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A Novel Construction of Mutually Orthogonal Three Disjoint Union of Certain Trees Squares

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Abstract: A family of decompositions $\{\mathscr{G}_0, \mathscr{G}_1, \ldots, \mathscr{G}_{k-1}\}$ of a complete bipartite graph $K_{n,n}$ is a set of k *mutually orthogonal graph squares* $(MOGS)$ if $\mathscr{G}_i$ and $\mathscr{G}_j$ are orthogonal for all $i, j \in \{0, 1, \ldots, k-1\}$ and $i \neq j$. For any subgraph G of $K_{n,n}$ with n edges, $N(n,G)$ denotes the maximum number k in a largest possible set $\{\mathscr{G}_0, \mathscr{G}_1, \ldots, \mathscr{G}_{k-1}\}$ of $MOGS$ of $K_{n,n}$ by G. In this paper we compute some new extensions of the well-known $N(n,G) \geq 3$, using a novel approach, where G represents disjoint unions of certain small trees subgraphs of $K_{n,n}$.

Keywords: Orthogonal graph squares; Orthogonal double cover; Mutually orthogonal Latin squares

1 Introduction

In this paper, $K_{m,n}$ denotes to the complete bipartite graph with partition sets of sizes m and n. Furthermore, P_n for the path on n vertices, sG for s disjoint copies of G and K_n for the complete graph on n vertices. We have shown that, there exist mutually orthogonal graph squares $(MOGS)$ of complete bipartite graphs by disjoint union of graphs using orthogonal squares. For thus reason, $(MOGS)$ is referred to as an extended mutually orthogonal Latin squares $(MOLS)$. It is well-known that orthogonal Latin squares exist for every $n \notin \{2,6\}$. A family of k orthogonal Latin squares of order n is a set of k Latin squares any two of which are orthogonal. It is customary to denote $N(n) =$Max$\{k : \exists k \quad MOLS\}$ by the maximal number of squares in the largest possible set of $MOLS$ of side n. In [1] display the fundamental result of $(MOLS)$, That is $N(n,nK_2) = N(n) = n-1$, where n is a prime power. An edge decomposition of $K_{n,n}$ by $nK_2 \simeq nK_{1,1}$ is equivalent to a Latin square of side n, two edge decompositions $\mathscr{G}$ and $\mathscr{F}$ of $K_{n,n}$ by $nK_{1,1}$ are orthogonal if and only if the corresponding Latin squares of side n are orthogonal; thus $N(n,nK_{1,1}) = N(n)$. The computation of $N(n)$ is one of the most complicated problems in combinatorial designs; see the survey articles by Abel et al. [2] and Colbourn and Dinitz in [3]. It is clear that $N(n,G)$ is a natural generalization of $N(n)$. Many authors studied $MOGS$ of $K_{n,n}$ by G, where $G \neq nK_2$ (see the survey articles [4],[5],[6]). The two sets $\{0_0, 1_0, \ldots, (n-1)_0\}$ and $\{0_1, 1_1, \ldots, (n-1)_1\}$ denote the vertices of the partition sets of $K_{n,n}$. If there is no chance of confusion, we will write (x,y) instead of $\{x_0, y_1\}$ for the edge between the vertices x_0 and y_1, see any row in Figure 1.

2 Materials and Discussion

In the following, We now provide the basic definitions of a $G-$Square over additive group $\mathbb{Z}_n$. We will represent the graph G_f by function $f : \mathbb{Z}_n \to \mathbb{Z}_n$.. We define $E(G_f) = \{(x, f(x)) : x \in \mathbb{Z}_n\}$. Note that the page of G_f has the property that $\upsilon(x) = 1$ (degree of x)for all $x \in \mathbb{Z}_n$.. That is, it represents unions of stars which has the same direction. In El-Shanawany (see [6]) give the formal definitions of the terms of subgraph of $K_{n,n}$ induced by a function over additive group $\mathbb{Z}_n$ as the follow,

Definition 1.*Let G be a subgraph of $K_{n,n}$. Let $f : \mathbb{Z}_n \to \mathbb{Z}_n$. Then G is called $f-$starter if $E(G) = \cup_{x \in \mathbb{Z}_n}(x; f(x))$.*

Definition 2.*Let G be a $f-$starter graph, and let $\beta \in \mathbb{Z}_n$.Then the graph $G_f + \beta$ with edge*

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$E(G_f + \beta) = \{(x, f(x)+\beta) : (x, f(x)) \in E(G_f)\}$ *is called the* $(\beta, f)-$*translate of* G_f.

