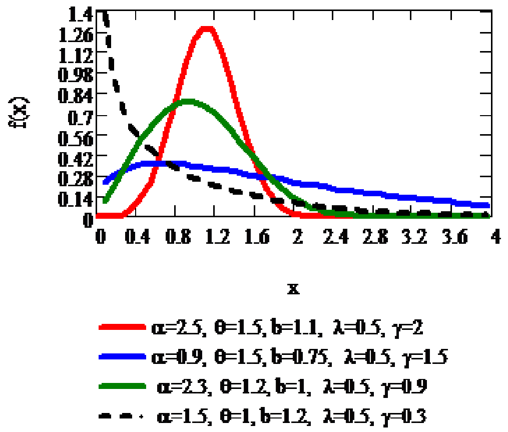

Fig. 1 Some possible shapes of the EGPGW density function

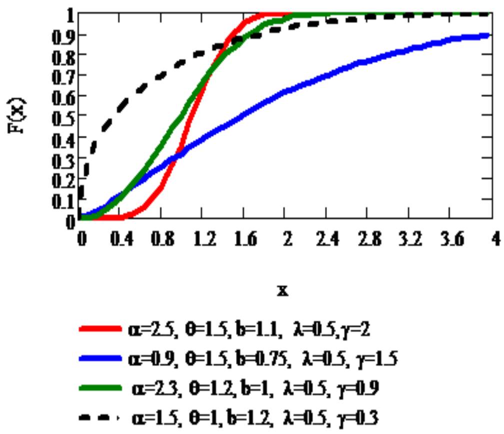

Fig. 2 Some possible shapes of the EGPGW cumulative density function

3.3 Random variables generation

The closed form of the quantile function of the *EGPGW* distribution makes the simulation from this distribution easier. Therefore, the random variables of *EGPGW* distribution are directly generated from the following function

$$X = \lambda^{-\frac{1}{\alpha}} \left\{ \left[1 - \frac{ln\left(1 - u^{\frac{1}{\gamma}}\right)}{b} \right]^{\frac{1}{\theta}} - 1 \right\}^{\frac{1}{\alpha}} \tag{11}$$

where $\alpha, \lambda, \theta, b$ and γ are known parameters and u is generated number from the uniform distribution $(0, 1)$.

3.4 The Moments

If X has the *EGPGW* distribution, then the r-th moment of X for integer value of $\frac{r}{\alpha}$ is

$$\mu_r' = b\,\gamma\, e^b \lambda^{-\frac{r}{\alpha}} \sum_{j=0}^{\frac{r}{\alpha}} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{j+\frac{r}{\alpha}+k+s}}{b^{\frac{j}{\theta}+s+1}} \binom{\frac{r}{\alpha}}{j} \binom{i}{s} \frac{(bk)^i}{i!} \frac{\Gamma(\gamma)}{\Gamma(\gamma-k)\;k!} \Gamma\left(\frac{j}{\theta}+s+1,\; b\right) \tag{12}$$

Proof. The r-th moment is defined as follows

$$\mu_r' = E\left(X^r\right) = \int_0^{\infty} x^r f(x) dx \tag{13}$$

Substituting $f(x)$ from (6) in (13) yields

$$\mu_r' = \alpha\theta\lambda b\gamma \int_0^\infty x^{r+\alpha-1} \left(1+\lambda x^\alpha\right)^{\theta-1} e^{b\left(1-\left(1+\lambda x^\alpha\right)^\theta\right)} \left[1-\exp\left\{b\left[1-\left(1+\lambda x^\alpha\right)^\theta\right]\right\}\right]^{\gamma-1} dx \tag{14}$$

by taking $u^{\frac{1}{\theta}} = 1+\lambda x^\alpha$, the above expression reduce to

$$\mu_r' = b\gamma e^b \lambda^{-\frac{r}{\alpha}} \int_1^\infty \left(u^{\frac{1}{\theta}}-1\right)^{\frac{r}{\alpha}} e^{-bu} \left[1-e^{b(1-u)}\right] du \tag{15}$$

by applying the binomial, the following expansions namely: $\left(1-e^{b(1-u)}\right)^{\gamma-1} = \sum_{k=0}^\infty (-1)^k \frac{\Gamma(k)}{\Gamma(\gamma-k)\,k!} e^{bk(1-u)}$, $e^{b(1-u)} \sum_{i=0}^\infty \frac{(bk)^i (1-u)^i}{i!}$ and $(1-u)^i = \sum_{s=0}^\infty (-1)^s \binom{i}{s} u^s$, then (15) becomes

$$\mu_r' = b\gamma e^b \lambda^{-\frac{r}{\alpha}} \sum_{j=0}^{r/\alpha} \sum_{k=0}^\infty \sum_{i=0}^\infty \sum_{s=0}^\infty \frac{(-1)^{j+\frac{r}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}} \binom{r/\alpha}{j} \binom{i}{s} \frac{(bk)^i}{i!} \frac{\Gamma(\gamma)}{\Gamma(\gamma-k)k!} \int_1^\infty u^{\frac{j}{\theta}+s} e^{-bu} du \tag{16}$$

by integrating the incomplete gamma function $\int_1^\infty u^{\frac{j}{\theta}+s} e^{-bu} du = \Gamma\left(\frac{j}{\theta}+s+1, b\right) / b^{\frac{j}{\theta}+s+1}$, we get the r-th moment of X as follows

$$\mu_r' = b\gamma e^b \lambda^{-\frac{r}{\alpha}} \sum_{j=0}^{r/\alpha} \sum_{k=0}^\infty \sum_{i=0}^\infty \sum_{s=0}^\infty \frac{(-1)^{j+\frac{r}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}} \binom{r/\alpha}{j} \binom{i}{s} \frac{(bk)^i}{i!} \frac{\Gamma(\gamma)}{\Gamma(\gamma-k)k!} \Gamma\left(\frac{j}{\theta}+s+1, b\right) \tag{17}$$

If, $b=1$ we get the moments of *PGW* distribution as follows

$$\mu_r' = \lambda^{-\frac{r}{\alpha}} e \sum_{j=0}^{\frac{r}{\alpha}} (-1)^{\frac{r}{\alpha}+j} \binom{\frac{r}{\alpha}}{j} \Gamma\left(\frac{j}{\theta}+1, 1\right)$$

which agrees with Nikulin and Haghighi [2]. Also, if $\alpha = b = 1$ we get the moments of Nadarajah-Haghighi distribution as follows

$$\mu_r' = \lambda^{-r} e \sum_{j=0}^{r} (-1)^{r+j} \binom{r}{j} \Gamma\left(\frac{j}{\theta}+1,\ 1\right)$$

