Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/120231

Alpha-Power of the Power Ailamujia Distribution: Properties and Applications

R. S. Gomaa[1,], E. A. Hebeshy[2], M. M. El Genidy[2] and B. S. El-Desouky[1]*

[1]Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
[2]Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, Egypt

Received: 2 Sep. 2022, Revised: 22 Oct. 2022, Accepted: 15 Nov. 2022.
Published online: 1 May 2023

Abstract: This paper deals with a novel distribution defined as Alpha Power of the Power Ailamujia distribution (APPA). Using the power transformation technique, which incorporates an extra parameter of the distribution, the proposed distribution is obtained. The quantile function, moments, moment generating function, characteristics function, mode, median, order statistics, Shannon's entropy, survival measures and other properties have been studied for the newly developed distribution. The behavior of probability density function (pdf), cumulative distribution function (cdf), survival function and hazard rate function are illustrated through various plots. The method of maximum likelihood estimation has been used to estimate the parameters of this distribution. Finally, the APPA distribution is more suitable than other competing distributions, according to four real data, including two COVID-19 data in two countries that were taken into consideration to assess the utility of the established distribution.

Keywords: COVID-19 pandemic, Power Ailamujia distribution, Alpha power transformation technique, Lambert function, Maximum likelihood estimation.

1 Introduction

The last few decades have seen a rise in the generalization of standard distributions in statistical literature. Generally, adding a new parameter involves combining the current distributions or generating new distributions using generators (see [1]). The purpose of this change is to increase the modelling flexibility of statistical distributions for data sets that arise in various disciplines.

More recently, several authors have used an Ailamujia distribution due to its importance in modeling lifetime and designing the maintenance time in [2]. Researchers advanced a strategy for including a new parameter in established statistical distributions in [3, 4]. Size biased Ailamujia distribution was proposed by Rather et al. [5], and it used to analyze data from medical and engineering. Further, [6] gave a survey on the techniques for producing univariate continuous models. In [7, 8], authors suggested the statistical characteristics and applications of the inverse analogue of Ailamujia distribution and provided the Bayesian estimation of this distribution applying various loss functions. The concept of continuous distributions' T-X family in which cdf of beta distribution was substituted by a function of cdf satisfying specific requirements and used the pdf of any continuous variable in place of the pdf was presented in [9]. Moreover, a comparison between Alpha Power Transformed Aradhana (APTA) distribution and other models such as Two Parameters Aradhana, Power Aradhana, Length Biased (LB) Garima, Exponential and Garima distributions was proposed in [10, 11].

Alpha power transformation (APT), a new technique that combines skewness to the baseline distribution by introducing extra parameters in continuous model was proposed in [12]. The APT is defined as follow

Definition 1.1. Let *g(x)* and *G(x)* be the pdf and cdf of any continuous random variable *X*, respectively, and then the pdf of the APT family with parameter Θ is defined as

$$g_{APT}(x) = \begin{cases} \dfrac{\log\Theta}{\Theta-1}\Theta^{G(x)}g(x), & if\ \Theta>0, \Theta\neq 1, \\ g(x) & , if\ \Theta = 1. \end{cases} \tag{1}$$

The cumulative distribution function is

$$G_{APT}(x) = \begin{cases} \dfrac{\Theta^{G(x)}-1}{\Theta-1}, & if\ \Theta>0, \Theta\neq 1, \\ G(x) & , if\ \Theta = 1. \end{cases} \tag{2}$$

The preceding generator was used in [13, 1] to convert two parameters Weibull distribution into three parameters AP Weibull

[*]Corresponding author e-mail: dr.rsg12@yahoo.com

distribution. In addition to applied the generator to transform one parameter exponential distribution into two parameter AP exponential distribution, researchers have studied the transformation to obtain APT distributions including APT generalized exponential [14], APT extended exponential [15], APT Lindly [16] and APT inverse Lindly [17].

On the other hand, the power Ailamujia (PA) distribution is a new two parameters lifetime distribution. It is an extension of the Ailamujia distribution in which adding a shape parameter β > 0 to the former Ailamujia distribution through the use of the power transform in order to increase its overall flexibility (see [18, 19, 20]).

Definition 1.2. The probability density function (pdf) of the PA distribution with parameters Θ and β is expressed as

$$f(x,\Theta,\beta)=4\Theta^2\beta x^{2\beta-1}e^{-2\Theta x^{\beta}}, x>0,\Theta,\beta>0. \quad (3)$$

The corresponding cumulative distribution function (cdf) is given as

$$F(x,\Theta,\beta)=1-\left(1+2\Theta x^{\beta}\right)e^{-2\Theta x^{\beta}}, x>0,\Theta,\beta>0. \quad (4)$$

Remark 1.1. If we set β = 1 in Eqs. (3) and (4), then the PA distribution is reduced to the former Ailamujia distribution.

This research aims to introduce a novel and flexible distribution, which is called alpha power of the power Ailamujia (APPA) distribution, by adding a new parameter to the basic power Ailamujia distribution in order to get a good fit. Section 2 includes many characteristics of the APPA distribution and expressions for pdf, cdf, survival and hazard rate functions. Section 3 contains the APPA model's statistical properties including moments, quantile function, median, mode, moment generating function, characteristic function, Shannon's entropy and order statistics. Method of maximum likelihood estimation of parameters for APPA model is obtained in Section 4. Section 5 provides simulation results for the APPA distribution. Four real data analyses are explained in Section 6 to assess the efficiency of the suggested distribution. The final Section provides the conclusions.

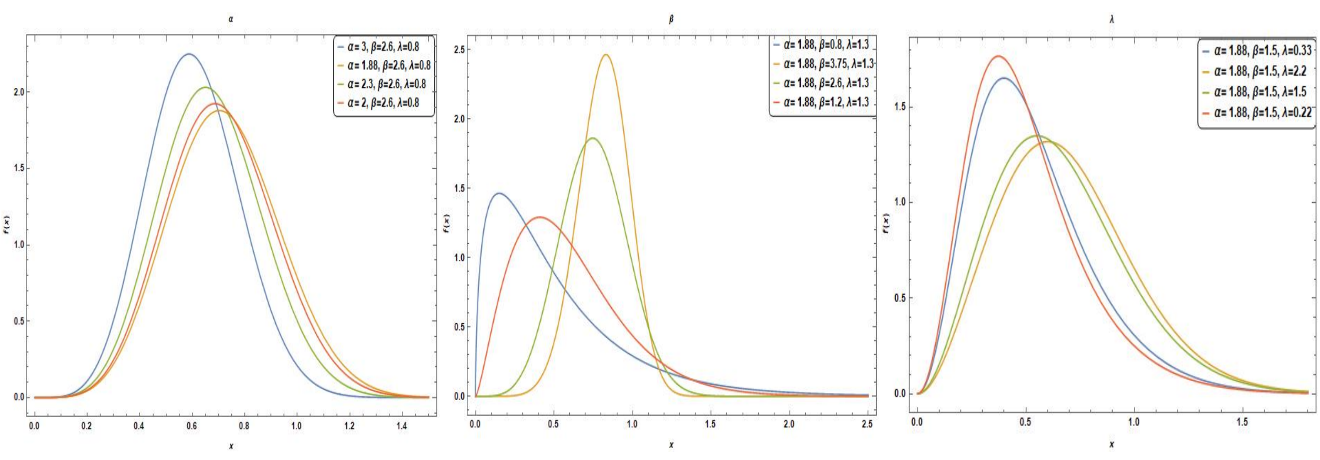
2 Alpha Power of the Power Ailamujia (APPA) Distribution

Definition 2.1. The random variable X has the following cumulative distribution function (cdf) and probability density function (pdf) of APPA distribution and are respectively given by

$$F_{APPA}(x)=\begin{cases}\dfrac{\lambda^{\left(1-\left(1+2\alpha x^{\beta}\right)e^{-2\alpha x^{\beta}}\right)}-1}{\lambda-1} & , if\ x>0,\alpha,\beta,\lambda>0,\lambda\neq1,\\ 1-\left(1+2\alpha x^{\beta}\right)e^{-2\alpha x^{\beta}} & , if\ \lambda=1.\end{cases} \quad (5)$$