Definition 3.*If* G *is a* $f-$ *starter graph, then the union of all translates of* G_f *forms an edge decomposition of* $K_{n,n}$ *i.e.* $\cup_{\beta\in\mathbb{Z}_n} E\left(G_f+\beta\right) = E\left(K_{n,n}\right)$.

In the next, we introduce now the formal basic definitions of a $G-$Square over additive group $\mathbb{Z}_n$.

Definition 4. *(see [6]) Let* G *be a subgraph of* $K_{n,n}$. *A square matrix* L *of order* n *is called a* $G-$*square if every element in* $\mathbb{Z}_n$ *occur exactly* n *times and the graphs* $G^{\alpha}, \alpha \in \mathbb{Z}_n$ *with* $E(G^{\alpha}) = \{(x,y) : L(x,y) = \alpha; x,y \in \mathbb{Z}_n\}$ *are isomorphic to graph* G.

For an edge decomposition G^i we may associate bijectively a $n\times\ n-$square with entries belonging to $\mathbb{Z}_n$ denoted by $L_i = L_i(x,y), 0 \le i \le k-1; x,y \in \mathbb{Z}_n$ with

$$L_i(x,y) = \alpha \Leftrightarrow (x,y) \in E(G^{\alpha,i}), \alpha \in \mathbb{Z}_n. \qquad (1)$$

Similar to Definition4, we define:

Definition 5.*(see [6]) Let* i, j *be different positive integers. Two square matrices* L_i *and* L_j *of order* n *are said to be orthogonal if for any ordered pair* (a,b), *there is exactly one position* (x,y) *for* $L_i(x,y) = a$, *and* $L_j(x,y) = b$.*That is, the two graph squares have the property that, when superimposed, every ordered pair occurs exactly once.*

In [6] El-Shanawany presented an immediate result of the Definition 4, $N(3, K_2 \cup K_{1,2}) = 3$. Define the 3 mutually orthogonal $(K_2 \cup K_{1,2})-$squares of order 3 (i.e. 3 mutually orthogonal decompositions $(MODs)\ of\ \ K_{3,3}$ by $K_2 \cup K_{1,2})$ as follows:

$$M_0 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

As an immediate consequence of the Definition 4 and the Equation 1, we will illustrate the following example.

*Example 1.*The subgraph $G \simeq K_2 \cup K_{1,2}$ of $K_{3,3}$ is a $f-$starter graph G_f induced by the function $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ defined by $f(x) = x^2 + sx$, for all $s, x \in \mathbb{Z}_3$ as shown in figure 1. Note that every row in figure 1 represents edge decompositions of $K_{3,3}$ by $(K_2 \cup K_{1,2})$. That is equivalent M_s squares, $s = 0,1,2$.

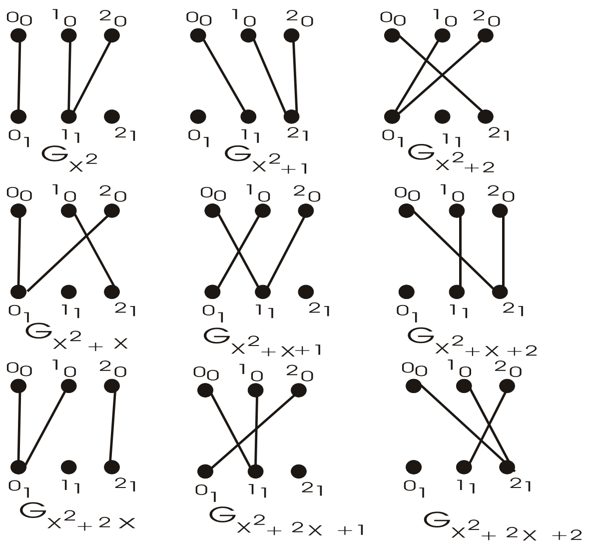

Fig. 1: MODs $G_f = K_1 \cup K_{1,2}$, of $K_{3,3}$ induced by the function f w.r.t. $\mathbb{Z}_3$.

3 Results and discussion

In this section, we use starter functions method to give some new direct constructions for $N(n,G) = k \ge 3$, where G represent a disjoint unions of certain small trees of $K_{n,n}$.