In particular, the first and second moments and the variance of X can be obtained, respectively, from (12) as follow

$$\begin{aligned} \mu_1' = E(X) &= b\gamma e^b \lambda^{-\frac{1}{\alpha}} \sum_{j=0}^{r/\alpha} \sum_{k=0}^\infty \sum_{i=0}^\infty \sum_{s=0}^\infty \frac{(-1)^{j+\frac{2}{\alpha}+k+s}}{b^{\frac{j}{\theta}+s+1}} \binom{1/\alpha}{j} \binom{i}{s} \frac{(bk)^i}{i!} \frac{\Gamma(\gamma)}{\Gamma(\gamma-k)k!} \\ &\quad \times \Gamma\left(\frac{j}{\theta}+s+1, b\right), \\ \mu_2' = E\left(X^2\right) &= b\gamma e^b \lambda^{-\frac{2}{\alpha}} \sum_{j=0}^{r/\alpha} \sum_{k=0}^\infty \sum_{i=0}^\infty \sum_{s=0}^\infty \frac{(-1)^{j+\frac{2}{\alpha}+k+s}}{b^{\frac{j}{\theta}+s+1}} \binom{2/\alpha}{j} \binom{i}{s} \frac{(bk)^i}{i!} \frac{\Gamma(\gamma)}{\Gamma(\gamma-k)k!} \\ &\quad \times \Gamma\left(\frac{j}{\theta}+s+1, b\right) \end{aligned} \tag{18}$$

$$Var\left(X\right) = \mu_2' - \left[\mu_1'\right]^2. \tag{19}$$

The non-central moments in (12) can be also used to calculated the central moments μ_r and the cumulants k_r as following $\mu_r = \sum_{k=0}^r (-1)^k \binom{r}{k} \mu_1'^k \mu_{r-k}' and k_r = \mu_r' - \sum_{k=1}^{r-1} \binom{r-1}{k-1} k_r \mu_{r-k}'$, where the cumulants k_r are defined as quantities that provide an alternative to the distribution moments. The skewness γ_1 and kurtosis γ_2 can be calculated based on cumulants in the forms $\gamma_1 = k_3/k_2^{3/2}$ and k_4/k_2^2, respectively.

3.5 The moment generating function

If $X \sim EGPGW$ distribution, then for any integer value of $\frac{r}{\alpha}$ the moment generating function is

$$M_x(t) = b\gamma e^b \lambda^{-\frac{r}{\alpha}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{j+\frac{r}{\alpha}+k+s}\ t^r}{b^{\frac{j}{\theta}+s+1}\ r!} \binom{\frac{r}{\alpha}}{j}\binom{i}{s}\frac{(bk)^i}{i!}\frac{\Gamma(\gamma)}{\Gamma(\gamma-k)\ k!}\Gamma\left(\frac{j}{\theta}+s+1,\ b\right) \tag{20}$$

Proof. The moment generating function is defined as follows $M_x(t) = \int_0^\infty e^{tx} f(x)dx$ Using exponential function formula $e^{tx} = \sum_{r=0}^{\infty}\frac{(tx)^r}{r!}$, we get

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^\infty x^r f(x)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r' \tag{21}$$

Substituting (12) in to (21) yields the mgf of *EGPGW* distribution as in (20). □

3.6 Incomplete moments

The c-th incomplete moment of X, is defined as follows

$$m_r(s) = E\left(X^r \mid X > c\right) = \int_c^\infty x^r f(x)dx \tag{22}$$

Hence, by inserting (12) in (22) and after some manipulate, we get the m incomplete moment of $EGPGW$ distribution as follows

$$\begin{aligned} m_r(c) = b\gamma\lambda^{-\frac{r}{\alpha}} e^b \sum_{j=0}^{\frac{r}{\alpha}}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{j+\frac{r}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}} \binom{r/\alpha}{j}\binom{i}{s}\frac{(bk)^i}{i!}\frac{\Gamma(\gamma)}{\Gamma(\gamma-k)k!} \\ \times\Gamma\left(\frac{j}{\theta}+s+1, b\left(1+\lambda c^\alpha\right)^\theta\right)/b^{\frac{j}{\theta}+1} \end{aligned} \tag{23}$$

Proof. Substituting (6) into (22), yields

$$m_r(c) = \alpha\theta\lambda b\gamma\int_c^\infty x^{r+\alpha-1}\left(1+\lambda x^\alpha\right)^{\theta-1} e^{b\left(1-(1+\lambda x^\alpha)^\theta\right)}\left[1-\exp\left\{b\left[1-\left(1+\lambda x^\alpha\right)^\theta\right]\right\}\right]^{\gamma-1} dx \tag{24}$$

By setting $u = \left(1+\lambda x^\alpha\right)^\theta$, the above expression reduce to

$$m_r(c) = b\gamma e^b\lambda^{-\frac{r}{\alpha}}\int_{(1+\lambda c^\alpha)^\theta}^{\infty}\left(u^{\frac{1}{\theta}}-1\right)^{\frac{r}{\alpha}} e^{-bu}\left[1-e^{b(1-u)}\right] du, \tag{25}$$

Applying the previous binomial expansions used in (16), then, (25) becomes

$$m_r(c) = b\gamma\lambda^{-\frac{r}{\alpha}} e^b \sum_{j=0}^{r/\alpha}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{j+\frac{r}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}}\binom{r/\alpha}{j}\binom{i}{s}\frac{(bk)^i}{i!}\frac{\Gamma(\gamma)}{\Gamma(\gamma-k)k!}\int_{(1+\lambda c^\alpha)}^{\infty} u^{\frac{j}{\theta}+s} e^{-bu} du \tag{26}$$

By integrating the incomplete gamma function $\int_{(1+\lambda c^\alpha)^\infty u^{\frac{j}{\theta}+s}} e^{-bu}du = \Gamma\left(\frac{j}{\theta}+s+1, b\left(1+\lambda c^\alpha\right)^\theta\right)/b^{\frac{j}{\theta}+1}$, we get the r th upper incomplete moment of $EGPGW$ distribution as follows

$$\begin{aligned} m_r(c) = b\gamma\lambda^{-\frac{r}{\alpha}} e^b \sum_{j=0}^{\frac{r}{\alpha}}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{j+\frac{r}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}}\binom{\frac{r}{\alpha}}{j}\binom{i}{s}\frac{(bk)^i}{i!} \\ \times\frac{\Gamma(\gamma)}{\Gamma(\gamma-k)\ k!}\Gamma\left(\frac{j}{\theta}+s+1,\ b\left(1+\lambda c^\alpha\right)^\theta\right)/b^{\frac{j}{\theta}+1} \end{aligned}$$

In particular, the first incomplete moments of the $EGPGW$ distribution can be obtained by putting $(r = 1)$ in (23), as follows

$$m_1(c) = b\,\gamma\,\lambda^{-\frac{1}{\alpha}} e^b \sum_{j=0}^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{j+\frac{1}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}} \binom{\frac{1}{\alpha}}{j} \binom{i}{s} \frac{(bk)^i}{i!} \times \frac{\Gamma(\gamma)}{\Gamma(\gamma-k)\;k!} \Gamma\left(\frac{j}{\theta}+s+1,\; b\left(1+\lambda c^{\alpha}\right)^{\theta}\right) / b^{\frac{j}{\theta}+1} \tag{27}$$

3.7 Mean deviations

The mean deviations about the mean $(\delta_1(X))$ and about the median $(\delta_2(X))$ of X are given, respectively, by

$$\begin{aligned} \delta_1(X) &= E\left(\left|X-\mu_1'\right|\right) = \int_0^{\infty} \left|X-\mu_1'\right| f(x)dx \\ &= 2\mu_1' F\left(\mu_1'\right) - 2\mu_1' + 2\int_{\mu_1}^{\infty} x f(x)dx \\ &= 2\mu_1' F\left(\mu_1'\right) - 2H_1\left(\mu_1'\right) \end{aligned} \tag{28}$$

and

$$\begin{aligned} \delta_2(X) &= E(|X-M|) = \int_0^{\infty} |X-M| f(x)dx \\ &= 2MF(M) - M - \mu_1' + 2\int_M^{\infty} x f(x)dx \\ &= \mu_1' - 2H_1(M) \end{aligned} \tag{29}$$

where $\mu_1' = E(X)\,, M = \text{median}\,(X)\,, F\left(\mu_1'\right)$ from (5) and $H\left(\mu_1'\right)$ is the cth lower incomplete moment as follows

$$H_1(s) = \int_0^{\mu_1'} x f(x)dx = b\gamma\lambda^{-\frac{r}{\alpha}} e^b \sum_{j=0}^{\frac{r}{\alpha}} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{j+\frac{1}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}} \binom{1/\alpha}{j} \binom{i}{s} \frac{(bk)^i}{i!} \times \frac{\mathit{\Gamma}(\gamma)}{\mathit{\Gamma}(\gamma-k)k!} \left[\mathit{\Gamma}\left(\frac{j}{\theta}+s+1, b\right) - \mathit{\Gamma}\left(\frac{j}{\theta}+s+1, b\left(1+\lambda s^{\alpha}\right)^{\theta}\right)\right] \tag{30}$$