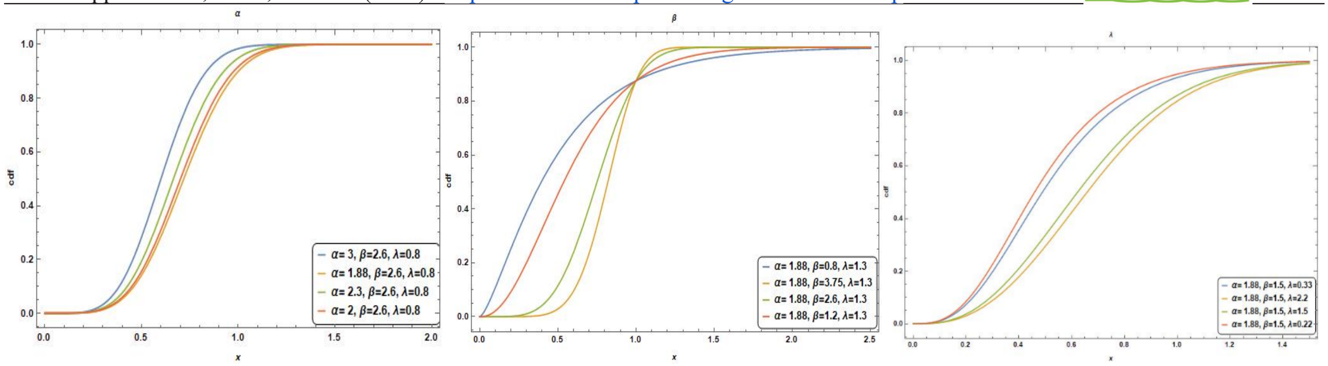
$$f_{APPA}(x)=\begin{cases}4\alpha^2\beta\left(\dfrac{log\lambda}{\lambda-1}\right)\lambda^{\left(1-\left(1+2\alpha x^{\beta}\right)e^{-2\alpha x^{\beta}}\right)}e^{-2\alpha x^{\beta}}x^{2\beta-1} & , if\ \alpha,\beta,\lambda>0,\lambda\neq1,\\ 4\alpha^2\beta x^{2\beta-1}e^{-2\alpha x^{\beta}} & , if\ \lambda=1,\\ 0 & , otherwise.\end{cases} \quad (6)$$

The APPA pdf and cdf are displayed in Fig. (1) and Fig. (2), respectively. Clearly, pdf is positively skewed, symmetric and slightly negatively skewed for various values of parameters.



(a) (b) (c)

Fig. 1: Plots (a), (b) and (c) illustrates various shapes of the pdf of APPA distribution.



(d) (e) (f)

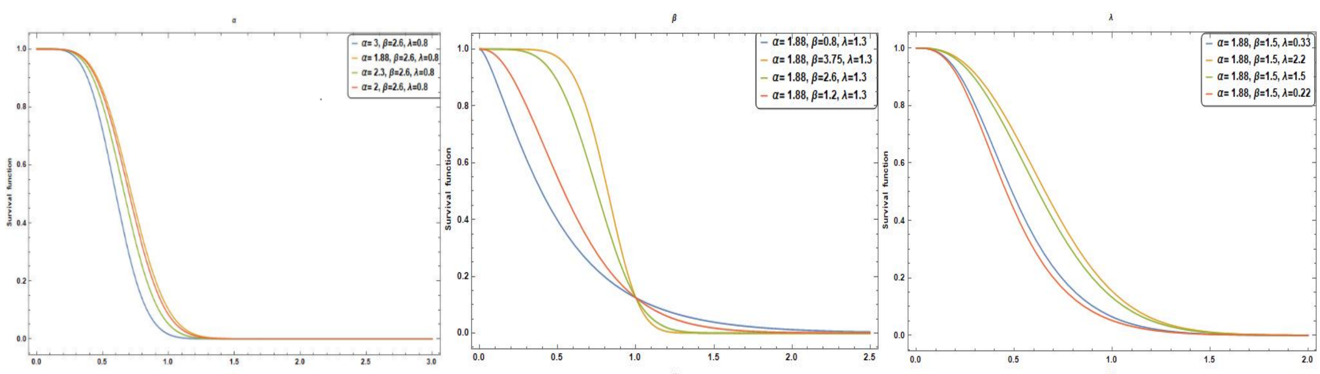
Fig. 2: Plots (d), (e) and (f) illustrates various shapes of the cdf of APPA distribution.

Definition 2.2. The reliability (Survival) function and hazard rate function (HF) of APPA distribution are obtained, respectively, as follows:

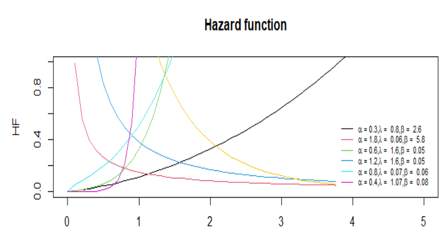
$$S_{APPA}(x) = \begin{cases} \dfrac{\lambda - \lambda^{\left(1-(1+2\alpha x^{\beta})e^{-2\alpha x^{\beta}}\right)}}{\lambda - 1} & ,\lambda \neq 1, \\ (1+2\alpha x^{\beta})e^{-2\alpha x^{\beta}} & ,\lambda = 1. \end{cases} \tag{7}$$

$$HF = H_{APPA}(x) = \begin{cases} \dfrac{4\alpha^2 \beta (log\lambda)\lambda^{\left(1-(1+2\alpha x^{\beta})e^{-2\alpha x^{\beta}}\right)} e^{-2\alpha x^{\beta}} x^{2\beta-1}}{\lambda - \lambda^{\left(1-(1+2\alpha x^{\beta})e^{-2\alpha x^{\beta}}\right)}} & ,\lambda \neq 1, \\ \dfrac{4\alpha^2 \beta x^{2\beta-1}}{(1+2\alpha x^{\beta})} & ,\lambda = 1. \end{cases} \tag{8}$$

The survival function and HF of the APPA distribution are displayed in Fig. (3). Clearly that, the hazard rate function for various parameters exhibits increasing, decreasing, bathtub and J- shaped.



(g) **(h)** **(i)**



(j)

Fig. 3: Plots (g), (h), (i) and (j) illustrates various shapes of the survival function and the HF of APPA distribution.

3 Statistical Properties of APPA Model

3.1 Expansion of the Density Function of APPA Distribution

The pdf in Eq. (6) can be rewritten by making the substitution $x^{\beta} = t$ as follows

$$f_{APPA}(x) = f_{APPA}\left(t^{\frac{1}{\beta}}\right) = 4\alpha^2\beta\,\lambda\left(\frac{log\lambda}{\lambda-1}\right)\lambda^{-(1+2\alpha t)e^{-2\alpha t}}e^{-2\alpha t}t^{2-\frac{1}{\beta}},$$

by using the following series representation $\alpha^{-z} = \sum_{k=0}^{\infty}\frac{(-\log\alpha)^k z^k}{k!}$ for $\lambda^{-(1+2\alpha t)e^{-2\alpha t}}$, then

$$f_{APPA}\left(t^{\frac{1}{\beta}}\right) = 4\alpha^2\beta\,\lambda\left(\frac{log\lambda}{\lambda-1}\right)\sum_{k=0}^{\infty}\frac{(-\log\lambda)^k}{k!}(1+2\alpha t)^k\,e^{-(2\alpha k+2\alpha)t}\,t^{2-\frac{1}{\beta}},$$

finally, we get the expansion of $f_{APPA}(x)$ by using the binomial theory of $(1+2\alpha t)^k$ as follows

$$f_{APPA}\left(t^{\frac{1}{\beta}}\right) = 4\alpha^2\beta\,\lambda\left(\frac{log\lambda}{\lambda-1}\right)\sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k}{n}\frac{(-log\,\lambda)^k}{k!}(2\alpha)^n\,e^{-(2\alpha k+2\alpha)t}\,t^{n-\frac{1}{\beta}+2}, \tag{9}$$

where $t>0, \alpha,\beta,\lambda>0, \lambda\neq 1$, $0<k<\infty$ and $0<n<k<\infty$.