Let p a prime number, Let $f-$starter function of subgraph G of $K_{p,p}$ with p edges, $N(p, G_f)$ denotes the maximum number k in a largest possible set $\{\mathscr{G}_0, \mathscr{G}_1, ..., \mathscr{G}_{k-1}\}$ of $MOGS$ of $K_{p,p}$ by G_f. For all $x,y \in \mathbb{Z}_p$. Let $L_s(x,y) = j$, where $y = f(x) + sx + j$, for all $0 \le s \le k-1,\ j \in \mathbb{Z}_p$. Since $j = y - f(x) - sx$, then we can write

$$L_s(x,y) = y - f(x) - sx. \qquad (2)$$

It is easily verified that for all different $0 \le s, r\ \le k-1$ the pair $(L_s; L_r)$ is orthogonal for all $x,y \in \mathbb{Z}_p$ under the condition:

$$(L_s(x,y), L_r(x,y)) = (y - f(x) - sx\ , y - f(x) - rx). \qquad (3)$$

Theorem 1.$N(11, K_{1,3} \cup 2K_{1,2} \cup 4K_2) \ge 10$

*Proof.*Let $f(x) = (x+1)^4$ be the starter function of the subgraph $(K_{1,3} \cup 2K_{1,2} \cup 4K_2)$ of $K_{11,11}$. From the equation 1, 2, we have $(K_{1,3} \cup 2K_{1,2} \cup 4K_2)-$Squares L_s of order 11 which is defined as follows:

$$L_s(x,y) = y - f(x) - sx, \text{ for all } 1 \le s \le 10. \qquad (4)$$

That is mean, there exist 10 $MODs$ of $K_{11,11}$ by $(K_{1,3} \cup 2K_{1,2} \cup 4K_2)$. Applying Definition 5, it is easily to see that for all different $1 \le k, r\ < 10$ the pair$(L_k; L_r)$ is orthogonal under the condition

$$(L_k(x,y), L_r(x,y)) = (y - f(x) - kx\ , y - f(x) - rx)\ , \forall x,y \in \mathbb{Z}_{11}$$

We prove that the page obtained from the entries in L_1 equal to zero is isomorphic to $\left(K_{1,3} \cup 2K_{1,2} \cup 4K_2\right)$. Also, a similar argument can be applied to the other pages in $L_1, L_2, \ldots, L_{10}$. It is clear that every row contains one zero, there is exactly 1 column has 3 zeros, 2 columns have 2 zeros, 4 columns have one zero, 4 columns have no zeros. That is, for all $x \in \mathbb{Z}_{11}$, all vertices x_0 have degree one. There is exactly 1 vertex x_1 has degree 3, exactly 2 vertices x_1 have degree 2, 4 vertices have degree one, and exactly 4 vertices have degree zero.

As a direct construction of this theorem for $s = 1, 2$ in 4 is the following to Squares L_s of order 11.

$$L_1 = \begin{bmatrix} 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 \\ 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 \\ 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 \\ 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 \end{bmatrix},$$

Theorem 2. $N(13, K_{1,4} \cup 3K_{1,2} \cup 3K_2) \geq 4$.

Proof. Let $f(x) = (x+1)^4$ be the starter function of the subgraph $(K_{1,4} \cup 3K_{1,2} \cup 3K_2\ )$ of $K_{13,13}$. From the equation 1, 2, we have $(K_{1,4} \cup 3K_{1,2} \cup 3K_2\ ) -$Squares L_s of order 13 which is defined as follows:

$$L_s(x,y) = y - f(x) - sx, \text{ for all } s \in \{1,5,8,12\}. \quad (5)$$

that is mean, there exist 4 *MODs* of $K_{13,13}$ by $(K_{1,4} \cup 3K_{1,2} \cup 3K_2)$. Applying Definition 5, it is easily to see that for all different $k, r \in \{1,5,8,12\}$. the pair$(L_k; L_r)$ is orthogonal under the condition

$$(L_k(x,y), L_r(x,y)) = (y - f(x) - kx\ , y - f(x) - rx)\ , \ \forall x,y \in \mathbb{Z}_{13}$$

We prove that the page obtained from the entries in L_1 equal to zero is isomorphic to $\left(K_{1,4} \cup 3K_{1,2} \cup 3K_2\right)$. Also, a similar argument can be applied to the other pages in L_1, L_5, L_8, L_{12}. It is clear that every row contains one zero, there is exactly one column has 4 zeros, three columns have two zeros, three columns have one zero, and six columns have no zeros. That is, for all $x \in \mathbb{Z}_{13}$, all vertices x_0 have degree one. There is exactly one vertex x_1 has degree 4, exactly 3 vertices x_1 have degree two, exactly 3 vertices have one degree, and exactly 6 vertices have degree zeros.

As a direct construction of this theorem for $s = 1, 5$ in 5 is the following to Squares L_s of order 13.

$$L_1 = \begin{bmatrix} 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 \\ 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 \end{bmatrix},$$

$$L_5 = \begin{bmatrix} 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 \\ 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

Theorem 3. $N(13, K_{1,3} \cup 2K_{1,2} \cup 6K_2) \geq 6$.