3.8 Bonferroni and Lorenz curves

Bonferroni (1930) introduced Bonferroni and Lorenz curves which have many applications in many fields, especially in economics to study income and poverty and in other fields such reliability, demography, insurance and medicine. These curves are defined as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x f(x)dx = \frac{\mu_1' - m_1(q)}{p\mu_1'} \tag{31}$$

and

$$L(p) = \frac{1}{\mu_1'} \int_0^q x f(x)dx = \frac{\mu_1' - m_1(q)}{\mu_1'} \tag{32}$$

respectively, where $\mu_1' = E\;(X)$ and $q = Q\,(p)$ are calculated from (7) for a given probability (P), and $m_1\,(q)$ is the first incomplete moment from (23).

$$B(p) = \frac{1}{p} - b\,\gamma\,\lambda^{-\frac{1}{\alpha}} e^b \sum_{j=0}^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{j+\frac{1}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}} \binom{\frac{1}{\alpha}}{j} \binom{i}{s} \frac{(bk)^i}{i!} \left[\frac{\Gamma(\gamma)}{\Gamma(\gamma-k)\;k!}\right] \times \frac{\left[\Gamma\left(\frac{j}{\theta}+s+1,\; b\left(1+\lambda q^{\alpha}\right)^{\theta}\right)\right]}{p\mu_1'} \tag{33}$$

and

$$L(p) = 1 - b\,\gamma\,\lambda^{-\frac{1}{\alpha}} e^{b} \sum_{j=0}^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{j+\frac{1}{\alpha}+k+s}}{b^{\frac{j}{\theta}+1}} \binom{\frac{1}{\alpha}}{j} \binom{i}{s} \frac{(bk)^{i}}{i!} \times \left[\frac{\Gamma(\gamma)}{\Gamma(\gamma-k)\;k!}\right] \frac{\left[\Gamma\left(\frac{j}{\theta}+s+1,\; b\left(1+\lambda q^{\alpha}\right)^{\theta}\right)\right]}{\mu_1'} \tag{34}$$

4 Reliability Analysis

In this section, the survival $s(t)$, failure rate $h(t)$, reversed hazard $r(t)$ and the cumulative failure rate $H(t)$ functions of $EGPGW$ distribution are derived.

4.1 The survival function

The survival function $s(x)$ of the $EGPGW$ distribution can be derived using the cumulative distribution function in (5) as follows

$$s(x) = 1 - \left[1 - exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\right]^{\gamma}, \quad x > 0 \tag{35}$$

4.2 The hazard function

For a continuous distribution with pdf $f(x)$ and cdf $F(x)$, the hazard rate function for any time is defined as follows

$$h(t) = \lim_{\Delta t \to 0} \frac{P(T < t + \Delta x | T \rangle t)}{\Delta t} = \frac{f(t)}{1 - F(t)}$$

Subsequently, the hazard rate for any time of the $EGPGW$ distribution can be determined using the cdf and pdf in Eqs. (5), (6) as follow

$$h(x) = \alpha\lambda\theta b\gamma\, x^{\alpha-1}(1+\lambda x^{\alpha})^{\theta-1} exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\} \times \frac{\left[1-exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\right]^{\gamma-1}}{1-\left[1-exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\right]^{\gamma}}, \quad x > 0 \tag{36}$$

For some selected parameters values, the plots of the hazard function of $EGPGW$ distribution are displayed in Fig.3 These plots show the flexibility of hazard rate function that makes the $EGPGW$ hazard rate function useful and suitable for non-monotone hazard behaviors that are more likely to be observed in real life situations.

4.3 The reversed hazard and cumulative hazard rate functions

The reversed hazard $r(t)$ and the cumulative hazard rate $H(t)$ functions of $EGPGW$ distribution are given, respectively, as follow

$$r(t) = \alpha\lambda\theta b\gamma\, x^{\alpha-1}(1+\lambda x^{\alpha})^{\theta-1} exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\left[1-exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\right]^{-1}, \quad t > 0, \tag{37}$$

and

$$H(t) = -ln\left[1-\left[1-exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\right]^{\gamma}\right], \quad t > 0. \tag{38}$$