3.2 Moments of APPA Distribution

Theorem 3.2.1.

For $x>0$ and let X be a random variable follow APPA distribution, then the rth moment is given by

$$\mu_r^{'} = \frac{\lambda}{(2\alpha)^{\frac{r}{\beta}}}\left(\frac{\log\lambda}{\lambda-1}\right)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^k}{n!\,(k-n)!}\frac{\Gamma\left(n+\frac{r}{\beta}+2\right)}{(k+1)^{n+\frac{r}{\beta}+2}}, \tag{10}$$

where $\alpha,\beta,\lambda>0$ and $\lambda\neq 1$.

Proof. From Eq. (6), we have

$$\mu_r^{'} = E(X^r) = \int_0^{\infty} x^r\,f(x;\,\alpha,\beta,\lambda)dx,$$

$$= 4\alpha^2\beta\left(\frac{\log\lambda}{\lambda-1}\right)\int_0^{\infty} x^{r+2\beta-1}\,e^{-2\alpha x^{\beta}}\lambda^{\left(1-\left(1+2\alpha x^{\beta}\right)e^{-2\alpha x^{\beta}}\right)}dx.$$

Making the substitution $x^{\beta} = t$ and using Eq. (9), then we get Eq. (10).

Remark 3.2.1.

By substituting r = 1 and 2, the first two moments of the APPA distribution about origin are obtained

$$\mu_1^{'} = \frac{\lambda}{(2\alpha)^{\frac{1}{\beta}}}\left(\frac{\log\lambda}{\lambda-1}\right)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^k}{n!\,(k-n)!}\frac{\Gamma\left(n+\frac{1}{\beta}+2\right)}{(k+1)^{n+\frac{1}{\beta}+2}}. \tag{11}$$

$$\mu_2^{'} = \frac{\lambda}{(2\alpha)^{\frac{2}{\beta}}}\left(\frac{\log\lambda}{\lambda-1}\right)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^k}{n!\,(k-n)!}\frac{\Gamma\left(n+\frac{2}{\beta}+2\right)}{(k+1)^{n+\frac{2}{\beta}+2}}. \tag{12}$$

Remark 3.2.2.

The mean and the variance of APPA distribution are respectively given by

$$\mu = \mu_1^{'} = \frac{\lambda}{(2\alpha)^{\frac{1}{\beta}}}\left(\frac{\log\lambda}{\lambda-1}\right)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^k}{n!\,(k-n)!}\frac{\Gamma\left(n+\frac{1}{\beta}+2\right)}{(k+1)^{n+\frac{1}{\beta}+2}}. \tag{13}$$

$$\sigma^2 = \mu_2^{'} - \left(\mu_1^{'}\right)^2,$$

$$\sigma^2 = \frac{\lambda \log \lambda}{(2\alpha)^{\frac{2}{\beta}}(\lambda-1)^2}\left((\lambda-1)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^k}{n!\,(k-n)!}\frac{\Gamma\left(n+\frac{2}{\beta}+2\right)}{(k+1)^{n+\frac{2}{\beta}+2}} - \lambda\log\lambda\left(\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^k}{n!\,(k-n)!}\frac{\Gamma\left(n+\frac{1}{\beta}+2\right)}{(k+1)^{n+\frac{1}{\beta}+2}}\right)^2\right). \quad (14)$$

The plots in Fig. (4), it is apparent that mean and variance of APPA model have bounds.

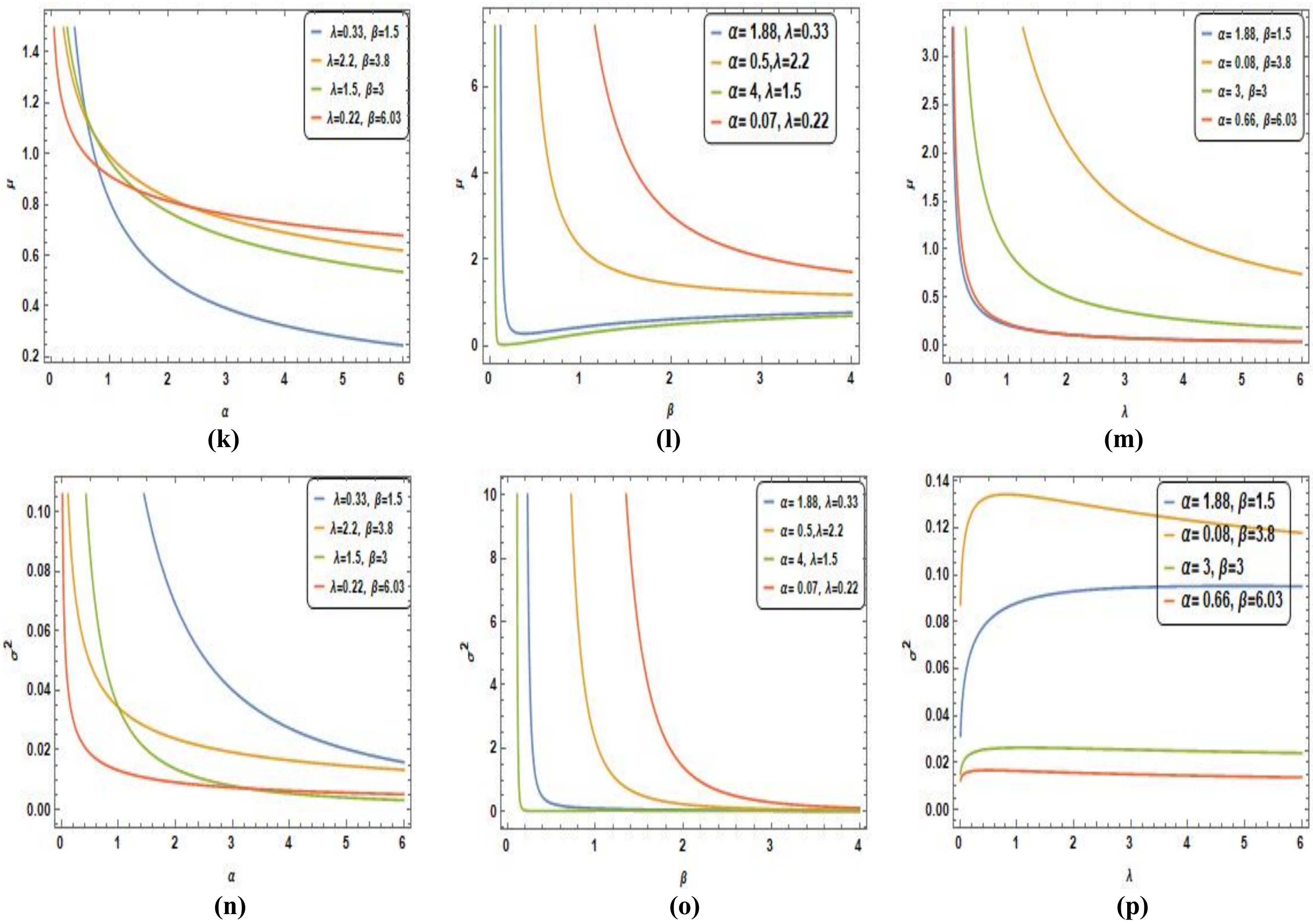


Fig. 4: Plots (k), (l), (m), (n), (o) and (p) illustrates various shapes of the mean and the variance of APPA distribution.

3.3 Moment Generating Function (M.G.F.) of APPA Distribution

Theorem 3. 3. 1.