Proof. Let $f(x) = (x^3+1)^5$ be the starter function of the subgraph $(K_{1,3} \cup 2K_{1,2} \cup 6K_2)$ of $K_{13,13}$. From the equation 1,2, we have $(K_{1,3} \cup 2K_{1,2} \cup 6K_2) -$Squares L_s of order 13 which is defined as follows:

$$L_s(x,y) = y - f(x) - sx, \text{ for all } s \in \{2,4,5,6,10,12\ \} \quad (6)$$

That is mean, there exist 6 *MODs* of $K_{13,13}$ by $(K_{1,3} \cup 2K_{1,2} \cup 6K_2)$. Applying Definition 5. It is easily to see that for all different $k, r \in \{2,4,5,6,10,12\ \}$ the pair $(L_k; L_r)$ is orthogonal under the condition

$$(L_k(x,y), L_r(x,y)) = (y - f(x) - kx\ , y - f(x) - rx)\ , \ \forall x,y \in \mathbb{Z}_{13}.$$

We prove that the page obtained from the entries in L_2 equal to zero is isomorphic to $\left(K_{1,3} \cup 2K_{1,2} \cup 6K_2\right)$. Also, a similar argument can be applied to the other pages in $L_2, L_4, L_5, .., L_{12}$. It is clear that every row contains one zero, there is exactly one column have 3 zeros, 2 columns have two zero, exactly 6 columns have one zeros, and 4 columns have no zeros. That is, for all $x \in \mathbb{Z}_{13}$, all vertices x_0 have degree one. There is exactly 1 vertex x_1 has degree 3, exactly 2 vertices x_1 have degree two, exactly 6 vertices x_1 have degree one, and 4 columns have degree zero.

As a direct construction of this theorem for $s = 2,4$ in 6 is the following to Squares L_s of order 13.

$$L_2 = \begin{bmatrix} 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 \end{bmatrix},$$

$$L_4 = \begin{bmatrix} 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 \\ 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 \\ 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 \\ 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Theorem 4. $N(17, K_{1,3} \cup 4K_{1,2} \cup 6K_2) \geq 16$.

Proof. Let $f(x) = (x+1)^4$ be the starter function of the subgraph $(K_{1,3} \cup 4K_{1,2} \cup 6K_2)$ of $K_{17,17}$. From the equation 1, 2, we have $(K_{1,3} \cup 4K_{1,2} \cup 6K_2)-$Squares L_s of order 17 which is defined as follows:

$$L_s(x,y) = y - f(x) - sx, \text{ for all } 1 \leq s \leq 16. \tag{7}$$

That is mean, there exist 16 *MODs* of $K_{17,17}$ by $(K_{1,3} \cup 4K_{1,2} \cup 6K_2)$. Applying Definition 5, it is easily to see that for all different $1 \leq k, r < 16$ the pair $(L_k; L_r)$ is orthogonal under the condition:

$$(L_k(x,y), L_r(x,y)) = (y - f(x) - kx\ , y - f(x) - rx)\ ,\ \forall x, y \in \mathbb{Z}_{17}$$

We prove that the page obtained from the entries in L_1 equal to zero is isomorphic to $(K_{1,3} \cup 4K_{1,2} \cup 6K_2)$. Also, a similar argument can be applied to the other pages in $L_1, L_2, L_3, .., L_{16}$. It is clear that every row contains one zero, there is exactly one column have 3 zeros, 4 columns have two zero, exactly 6 columns have one zeros, and 6 columns have no zeros. That is, for all $x \in \mathbb{Z}_{17}$, all vertices x_0 have degree one. There is exactly 1 vertex x_1 has degree 3, exactly 4 vertices x_1 have degree two, exactly 6 vertices x_1 have degree one, and 6 columns have degree zero.

As a direct construction of this theorem for $s = 1,2$ in 7 is the following to Squares L_s of order 16.

$$L_1 = \begin{bmatrix} 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 \\ 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 \\ 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 0 & 1 \end{bmatrix}.$$

4 conclusion

In conclusion, we can summarize our results in the following table:

n	G	$N(n,G)$
11	$K_{1,3} \cup 2K_{1,2} \cup 4K_2$	≥ 10
13	$K_{1,4} \cup 3K_{1,2} \cup 3K_2$	≥ 4
13	$K_{1,3} \cup 2K_{1,2} \cup 6K_2$	≥ 6
17	$K_{1,3} \cup 4K_{1,2} \cup 6K_2$	≥ 16

Furthermore, we conjecture that obtain superior outcomes to those in the above table.

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