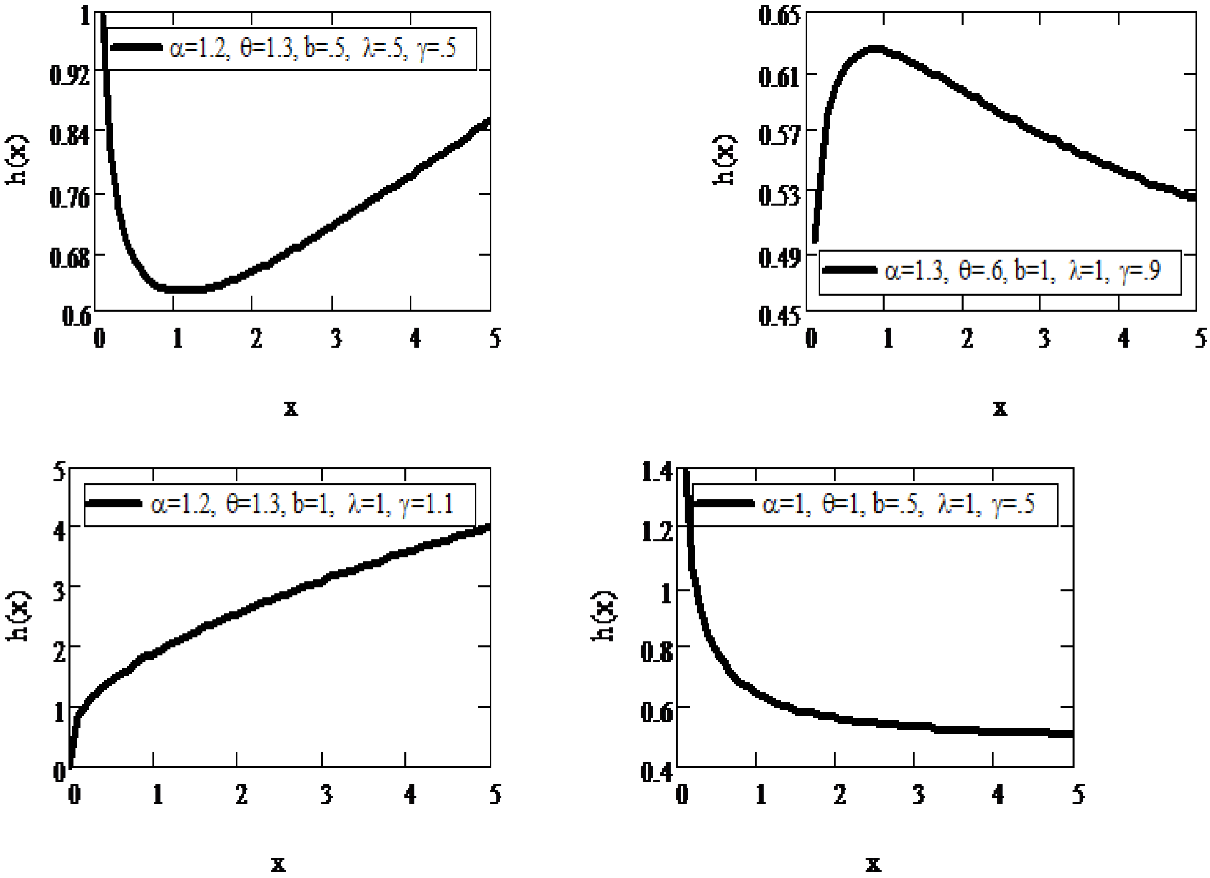

Fig. 3 Some possible shapes of the $EGPGW$ hazard rate function

5 Order Statistics

The order statistics arise naturally in many areas of statistical theory and practice which makes it one of the important statistical topics. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of a random sample drawn from a continuous distribution with cdf $F(x)$ and pdf $f(x)$, then the pdf of $X_{(k)}$is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!\,(n-k)!} f(x) \left[F(x)\right]^{k-1} \left[1 - F(x)\right]^{n-k}, \quad k = 1, 2, \ldots, n \tag{39}$$

Let X is a random variable of EGPGW distribution, then by substituting (5) and (6) into equation (39), we get the kth order statistics of EGPGW density function as follows

$$\begin{aligned} f_{k:n}(x) = {} & \frac{n!}{(k-1)!\,(n-k)!} \alpha\lambda\theta b\gamma\, x^{\alpha-1} \left(1+\lambda x^{\alpha}\right)^{\theta-1} exp\left\{b\left[1-\left(1+\lambda x^{\alpha}\right)^{\theta}\right]\right\} \\ & \times \left[1 - exp\left\{b\left[1-\left(1+\lambda x^{\alpha}\right)^{\theta}\right]\right\}\right]^{\gamma-1} \left[1 - exp\left\{b\left[1-\left(1+\lambda x^{\alpha}\right)^{\theta}\right]\right\}\right]^{\gamma(k-1)} \\ & \times \left[1 - \left[1 - exp\left\{b\left[1-\left(1+\lambda x^{\alpha}\right)^{\theta}\right]\right\}\right]^{\gamma}\right]^{n-k} \end{aligned} \tag{40}$$

The pdf of order statistics when $k = 1$ and when $k = n$ respectively, are

$$\begin{aligned} f_{1:n}(x) = {} & n\alpha\lambda\theta b\gamma\, x^{\alpha-1} \left(1+\lambda x^{\alpha}\right)^{\theta-1} exp\left\{b\left[1-\left(1+\lambda x^{\alpha}\right)^{\theta}\right]\right\} \\ & \times \left[1 - exp\left\{b\left[1-\left(1+\lambda x^{\alpha}\right)^{\theta}\right]\right\}\right]^{\gamma-1} \left[1 - \left[1 - exp\left\{b\left[1-\left(1+\lambda x^{\alpha}\right)^{\theta}\right]\right\}\right]^{\gamma}\right]^{n} \end{aligned} \tag{41}$$

and

$$f_{n:n}(x) = n\alpha\lambda\theta b\gamma\, x^{\alpha-1}(1+\lambda x^{\alpha})^{\theta-1} exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}$$
$$\times\left[1-exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\right]^{\gamma-1}\left[1-exp\left\{b\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\right]^{\gamma(n-1)} \tag{42}$$

6 Maximum Likelihood Estimation

This section discusses the maximum likelihood estimation (MLE) for the parameters $(\alpha, \lambda, \theta, b, \gamma)$ of the EGPGW distribution. Let $x_1, x_1, \ldots, x_n$ is a complete random sample of size n from the EGPGW distribution. Then the likelihood function (LF) is

$$L(\sigma|x) = (\alpha\lambda\theta b\gamma)^n \prod_{i=1}^{n} x_i^{\alpha-1}(1+\lambda x_i^{\alpha})^{\theta-1} exp\left\{b\left[1-(1+\lambda x_i^{\alpha})^{\theta}\right]\right\}$$
$$\times\left[1-exp\left\{b\left[1-(1+\lambda x_i^{\alpha})^{\theta}\right]\right\}\right]^{\gamma-1} \tag{43}$$

and the log-likelihood function (lnL) is

$$lnL = n\, ln(\alpha\lambda\theta b\gamma) + (\alpha-1)\sum_{i=1}^{n} lnx_i + (\theta-1)\sum_{i=1}^{n} ln(\lambda {x_i}^{\alpha}+1)$$
$$+ b - b\sum_{i=1}^{n}(\lambda {x_i}^{\alpha}+1)^{\theta} + (\gamma-1)\sum_{i=1}^{n} ln\left[1-exp\left\{b\left[1-(1+\lambda x_i^{\alpha})^{\theta}\right]\right\}\right] \tag{44}$$

The partial derivatives of the above equation for $\alpha, \lambda, \theta, b$and γ respectively are

$$\frac{\partial lnL}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln(x_i) - \theta\lambda b\sum_{i=1}^{n} ln(x_i){x_i}^{\alpha}\delta^{\theta-1} + (\theta-1)\lambda\sum_{i=1}^{n}\frac{ln(x_i){x_i}^{\alpha}}{\delta}$$
$$+ (\gamma-1)\sum_{i=1}^{n}\lambda\, b\,\theta {x_i}^{\alpha}\,\delta^{\theta-1}\,\frac{A}{[1-A]} \tag{45}$$

$$\frac{\partial lnL}{\partial\lambda} = \frac{n}{\lambda} - \theta b\sum_{i=1}^{n}{x_i}^{\alpha}\delta^{\theta-1} + (\theta-1)\sum_{i=1}^{n}\frac{{x_i}^{\alpha}}{\delta} + (\gamma-1)\sum_{i=1}^{n} b\,\theta {x_i}^{\alpha}\,\delta^{\theta-1}\,\frac{A}{[1-A]}, \tag{46}$$

$$\frac{\partial lnL}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} ln(\delta) - b\sum_{i=1}^{n} ln(\delta)(\delta)^{\theta} + (\gamma-1)\, b\sum_{i=1}^{n}\delta^{\theta}\, ln(\delta)\frac{A}{[1-A]}, \tag{47}$$

$$\frac{\partial lnL}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n}\delta^{\theta} + 1 - (\gamma-1)\sum_{i=1}^{n}\left[1-\delta^{\theta}\right]\frac{A}{[1-A]}, \tag{48}$$

$$\frac{\partial lnL}{\partial\gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} ln\left[1-A\right]. \tag{49}$$

Where $\delta = 1+\lambda x_i^{\alpha}$ and $A = e^{b(1-(1+\lambda x_i^{\alpha}))}$. The maximum likelihood estimators of $\alpha, \lambda, \theta,\ b$ and γ are the simultaneous solutions of the following nonlinear likelihood equations

$$\frac{\partial lnL}{\partial\alpha} = \frac{\partial lnL}{\partial\lambda} = \frac{\partial lnL}{\partial\theta} = \frac{\partial lnL}{\partial b} = \frac{\partial lnL}{\partial\gamma} = 0$$