For $x > 0$ and let X be a random variable follow APPA distribution, then the M.G.F. is obtained as

$$M_X(t) = \left(\frac{\lambda \log\lambda}{\lambda-1}\right)\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{t^r(-\log\lambda)^k}{r!\,n!\,(k-n)!}\frac{\Gamma\left(n+\frac{r}{\beta}+2\right)}{(2\alpha)^{\frac{r}{\beta}}(k+1)^{n+\frac{r}{\beta}+2}}, \quad (15)$$

it is convergent for $\alpha, \beta, \lambda > 0$ and $\lambda \neq 1$.

Proof. From Eq. (6), we have

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x;\ \alpha,\beta,\lambda)dx,$$

using Taylor's theorem, then

$$M_X(t)=\int_0^\infty\left\{1+tx+\frac{(tx)^2}{2!}+\frac{(tx)^3}{3!}+\cdots\right\}f(x;\ \alpha,\beta,\lambda)dx$$

$$=\int_0^\infty\sum_{r=0}^\infty\frac{t^r}{r!}x^r f(x;\ \alpha,\beta,\lambda)dx,$$

$$=\sum_{r=0}^\infty\frac{t^r}{r!}\int_0^\infty x^r f(x;\ \alpha,\beta,\lambda)dx.$$

By using Theorem 3. 2. 1., then we get Eq. (15).

3.4 Quantile Function of APPA Distribution

The APPA model of a random variable X can be expressed utilizing its quantile function (QF). The QF can be obtained as the distribution function's inverse, and it can be used to compute the distribution's median, mode, skewness, and kurtosis using well-known relationships.

Theorem 3. 4. 1.

For $x>0,\alpha,\beta,\lambda>0$ and $\lambda\neq 1$, the QF of APPA distribution is defined as

$$X_q=\left(\frac{1+W_{-1}\left(\left(\frac{\log(q(\lambda-1)+1)}{\log\lambda}-1\right)e^{-1}\right)}{-2\alpha}\right)^{\frac{1}{\beta}},\tag{16}$$

where q ~ uniform (0, 1) and W_{-1} denotes the negative branch of the Lambert W function ($W(z)e^{W(z)}=z$), and z is a complex number (see [19]).

Proof. Let $q=F(x)$, then

$$q=F_{APPA}(x)=\frac{\lambda^{\left(1-(1+2\alpha x^\beta)e^{-2\alpha x^\beta}\right)}-1}{\lambda-1}$$

$$-(1+2\alpha x^\beta)e^{-(1+2\alpha x^\beta)}=\left(\frac{log(q(\lambda-1)+1)}{log\,\lambda}-1\right)e^{-1},$$

from the above equation, we can see that $\left(-(1+2\alpha x^\beta)\right)$ is the Lambert W function of the real number $\left(\frac{log(q(\lambda-1)+1)}{log\,\lambda}-1\right)e^{-1}$. Moreover, for any $x>0,\ \alpha,\beta,\lambda>0\ and\ \lambda\neq 1$ we can check that $(1+2\alpha x^\beta)>0$ and $-(1+2\alpha x^\beta)e^{-(1+2\alpha x^\beta)}=\left(\frac{log(q(\lambda-1)+1)}{log\,\lambda}-1\right)e^{-1}\in\left(\frac{-1}{e},0\right)$. Then,

$$-(1+2\alpha x^\beta)=W_{-1}\left(\left(\frac{log(q(\lambda-1)+1)}{log\,\lambda}-1\right)e^{-1}\right);\ 0<q<1,$$

where W_{-1} denotes the well-known negative branch of the Lambert W function, then we get Eq. (16).

Remark 3. 4. 1.

The lower quartile, the median and the upper quartile of APPA distribution can be obtained by setting q= 0.25, 0.5 and 0.75, respectively. Then, the median (M_d) of APPA distribution is given by

$$M_d=x_{0.5}=\left(\frac{1+W_{-1}\left(\left(\frac{\log(0.5(\lambda-1)+1)}{\log\lambda}-1\right)e^{-1}\right)}{-2\alpha}\right)^{\frac{1}{\beta}}.\tag{17}$$

Also, the mode (Mo) of APPA distribution is obtained by using the empirical formula for median, then

$$M_o = 3\,M_d - 2\,\mu_1^{'},$$

from Eq. (17), gives

$$M_o = 3\left(\frac{1+W_{-1}\left(\left(\log\left(\frac{\lambda+1}{2\lambda}\right)-\log\lambda\right)e^{-1}\right)}{-2\alpha}\right)^{\frac{1}{\beta}} - \frac{2\lambda}{(2\alpha)^{\frac{1}{\beta}}}\left(\frac{\log\lambda}{\lambda-1}\right)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^k}{n!\,(k-n)!}\frac{\Gamma\left(n+\frac{1}{\beta}+2\right)}{(k+1)^{n+\frac{1}{\beta}+2}}. \quad (18)$$

In particular, we use a x_q to obtain the Galton skewness coefficient described as

$$S_k = \frac{x_{0.75} - 2x_{0.5} + x_{0.25}}{x_{0.75} - x_{0.25}}. \quad (19)$$

The following Moors kurtosis coefficient is used to evaluate the kurtosis as

$$K_u = \frac{x_{0.875} - x_{0.625} + x_{0.375} + x_{0.125}}{x_{0.75} - x_{0.25}}. \quad (20)$$

Figs. (5), (6) and (7), represents the plots of skewness and kurtosis for various values of parameters. Clearly, the APPA distribution is symmetric, positively skewed and slightly negatively skewed. In addition, it may also be mesokurtic, platykurtic, or leptokurtic.