To solve the previous equations, we can use iterative numerical techniques like the Newton-Raphson algorithm because, they cannot be analytically solved. Now, to construct confidence intervals for the parameters, the observed information matrix $J(\tau)$ can be used which is given by

$$J(\tau) = -\begin{bmatrix} \frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\partial\lambda} & \frac{\partial^2 lnL}{\partial\alpha\partial\theta} & \frac{\partial^2 lnL}{\partial\alpha\partial b} & \frac{\partial^2 lnL}{\partial\alpha\partial\gamma} \\ & \frac{\partial^2 lnL}{\partial\lambda^2} & \frac{\partial^2 lnL}{\partial\lambda\partial\theta} & \frac{\partial^2 lnL}{\partial\lambda\partial b} & \frac{\partial^2 lnL}{\partial\lambda\partial\gamma} \\ & & \frac{\partial^2 lnL}{\partial\theta^2} & \frac{\partial^2 lnL}{\partial\theta\partial b} & \frac{\partial^2 lnL}{\partial\theta\partial\gamma} \\ & & & \frac{\partial^2 lnL}{\partial b^2} & \frac{\partial^2 lnL}{\partial b\partial\gamma} \\ & & & & \frac{\partial^2 lnL}{\partial\gamma^2} \end{bmatrix}, \quad \tau = (\alpha, \lambda, \theta, b, \gamma)' \,. \tag{50}$$

The elements of $J(\tau)$ are given in the Appendix.

7 Real Data Illustration

In this section, to illustrate the usefulness of the *EGPGW* distribution in the analysis of reliability and survival data, we provided four different real datasets are:

The data set (I): Remission times data

The first real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients (Lee and Wang [[21]]): 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The data set (II): Stress-rupture life data

The second data set consists of 76 observations of the strengths of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. For previous studies with the data sets see Andrews and Herzberg [[22]] and Barlow, Toland [[23]]. These data are: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

The data set (III): Failure times of 50 devices

The third data set represents the failure times of 50 devices put under a life test (see Aarset [[24]]). These data are:0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0, 86.0.

The data set (IV): the survival times of 33 patients

The fourth real data set represents the survival times (in weeks) of 33 patients suffering from acute myelogenous leukaemia (see Feigl and Zelen [[25]]). The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

To ensure the proposed model is suitable for our datasets, we can use a useful device called the total time on test (TTT) plot, for more details see Aarset [[24]]. This device allows identifying the shape of hazard function graphically. Fig. 4 displays the TTT plots for our datasets. The TTT plot for stress-rupture data in 4 (a) is a concave curve which indicates, according to Aarset [[24]], that data has an increasing failure rate function. Whereas the TTT plot for the remission times data in Fig. 4 (b) is shown first concave and then convex and this indicates an upside-down bathtub failure rate function. Also, the TTT plot for Failure time's data in Fig. 4 (c) indicates a bathtub shaped hazard rate function. And the TTT plot for the survival times in fig. 4 (d) indicates a decreasing hazard rate function. These shapes reveal the suitability of the EGPGW distribution to fit our data sets.

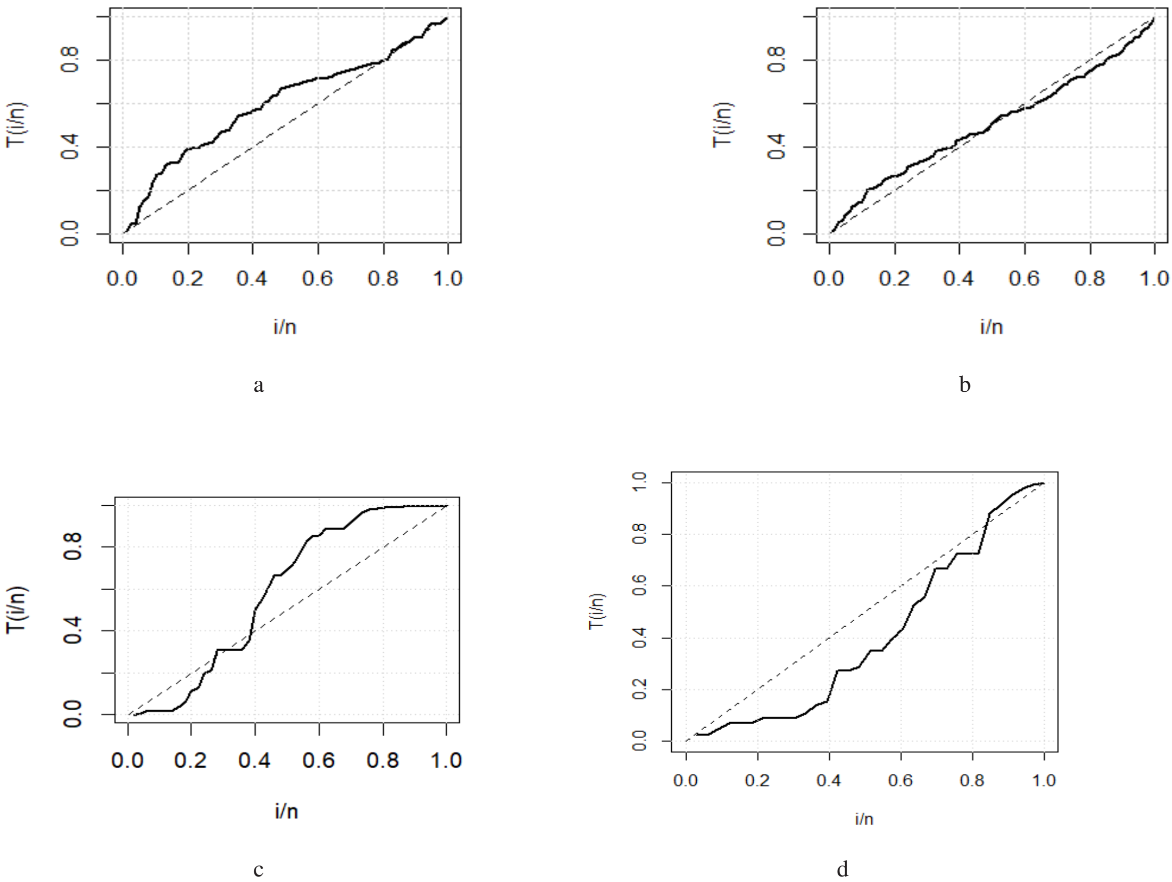

Fig. 4 TTT plots—(a) stress-rupture data; (b) remission times data; (c) Failure times data (d) survival times data

Now, We fit the $EGPGW$ distribution to these data and compare it with the other fitted models like generalized power Weibull (GPW), generalized power Weibull (EGPW), exponentiated Weibull (EW), Weibull (W), exponentiated exponential (EE), Nadarajah and Haghighi (NH) and exponential (E) distributions. The required numerical evaluations are performed using the R software. In order to compare the distributions and to verify the quality of the fits, we consider some of well-known Goodness-of-Fit statistics like, Cramér-von Mises (W^{*}), Anderson Darling (A^{*}), Kolmogorov-Smirnov (KS), Maximized Loglikelihood $(-L)$, Akaike Information Criterion (AIC). The model with a minimum value of Goodness-of-Fit statistics is the best model to fit the data.

The maximum likelihood estimates and their standard errors of the fitted models for the remission times, Stress-rupture life, Failure times and survival times datasets are displayed in Tables 2, 3, 4, 5 respectively. As well, the values of the W^{*}, A^{*},KS and $-L$ of fitted models for our datasets, are displayed in Tables 6, 7, 8, 9 respectively.