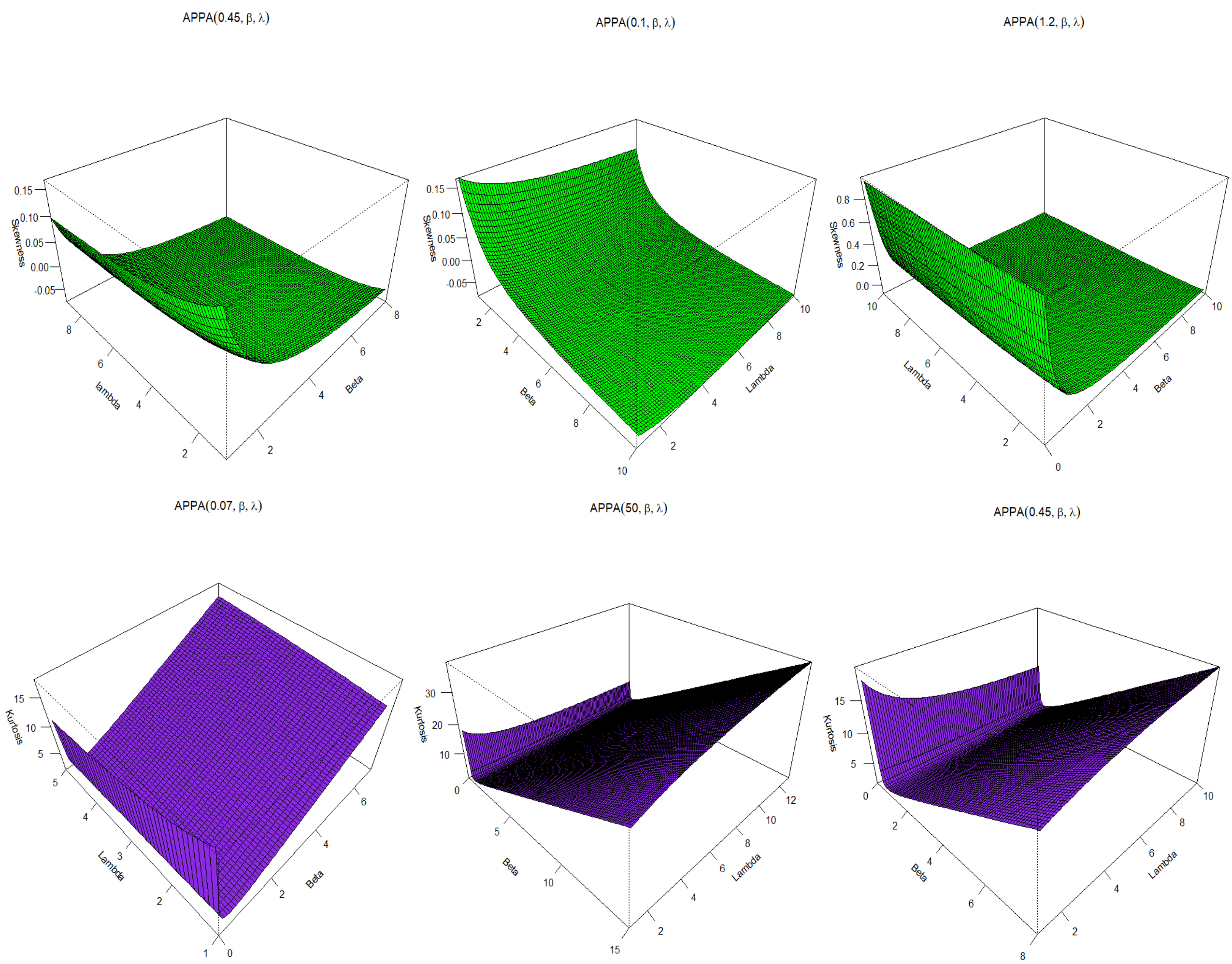


Fig. 5: Illustrate the plots of skewness and kurtosis for selected value of APPA model versus parameter α.

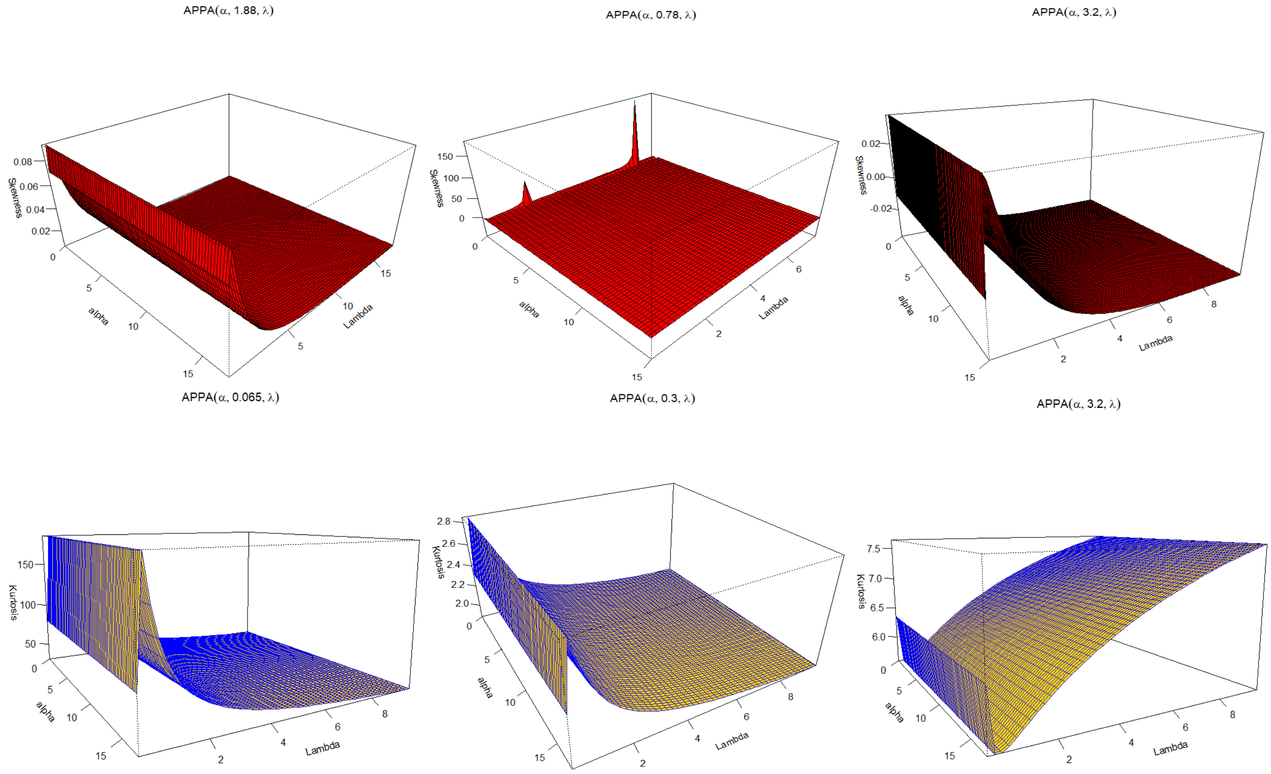


Fig. 6: Illustrate the plots of skewness and kurtosis for selected value of APPA model versus parameter β.

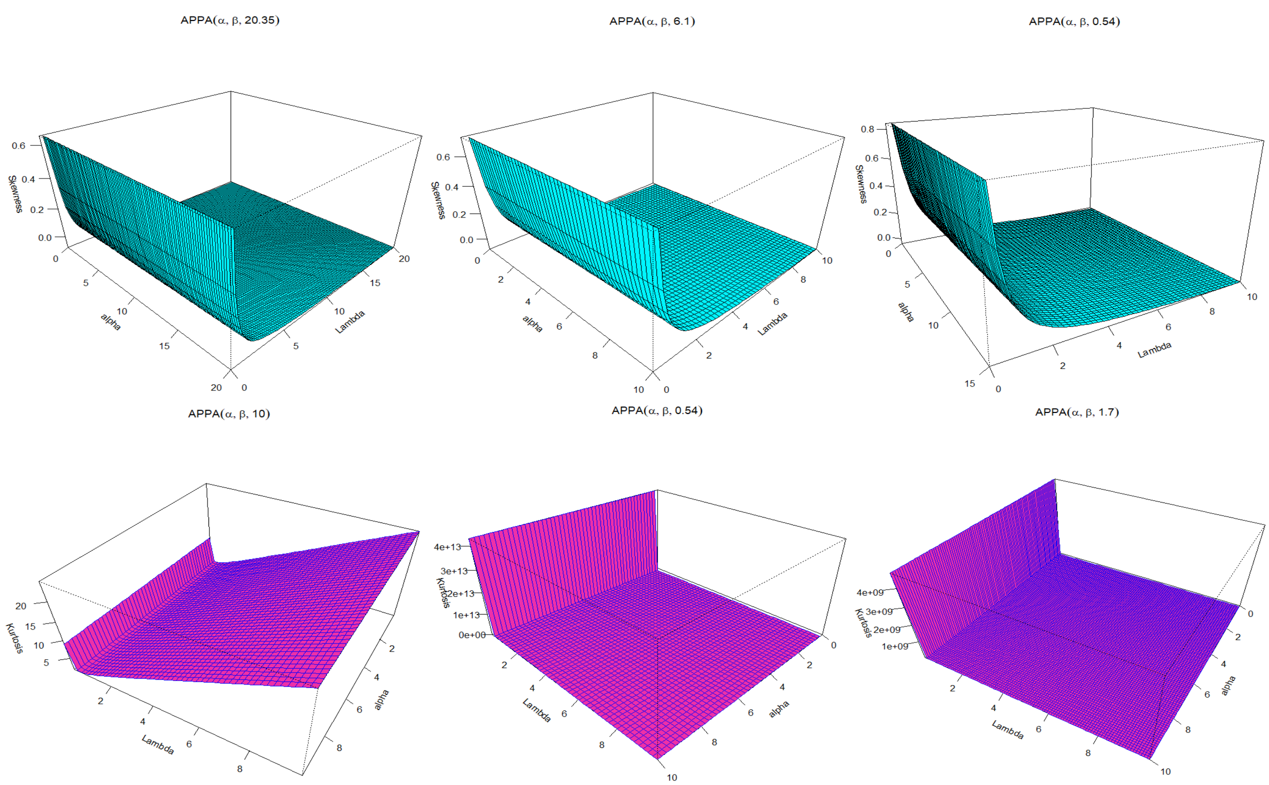


Fig. 7: Illustrate the plots of skewness and kurtosis for selected value of APPA model versus parameter λ.

3.5 Characteristic Function of APPA Distribution

Theorem 3. 5. 1.

For $x > 0$ and let X be a random variable follow APPA distribution, then the characteristic function is given by

$$\varphi_X(t) = \left(\frac{\lambda \log \lambda}{\lambda - 1}\right)\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(it)^r(-\log \lambda)^k}{r!\,n!\,(k-n)!}\frac{\Gamma\left(n+\frac{r}{\beta}+2\right)}{(2\alpha)^{\frac{r}{\beta}}\,(k+1)^{n+\frac{r}{\beta}+2}}, \quad (21)$$

where $\alpha, \beta, \lambda > 0$ and $\lambda \neq 1$.

Proof. Let

$$\varphi_X(t) = E(e^{itx}) = \int_0^{\infty} e^{itx}\, f(x;\ \alpha,\beta,\lambda)dx,$$

using Taylor's theorem, then

$$\varphi_X(t) = \int_0^{\infty}\left\{1 + itx + \frac{(itx)^2}{2!} + \frac{(itx)^3}{3!} + \cdots\right\} f(x;\ \alpha,\beta,\lambda)dx$$

$$= \int_0^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}x^r f(x;\ \alpha,\beta,\lambda)dx,$$

$$= \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^{\infty} x^r\, f(x;\ \alpha,\beta,\lambda)dx.$$

By using Theorem 3. 2. 1., then we get Eq. (21).

3.6 Shannon's Entropy of APPA Distribution

Entropy is a measure of the variance of uncertainty of a random variable and is used in various sciences. Also, it may be described as the average rate where the information is generated by a stochastic source of data.

Theorem 3. 6. 1. For $x > 0, \alpha, \beta, \lambda > 0$ and $\lambda \neq 1$, Shannon's entropy of APPA distribution is defined as

$$H(x;\ \alpha,\beta,\lambda) = \frac{\lambda \log\lambda}{\lambda - 1}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log \lambda)^k}{n!\,(k-n)!}\frac{\Gamma(n+2)}{(k+1)^{n+2}}\left(\log\left(\frac{\lambda-1}{4\alpha^2\beta\, \log\lambda}\right) + \left(2-\frac{1}{\beta}\right)\log 2\alpha - \left(1-\left(1+2\alpha x^{\beta}\right)e^{-2\alpha x^{\beta}}\right)\log \lambda + \left(\frac{n+2}{k+1}\right)\right). \quad (22)$$

Proof. Let

$$H(x;\ \alpha,\beta,\lambda) = -E(\log f(x;\ \alpha,\beta,\lambda))$$

$$= -E\left(\log\left(4\alpha^2\beta\left(\frac{\log\lambda}{\lambda-1}\right)x^{2\beta-1}e^{-2\alpha x^{\beta}}\lambda^{\left(1-\left(1+2\alpha x^{\beta}\right)e^{-2\alpha x^{\beta}}\right)}\right)\right)$$

$$= -E\left(\log 4\alpha^2\beta\left(\frac{\log\lambda}{\lambda-1}\right)\right) - (2\beta-1)E(\log x) + 2\alpha E\left(x^{\beta}\right) - \left(1-\left(1+2\alpha x^{\beta}\right)e^{-2\alpha x^{\beta}}\right)E(\log \lambda). \quad (23)$$

$$E(\log x) = \int_0^{\infty}(\log x)\, f(x;\ \alpha,\beta,\lambda)dx$$

$$= 4\alpha^2\beta\left(\frac{\log\lambda}{\lambda-1}\right)\int_0^{\infty}(\log x)\, x^{2\beta-1}e^{-2\alpha x^{\beta}}\lambda^{\left(1-\left(1+2\alpha x^{\beta}\right)e^{-2\alpha x^{\beta}}\right)}\, dx,$$

making substitution $2\alpha x^{\beta} = z$, then

$$E(\log x) = \frac{1}{\beta}\left(\frac{log\lambda}{\lambda-1}\right)\int_{0}^{\infty} \log\left(\frac{z}{2\alpha}\right)\, z\, e^{-z}\lambda^{(1-(1+z)e^{-z})}dz,$$

after solving the integral, then

$$E(\log x) = \frac{-\lambda \log 2\alpha}{\beta}\left(\frac{log\lambda}{\lambda-1}\right)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^{k}}{n!\,(k-n)!}\frac{\Gamma(n+2)}{(k+1)^{n+2}}. \quad (24)$$

$$E(x^{\beta}) = \int_{0}^{\infty} x^{\beta}\, f(x;\ \alpha,\beta,\lambda)dx$$

$$= \frac{\lambda}{2\alpha}\left(\frac{log\lambda}{\lambda-1}\right)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^{k}}{n!\,(k-n)!}\frac{\Gamma(n+3)}{(k+1)^{n+3}}. \quad (25)$$

$$E(\log\lambda) = \int_{0}^{\infty} (\log\lambda)\, f(x;\ \alpha,\beta,\lambda)dx$$

$$= \frac{\lambda(log\lambda)^{2}}{\lambda-1}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^{k}}{n!\,(k-n)!}\frac{\Gamma(n+2)}{(k+1)^{n+2}}. \quad (26)$$

$$E\left(\log 4\alpha^{2}\beta\left(\frac{log\lambda}{\lambda-1}\right)\right) = \int_{0}^{\infty}\left(\log 4\alpha^{2}\beta\left(\frac{log\lambda}{\lambda-1}\right)\right) f(x;\ \alpha,\beta,\lambda)dx,$$

$$= \frac{\lambda\, log\lambda}{\lambda-1}\log\left(\frac{4\alpha^{2}\beta\, log\lambda}{\lambda-1}\right)\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-\log\lambda)^{k}}{n!\,(k-n)!}\frac{\Gamma(n+2)}{(k+1)^{n+2}}. \quad (27)$$

Now substituting equations (24), (25), (26) and (27) in (23), then we get Eq. (22).

3.7 Order Statistics of APPA Distribution

Theorem 3. 7. 1.

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the order statistics obtained from a random samples $X_1, X_2, \ldots, X_n$ of size n of APPA distribution with pdf $f(x)$ and cdf $F(x)$, where $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ in which $X_{1:n} = min\{X_1, X_2, \ldots, X_n\}$ and $X_{n:n} = max\{X_1, X_2, \ldots, X_n\}$ are the first and last order statistics respectively. Then the pdf of k^{th} order statistics is obtained as

$$f_{X_k}(x) = \frac{n!}{(k-1)!\,(n-k)!}\left(\frac{\lambda^{\left(1-(1+2\alpha x^{\beta})e^{-2\alpha x^{\beta}}\right)}-1}{\lambda-1}\right)^{k-1}\left(\frac{\lambda-\lambda^{\left(1-(1+2\alpha x^{\beta})e^{-2\alpha x^{\beta}}\right)}}{\lambda-1}\right)^{n-k}$$
$$4\alpha^{2}\beta\left(\frac{log\lambda}{\lambda-1}\right)x^{2\beta-1}e^{-2\alpha x^{\beta}}\lambda^{\left(1-(1+2\alpha x^{\beta})e^{-2\alpha x^{\beta}}\right)}, \qquad k = 1,2,\ldots,n. \quad (28)$$

Proof. Let

$$f_{X_k}(x) = \frac{n!}{(k-1)!\,(n-k)!}\left(F(x)\right)^{k-1}\left(1-F(x)\right)^{n-k}f(x), k = 1,2,\ldots,n. \quad (29)$$

By substituting equations (5) and (6) in (29), then we get the probability of k^{th} order statistics of APPA distribution in Eq. (28).

Remark 3. 7. 1.