The Tables 6, 7, 8, 9 show that the proposed $EGPGW$ model gives smallest values for the Goodness-of-Fit statistics. The plots of fitted and empirical densities for the $EGPGW$ distribution and other fitted distributions of our datasets are displayed in Figs. 5 through 7, respectively. These plots also indicate that the $EGPGW$ distribution provides an adequate fit than other distributions.

Table 2 The estimates and the standard errors (in parentheses) for data set (I)

Model	Estimates				
	θ	λ	α	b	γ
EGPGW	0.1966 (0.1251)	0.0088 (0.0101)	2.5756 (0.8195)	0.5069 (0.1708)	2.0293 (3.1393)
EPGW	0.2412 (0.0585)	0.00716 (0.0073)	2.91333 (0.5087)	-	0.45194 (0.0956)
GPGW	0.2386 (0.2689)	0.0538 (0.0516)	1.4773 (0.2113)	4.5461 (9.0603)	-
PGW	0.4222 (0.1092)	0.1416 (0.0394)	1.5568 (0.2407)	-	-
EW	-	0.4538 (0.2398)	0.6544 (0.1346)	-	2.7963 (1.2631)
ENH	0.6371 (0.1172)	0.3446 (0.1752)	-	-	1.6892 (0.3647)
NH	0.9225 (0.1516)	0.1217 (0.0345)	-	-	-
EE	-	0.1212 (0.1488)	-	-	1.2179 (0.0136)
W	-	0.0939 (0.0191)	1.0478 (0.0675)	-	-
E	-	0.1068 (0.0094)	-	-	-

Table 3 The estimates and the standard errors (in parentheses) for data set (II)

Model	Estimates				
	θ	λ	α	b	γ
EGPGW	0.1420 (0.0321)	0.0092 (0.0066)	6.5173 (0.0367)	0.1965 (0.0295)	1.1817 (0.5353)
EPGW	0.1577 (0.0224)	0.0105 (0.0072)	6.4708 (0.0411)	-	0.2008 (0.0301)
GPGW	0.1520 (0.3317)	0.1320 (0.1295)	1.5942 (0.2450)	18.6347 (50.386)	-
PGW	0.6357 (0.2390)	0.6246 (0.3160)	1.5825 (0.0.2928)	-	-
EW	-	0.5796 (0.3003)	1.1013 (0.2629)	-	1.4427 (0.6436)
ENH	1.1086 (0.3552)	0.5792 (0.3479)	-	-	1.6228 (0.3638)
NH	2.0070 (0.7999)	0.1950 (0.1034)	-	-	-
W	-	0.3664 (0.0620)	1.3256 (0.1138)	-	-
EE	-	1.7095 (0.2826)	-	-	0.7028 (0.0921)
E	-	0.5104 (0.0585)	-	-	-

Table 4 **The estimates and the standard errors (in parentheses) for data set (III)**

Model	Estimates				
	θ	λ	α	b	γ
EGPGW	5.9934 (1.6721)	0.0021 (0.00003)	1.2987 (0.1353)	0.4078 (0.1168)	0.0656 (0.0883)
EPGW	3.4728 (2.0588)	0.0024 (0.0006)	1.1088 (0.1490)	-	0.6557 (0.1303)
GPGW	10.8630 (16.2942)	0.0232 (0.0770)	0.5872 (0.2732)	0.1101 (0.2031)	
PGW	7.2704 (4.004)	0.0045 (0.0022)	0.8074 (0.0986)	-	
EW	-	0.0029 (0.0010)	1.3965 (0.0832)	-	0.5356 (0.0876)
ENH	5.0595 (1.2794)	0.0026 (0.0007)	-	-	0.7243 (0.1146)
NH	4.2584 (1.4363)	0.0036 (0.0013)	-	-	-
EE	-	0.0187 (0.0036)	-	-	0.7801 (0.1351)
W	-	0.0272 (0.0137)	0.9476 (0.1174)	-	-
E	-	0.0219 (0.0031)	-	-	-

Table 5 **The estimates and the standard errors (in parentheses) for data set (IV)**

Model	Estimates				
	θ	λ	α	b	γ
EGPGW	0.0625 (0.0725)	1.8369 (5.1030)	17.3993 (20.1372)	0.4900 (0.1849)	0.010 (0.0155)
EPGW	0.0385 (0.0879)	19.5120 (65.3883)	6.7869 (15.6964)	-	2.0399 (1.2717)
GPGW	0.0373 (0.1342)	7.8281 (21.8268)	6.5271 (57.6467)	0.1239 (0.1026)	-
PGW	1.4058 (3.0889)	0.0475 (0.0905)	0.7200 (0.3198)	-	
EW	-	0.1680 (0.5769)	0.5998 (0.6057)	-	1.5748 (2.9553)
ENH	0.3151 (0.2651)	0.6516 (3.214)	-	-	1.8757 (3.5269)
NH	0.4897 (0.1482)	0.0998 (0.0701)	-	-	-
EE	-	0.6785 (0.1448)	-	-	0.0188 (0.0048)
W	-	0.0628 (0.0298)	0.7763 (0.1073)	-	-
E	-	0.0245 (0.0043)	-	-	-

Table 6 Goodness-of-fit statistics for data set (I)

Model	Kolmogorov-Smirnov K–S	P-value	W^*	A^*	-L	AIC
EGPGW	0.03131	0.9996	0.0143	0.0963	409.358	828.717
EPGW	0.03126	-0.9996	0.0159	0.1102	409.421	826.842
GPGW	0.03537	-0.9972	0.0221	0.1511	409.779	827.557
PGW	0.039	-0.9899	0.0352	0.2354	410.487	826.974
EW	0.04503	-0.9576	0.0437	0.2885	410.68	827.36
ENH	0.04419	-0.964	0.0421	0.2779	410.601	827.203
NH	0.09198	-0.229	0.1018	0.6137	414.226	832.451
EE	0.07249	-0.5118	0.1122	0.6741	413.078	830.155
W	0.07	-0.5573	0.1314	0.7864	414.087	832.174
E	0.08464	-0.3183	0.1193	0.7159	414.342	830.684

Table 7 Goodness-of-fit statistics for data set (II)

Model	Kolmogorov-Smirnov K–S	P-value	W^*	A^*	-L	AIC
EGPGW	0.0672	0.8589	0.0440	0.2780	119.533	249.066
EPGW	0.06340	0.8952	0.0458	0.2867	119.547	247.095
GPGW	0.0856	0.6033	0.0938	0.5587	121.551	251.102
PGW	0.0956	0.4628	0.1107	0.6549	121.973	249.946
EW	0.09876	0.4217	0.1168	0.6913	122.164	250.327
ENH	0.0980	0.4317	0.1170	0.6935	122.188	250.377
NH	0.1209	0.2000	0.1597	0.9302	124.738	253.475
W	0.1099	0.2953	0.1306	0.7672	122.525	249.049
EE	0.0943	0.4803	0.1167	0.6935	122.244	248.487
E	0.1663	0.0263	0.1193	0.7074	127.114	256.229

Table 8 Goodness-of-fit statistics for data set (III)