The pdf of the first order statistics X_1 and the n^{th} order statistics X_n of APPA distribution are respectively given by

$$f_{X_1}(x) = 4n\alpha^2\beta\left(\frac{log\lambda}{\lambda-1}\right)x^{2\beta-1}e^{-2\alpha x^\beta}\lambda^{\left(1-\left(1+2\alpha x^\beta\right)e^{-2\alpha x^\beta}\right)}\left(\frac{\lambda-\lambda^{\left(1-\left(1+2\alpha x^\beta\right)e^{-2\alpha x^\beta}\right)}}{\lambda-1}\right)^{n-1}. \quad (30)$$

$$f_{X_n}(x) = 4n\alpha^2\beta\left(\frac{log\lambda}{\lambda-1}\right)x^{2\beta-1}e^{-2\alpha x^\beta}\lambda^{\left(1-\left(1+2\alpha x^\beta\right)e^{-2\alpha x^\beta}\right)}\left(\frac{\lambda^{\left(1-\left(1+2\alpha x^\beta\right)e^{-2\alpha x^\beta}\right)}-1}{\lambda-1}\right)^{n-1}. \quad (31)$$

4 Parameters Estimation

4.1 Maximum Likelihood Method (MLE)

The parameters estimation of APPA model is derived by using the method of MLE. Let X_1, X_2, X_3, . . ., X_n be a random sample from APPA (α, β, λ). The likelihood function is then defined as

$$L(x;\alpha,\beta,\lambda) = \prod_{i=1}^{n} f(x;\ \alpha,\beta,\lambda)$$

$$= \prod_{i=1}^{n} 4\alpha^2\beta\left(\frac{\log\lambda}{\lambda-1}\right)x^{2\beta-1}e^{-2\alpha x^\beta}\lambda^{\left(1-\left(1+2\alpha x^\beta\right)e^{-2\alpha x^\beta}\right)}$$

$$= (2\alpha)^{2n}\beta^n\left(\frac{\log\lambda}{\lambda-1}\right)^n e^{-2\alpha\sum_{i=1}^n x_i^\beta}\lambda^{\left(n-\left(1+2\alpha\sum_{i=1}^n x_i^\beta\right)e^{-2\alpha\sum_{i=1}^n x_i^\beta}\right)}\sum_{i=1}^{n} x_i^{2\beta-1}, \quad (32)$$

where $x > 0$, $\alpha,\beta,\lambda > 0$, and $\lambda \neq 1$.

Thus, the log-likelihood function is

$$\ell = \log L(x;\alpha,\beta,\lambda) = 2n\log 2\alpha + n\log\beta + n\log\left(\frac{\log\lambda}{\lambda-1}\right) + (2\beta-1)\sum_{i=1}^{n}\log x_i - 2\alpha\sum_{i=1}^{n} x_i^\beta +$$

$$\left(n-\left(1+2\alpha\sum_{i=1}^{n} x_i^\beta\right)e^{-2\alpha\sum_{i=1}^n x_i^\beta}\right)\log\lambda\ . \quad (33)$$

The following equations are formed by taking the derivatives of the previous equation with respect to the three parameters and equating it to zero

$$\frac{\partial\ell}{\partial\alpha} = \frac{2n}{\alpha} - 2\sum_{i=1}^{n} x_i^\beta + 4\alpha\log\lambda\ e^{-2\alpha\sum_{i=1}^n x_i^\beta}\sum_{i=1}^{n} x_i^{2\beta} = 0. \quad (34)$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + 2\sum_{i=1}^{n}\log x_i - 2\alpha\sum_{i=1}^{n} x_i^\beta\sum_{i=1}^{n}\log x_i + 4\alpha^2\log\lambda\ e^{-2\alpha\sum_{i=1}^n x_i^\beta}\sum_{i=1}^{n}\log x_i\sum_{i=1}^{n} x_i^{2\beta} = 0. \quad (35)$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n(\lambda-1-\lambda\log\lambda)}{\lambda(\lambda-1)\log\lambda} + \frac{\left(n-\left(1+2\alpha\sum_{i=1}^n x_i^\beta\right)e^{-2\alpha\sum_{i=1}^n x_i^\beta}\right)}{\lambda} = 0. \quad (36)$$

The system of non-linear equations is numerically solved using the Newton-Raphson method to give estimates for the parameters α, β, and λ. Also, R packages can be applied to optimize log-likelihood.

5 Simulation Study

The efficiency of the MLE technique is assessed by simulation analysis utilizing various criteria, including mean square errors (MSEs), root mean square errors (RMSEs), and average bias (AB) values, as well as their convergent performance in finite samples [21].

It is possible to simulate the APPA model using two different sets of parameters ((α = 1.1, β = 0.3, λ = 0.02) and (α = 0.07, β = 0.55, λ = 0.009)). We consider different samples of sizes n= 50, 100, 200, 400, 500, 700, 800, 1000, and also conducted

1000 iterations to compute the ML estimates (MLEs), (MSEs), (RMSE) and (AB) by the following formulas

$$Bias_{\varphi}(n) = \frac{1}{1000}\sum_{k=1}^{1000}(\hat{\varphi}_k - \varphi). \quad (37)$$

$$MSE_{\varphi}(n) = \frac{1}{1000}\sum_{k=1}^{1000}(\hat{\varphi}_k - \varphi)^2. \quad (38)$$

$$RMSE_{\varphi}(n) = \sqrt{\frac{1}{1000}\sum_{k=1}^{1000}(\hat{\varphi}_k - \varphi)^2}. \quad (39)$$

Where $\varphi = (\alpha, \beta, \lambda)$ and $\hat{\varphi}_k = \left(\hat{\alpha}_k, \hat{\beta}_k, \hat{\lambda}_k\right)$ for $k = 1, 2, \cdots, 1000$.

Furthermore, R software has been used to investigate the goodness of estimates. Tables (1) and (2) are listed the observed outcomes and illustrate the MSE, RMSE, and AB values of the parameters (Ps) for various sample sizes.

According to Tables (1) and (2), the mean square error and bias for the MLEs, decrease as sample sizes rise such that MSE and all bias for all parameters approaches zero, this demonstrates the precision of estimation methodologies and meets the standard criteria for convergent qualities for MLEs.

The results indicate that the estimates are nearer with the actual values for various samples, confirming the efficiency of MLEs in estimating parameters.

Table 1: MLEs, MSEs and average biases for APPA distribution' simulation when (α = 1.1, β = 0.3, λ = 0.02).

n	Ps	MLEs	MSEs	RMSE	AB
50	α	2.272	0.369	0.608	0.368999
	β	0.329	0.0016	0.0410	0.0080
	λ	0.75	16.84	4.103668	0.5225
100	α	2.28	0.2534	0.503	0.2344
	β	0.368	0.0006	0.026	0.003
	λ	0.416	16.548	4.067	0.27134
200	α	1.72	0.208	0.4561	0.174
	β	0.302	0.0004	0.0208	0.00037
	λ	0.218	1.479	1.216	0.1184
400	α	1.42	0.127	0.357	0.079
	β	0.324	0.00028	0.016	- 0.001203
	λ	0.045	0.01164	0.107	0.0392
500	α	1.098	0.1138	0.337	0.0729
	β	0.307	0.00023	0.015	- 0.00133
	λ	0.012	0.0076	0.087	0.0341
800	α	1.2	0.069	0.264	0.0485
	β	0.31	0.00014	0.012	- 0.00182
	λ	0.20	0.003	0.055	0.0222
1000	α	1.1	0.062	0.25	0.0405
	β	0.29	0.00013	0.011	- 0.0006
	λ	0.02	0.0024	0.049	0.0169

Table 2: MLEs, MSEs and average biases for APPA distribution' simulation when (α = 0.07, β = 0.55, λ = 0.009).