Model	Kolmogorov-Smirnov K–S	P-value	W^*	A^*	-L	AIC
EGPGW	0.1541	0.1861	0.2159	1.4916	225.787	461.574
EPGW	0.2084	0.0261	0.3716	2.3410	234.497	476.995
GPGW	0.1675	0.1209	0.3201	2.0594	232.732	473.464
PGW	0.1977	0.0402	0.3952	2.4683	235.879	477.757
EW	0.2084	0.0260	0.4261	2.6343	237.311	480.622
ENH	0.2033	0.0320	0.3753	2.3612	234.5782	475.156
NH	0.1881	0.0581	0.3749	2.3584	237.182	478.365
EE	0.2043	0.0308	0.4837	2.9421	239.973	483.947
W	0.1930	0.0483	0.4948	3.0009	240.980	485.959
E	0.1914	0.0514	0.4861	2.9546	241.068	484.135

Table 9 **Goodness-of-fit statistics for data set (IV)**

Model	Kolmogorov-Smirnov K–S	P-value	W^*	A^*	-L	AIC
EGPGW	0.1064	0.8486	0.0591	0.4072	150.164	310.328
EPGW	0.1354	0.5805	0.1095	0.7058	153.466	314.932
GPGW	0.1289	0.6424	0.0688	0.4710	151.199	310.399
PGW	0.1366	0.5694	0.0950	0.6565	153.571	313.143
EW	0.1366	0.5692	0.0954	0.6455	153.562	313.123
ENH	0.1301	0.6315	0.1103	0.6949	153.746	313.492
NH	0.1393	0.5441	0.1002	0.6659	153.743	311.486
EE	0.1384	0.5521	0.0966	0.6691	153.652	311.303
W	0.1366	0.5689	0.0948	0.6508	153.587	311.173
E	0.2182	0.0864	0.0973	0.6730	155.450	312.900

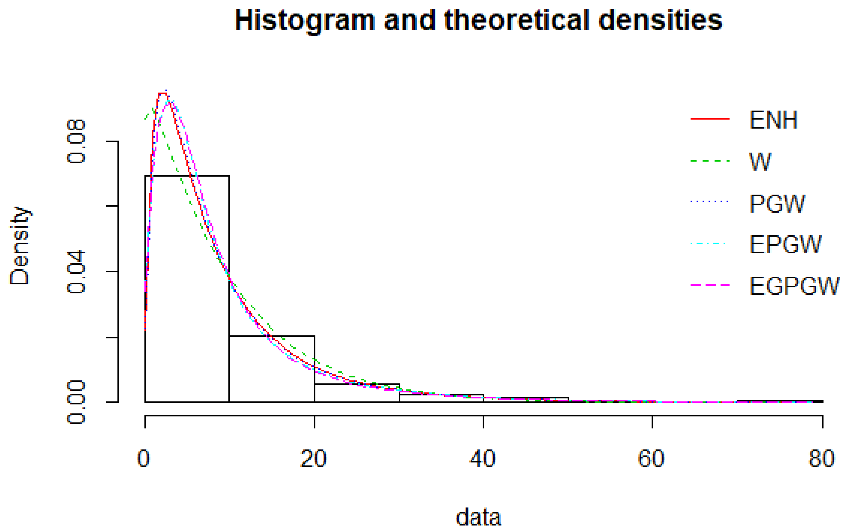

Fig. 5 Empirical and fitted pdfs for dataset (I)

8 Conclusion

This paper introduced a new five-parameter distribution called the exponentiated generalized power generalized Weibull distribution. This new distribution generalizes many known distributions like Weibull, Rayleigh, exponential, power generalized Weibull, Nadarajah-Haghighi distributions as sub-models. Hazard rate function, quantile function, order statistics, moments, incomplete moments, mean deviations and Bonferroni and Lorenz curves are derived to study the statistical properties of the proposed model. It has a variety of shapes for the hazard function such increasing, decreasing,

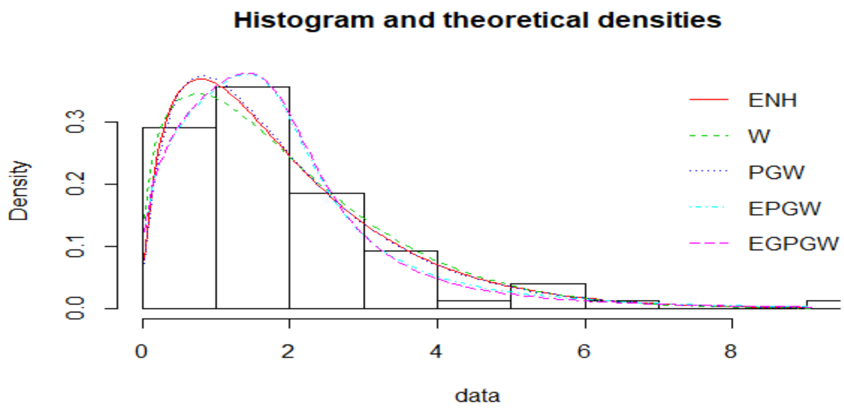

Fig. 6 Empirical and fitted pdfs for dataset (II)

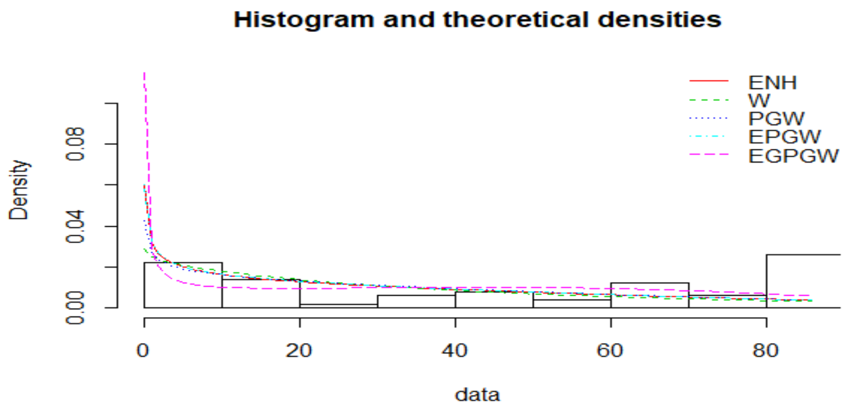

Fig. 7 Empirical and fitted pdfs for dataset (III)

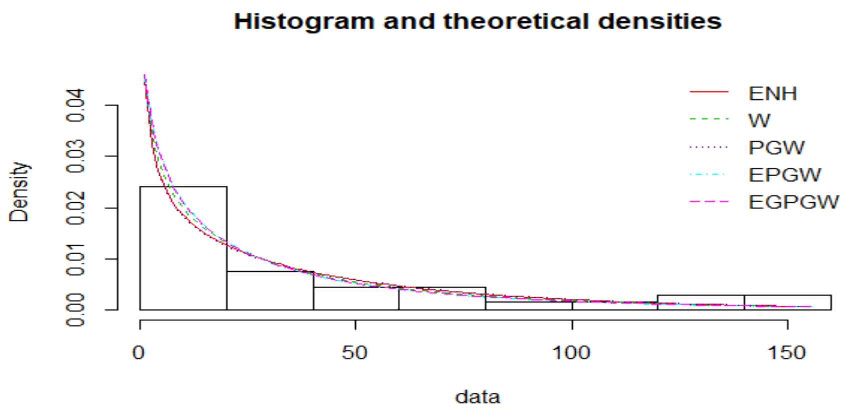

Fig. 8 Empirical and fitted pdfs for dataset (IV)

bathtub, unimodal and constant shapes. The distribution parameters are estimated using maximum likelihood method. The results of realistic applications showed that the proposed distribution represented data sets better than other distributions that are commonly used for fitting this type of data.

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APPENDIX

Elements of $J(\tau)$

$$\frac{\partial^2 lnL}{\partial\alpha^2}=\frac{-n}{\alpha^2}-\lambda\theta b\sum_{i=1}^{n}\left\{x_i^\alpha\, ln\,(x_i)^2\ \delta^{\theta-1}\left[1+\frac{\theta\,\lambda\,x_i^\alpha ln\,(x_i)}{\delta}-\frac{\lambda\,(x_i^\alpha)}{\delta}\right]\right\}$$
$$+\lambda(\theta-1)\sum_{i=1}^{n}\left\{\frac{x_i^\alpha\, ln\,(x_i)^2}{\delta}\left[1-\frac{\lambda\,(x_i^\alpha)}{\delta}\right]\right\} \tag{1}$$
$$+(\gamma-1)\ \lambda\theta b\sum_{i=1}^{n}\left\{x_i^\alpha\, ln\,(x_i)^2\ \delta^{\theta-1}\,\frac{A}{1-A}\left[1+\frac{\lambda\theta x_i^\alpha}{\delta}-\frac{\lambda\,(x_i^\alpha)}{\delta}-\lambda\theta b\frac{x_i^\alpha\,\delta^\theta}{\delta}\left(1+\frac{A}{1-A}\right)\right]\right\}$$

$$\frac{\partial^2 lnL}{\partial\lambda^2}=\frac{-n}{\lambda^2}-(\theta-1)\sum_{i=1}^{n}\frac{(x_i^\alpha)^2}{\delta^2}-\theta\, b\sum_{i=1}^{n}\delta^{\theta-2}(x_i^\alpha)^2(1-\theta)$$
$$+(\gamma-1)\,\theta b\sum_{i=1}^{n}\left\{(x_i^\alpha)^2\,\delta^{\theta-2}\,\frac{A}{1-A}\left[\theta-1-\theta b\delta^\theta\left(1+\frac{A}{1-A}\right)\right]\right\} \tag{2}$$

$$\frac{\partial^2\ell}{\partial\theta^2}=\frac{-n}{\theta^2}-b\sum_{i=1}^{n}\delta^2\ ln\ (\delta)^2+(\gamma-1)\sum_{i=1}^{n}b\,\delta^\theta\ ln\ (\delta)^2\frac{A}{1-A}\left[1-b\delta^\theta\left(1+\frac{A}{1-A}\right)\right] \tag{3}$$

$$\frac{\partial^2 lnL}{\partial b^2}=\frac{-n}{b^2}+(\gamma-1)\sum_{i=1}^{n}\left((1-\delta)^\theta\right)^2\frac{A}{1-A}\left[1+\frac{A}{1-A}\right] \tag{4}$$

$$\frac{\partial^2 lnL}{\partial\gamma^2}=\frac{-n}{\gamma^2} \tag{5}$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial\lambda}=(\theta-1)\sum_{i=1}^{n}\frac{x_i^\alpha\, ln\,(x_i)}{\delta}\left(1-\frac{\lambda x_i^\alpha}{\delta}\right)-\lambda\theta b\sum_{i=1}^{n}(x_i^\alpha)^2\,\delta^{\theta-2}\,ln\,(x_i)\left[\theta+\frac{\delta}{x_i^\alpha}-1\right]$$
$$+(\gamma-1)\,\lambda\theta b\sum_{i=1}^{n}(x_i^\alpha)^2\,\delta^{\theta-2}\,ln\,(x_i)\,\frac{A}{1-A}\left[\theta+\frac{\delta}{\lambda x_i^\alpha}-1-\theta b\delta^\theta\left(1+\frac{A}{1-A}\right)\right] \tag{6}$$

$$\frac{\partial^2\ln L}{\partial\alpha\partial\theta}=-\lambda b\sum_{i=1}^{n}\left\{x_i^\alpha\ln(x_i)\,\delta^{\theta-1}[\theta\ln(\delta)+1]\right\}+\lambda\sum_{i=1}^{n}\frac{x_i^\alpha\ln(x_i^\alpha)}{\delta}$$
$$-(\gamma-1)\lambda\theta b\sum_{i=1}^{n}\left\{\delta^{\theta-1}\ x_i^\alpha\ ln\,(x_i)\,\frac{A}{1-A}\left[1+\frac{1}{\theta ln\,(\delta)}-b\delta^\theta\left(1+\frac{A}{1-A}\right)\right]\right\} \tag{7}$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial b}=\lambda\theta\sum_{i=1}^{n}x_i^\alpha\ ln\,(x_i)\ \delta^{\theta-1}$$
$$+(\gamma-1)\ \theta\lambda\sum_{i=1}^{n}\left\{x_i^\alpha\ ln\,(x_i)\,\delta^{\theta-1}\,\frac{A}{1-A}\left[1+b\ (1-\delta^\theta)\left(1-\frac{A}{1-A}\right)\right]\right\} \tag{8}$$

$$\frac{\partial^2 lnL}{\partial\alpha\,\partial\gamma}=\lambda\theta b\sum_{i=1}^{n}x_i^\alpha ln\,(x_i)\ \delta^{\theta-1}\ \frac{A}{1-A} \tag{9}$$

$$\frac{\partial^2 lnL}{\partial\lambda\partial\theta}=-\sum_{i=1}^{n}\delta^\theta\ ln\,(\delta)+(\gamma-1)\,b\sum_{i=1}^{n}\delta^\theta\ ln\,(\delta)\,\frac{A}{1-A}$$
$$+(\gamma-1)\ b\ \sum_{i=1}^{n}\delta^\theta\ ln\,(\delta)\,\frac{A}{1-A}\left[1+b\ (1-\delta^\theta)\left(1-\frac{A}{1-A}\right)\right] \tag{10}$$

$$\frac{\partial^2 lnL}{\partial\lambda\partial b} = -\theta \sum_{i=1}^{n} x_i^{\alpha} \ \delta^{\theta-1} \ + (\gamma-1) \ \theta \sum_{i=1}^{n} \left\{ x_i^{\alpha} \delta^{\theta-1} \frac{A}{1-A} \left[1 + b \ \left(1-\delta^{\theta}\right) \left(\left(1 - \frac{A}{1-A} \right) \right) \right] \right\} \tag{11}$$

$$\frac{\partial^2 lnL}{\partial\lambda\partial\gamma} = \theta b \sum_{i=1}^{n} x_i^{\alpha} \ \delta^{\theta-1} \frac{A}{1-A} \tag{12}$$

$$\frac{\partial^2 lnL}{\partial\theta\partial b} = -\sum_{i=1}^{n} \delta^{\theta} \ ln\left(\delta\right) \ + (\gamma-1) \ \sum_{i=1}^{n} \delta^{\theta} \ ln\left(\delta\right) \frac{A}{1-A} \left[1 + b \ \left(1-\delta\right)^{\theta} \left(1 - \frac{A}{1-A} \right) \right] \tag{13}$$

$$\frac{\partial^2 lnL}{\partial\theta\partial\gamma} = \theta b \sum_{i=1}^{n} \ \delta^{\theta} \ ln\left(\delta\right) \frac{A}{1-A} \tag{14}$$

$$\frac{\partial^2 lnL}{\partial b\partial\gamma} = \sum_{i=1}^{n} \left[\delta^{\theta} \ - 1 \right] \frac{A}{1-A} \tag{15}$$

Where $\delta = 1 + \lambda x_i^{\alpha}$ and $A = e^{b(1-(1+\lambda x_i^{\alpha}))}$.