n	Ps	MLEs	MSEs	RMSE	AB
50	α β λ	0.133 0.57 0.491	0.00145 0.00569 0.0303	0.038132 0.075 0.1741	0.01625 0.0251 0.0883
100	α β λ	0.06 0.56 0.02	0.00087 0.0028 0.0182	0.0295 0.0533 0.135	0.0066 0.0073 0.0385
200	α β λ	0.068 0.555 0.012	0.00042 0.0016 0.0066	0.021 0.0404 0.0814	0.0059 0.0033 0.0347
400	α β λ	0.07 0.561 0.01	0.003 0.0011 0.004	0.017 0.033 0.0636	0.0047 0.001 0.0268
500	α β λ	0.0724 0.5460 0.0096	0.00023 0.00098 0.0025	0.0151 0.031 0.05	0.00359 0.00098 0.021
700	α β λ	0.070 0.53 0.0096	0.00018 0.00081 0.0019	0.0134 0.028 0.0445	0.003 0.0007 0.017
1000	α β λ	0.75 0.547 0.013	0.00011 0.00056 0.00098	0.0106 0.023 0.031	0.0018 0.00058 0.0112

6 Data Analysis

Four real data sets have been analyzed to demonstrate the performance of the APPA distribution. The proposed distribution is fitted to more important fields of survival times of COVID-19 data with two countries including Italy and United Kingdom. In addition, the vinyl chloride data and fourth data for the March precipitation in Minneapolis/St Paul. Several criteria are considered to compare the efficiency of the APPA distribution such as Kolmogorov Smirnov (K-S) test and its p-value, Cramér–von (W*) Mises distance values, Anderson-Darling (A*), Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The required numerical estimates are generated by using the Package of R software. The data sets are given as

Data Set 1: Next COVID-19 data are from Italy and cover 111 days between 1 April and 20 July 2020, as suggested in [22, 23]. This data created by dividing daily new deaths on daily new cases.

0.2070, 0.1520, 0.1628, 0.1666, 0.1417, 0.1221, 0.1767, 0.1987, 0.1408, 0.1456, 0.1443, 0.1319, 0.1053, 0.1789, 0.2032, 0.2167, 0.1387, 0.1646, 0.1375, 0.1421, 0.2012, 0.1957, 0.1297, 0.1754, 0.1390, 0.1761, 0.1119, 0.1915, 0.1827, 0.1548, 0.1522, 0.1369, 0.2495, 0.1253, 0.1597, 0.2195, 0.2555, 0.1956, 0.1831, 0.1791, 0.2057, 0.2406, 0.1227, 0.2196, 0.2641, 0.3067, 0.1749, 0.2148, 0.2195, 0.1993, 0.2421, 0.2430, 0.1994, 0.1779, 0.0942, 0.3067, 0.1965, 0.2003, 0.1180, 0.1686, 0.2668, 0.2113, 0.3371, 0.1730, 0.2212, 0.4972, 0.1641, 0.2667, 0.2690, 0.2321, 0.2792, 0.3515, 0.1398, 0.3436, 0.2254, 0.1302, 0.0864, 0.1619, 0.1311, 0.1994, 0.3176, 0.1856, 0.1071, 0.1041, 0.1593, 0.0537, 0.1149, 0.1176, 0.0457, 0.1264, 0.0476, 0.1620, 0.1154, 0.1493, 0.0673, 0.0894, 0.0365, 0.0385, 0.2190, 0.0777, 0.0561, 0.0435, 0.0372, 0.0385, 0.0769, 0.1491, 0.0802, 0.0870, 0.0476, 0.0562, 0.0138.

Data Set 2: The United Kingdom's COVID-19 [24]. This data covers 82 days, from May 1 to July 16, 2021, and constructed by using daily new deaths (ND), daily cumulative deaths (CD), and daily cumulative cases (CC) as:

$$x_i = \frac{ND_i}{CC_i - CD_{i-1}} \times 1000. \quad (40)$$

0.0023, 0.0023, 0.0023, 0.0046, 0.0065, 0.0067, 0.0069, 0.0069, 0.0091, 0.0093, 0.0093, 0.0093, 0.0111, 0.0115, 0.0116, 0.0116, 0.0119, 0.0133, 0.0136, 0.0138, 0.0138, 0.0159, 0.0161, 0.0162, 0.0162, 0.0162, 0.0163, 0.0180, 0.0187, 0.0202, 0.0207, 0.0208, 0.0225, 0.0230, 0.0230, 0.0239, 0.0245, 0.0251, 0.0255, 0.0255, 0.0271, 0.0275, 0.0295, 0.0297, 0.0300, 0.0302, 0.0312, 0.0314, 0.0326, 0.0346, 0.0349, 0.0350, 0.0355, 0.0379, 0.0384, 0.0394, 0.0394, 0.0412, 0.0419, 0.0425, 0.0461, 0.0464, 0.0468, 0.0471, 0.0495, 0.0501, 0.0521, 0.0571, 0.0588, 0.0597, 0.0628, 0.0679, 0.0685, 0.0715, 0.0766, 0.0780, 0.0942, 0.0960, 0.0988, 0.1223, 0.1343, and 0.1781.

The descriptive statistics for COVID-19 data in both Italy and United Kingdom are presented in Table (3).

Table 3: Descriptive statistics for APPA distribution for two COVID-19 data in Italy and United Kingdom.

COVID-19 Data	Median	Mean	Skewness	Kurtosis
Italy	0.1628	0.1668	0.762351	1.812914
United Kingdom	0.02730	0.03571	1.984259	4.97806

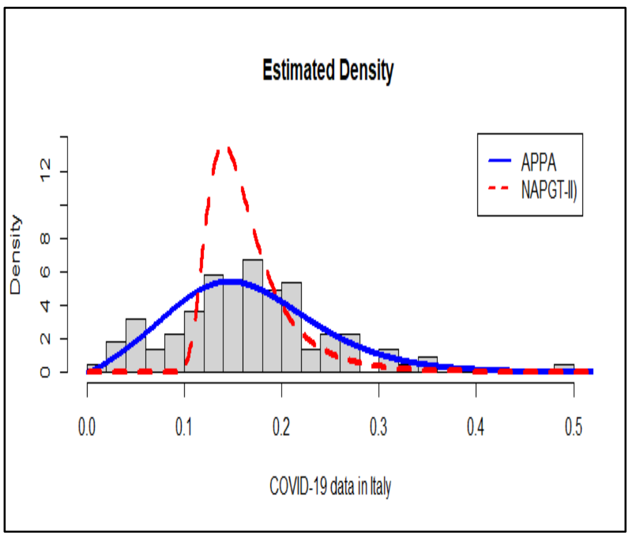
For data set 1, the MLEs of the parameters with standard errors, K-S, P-value, W* and A* are computed and displayed in Tables (4) and (5). The APPA distribution is compared with other some competitive models as, Kumaraswamy inverted Topp–Leone **(**KITL), inverted Topp–Leone (ITL), inverse Weibull (IW), inverse Lomax (IL), inverse Kumaraswamy (IK), Topp Leone inverted Kumaraswamy (TLIK), novel alpha power Gumbel Type II (NAPGT-II), New Alpha Power Exponential (NAPE), Exponentiated Gumbel Type-II (EGT-II), Weibull and Gumbel Type-Two (GT-II) distributions [25, 26, 27, 28]. Fig. (8) shows the plots of the estimated density and the empirical CDF and Fig. (9) shows the P-P plot and the Q-Q plot of the APPA distribution for COVID-19 data in Italy. It is clear from Table (5), Fig. (8) and Fig. (9) that APPA model is the best for modeling COVID-19 data in Italy where it has the greatest P-value and the least K-S, W* and A* values. Furthermore, total time on test transform (TTT) is presented in Fig. (10), which takes convex shape followed by a concave shape. This corresponds to a bathtub shaped HF. Also, the boxplot of the APPA model in Italy is displayed in Fig. (10).

Table 4: MLEs of parameters and standard errors for COVID-19 data in Italy.

Italy	MLEs	Standard Errors
APPA	$\alpha = 12.03915$ $\beta = 1.19744$ $\lambda = 10.89357$	2.27006 0.17388 12.69443
ITL	$\upsilon = 43.6078$	4.1391
KIT	$\delta = 1.3430$ $\vartheta = 20.4473$ $\upsilon = 4.4464$	0.1180 28.9206 5.1612
IW	$\vartheta = 1.3507$ $\upsilon = 0.0483$	0.0818 0.0115
IL	$\delta = 17.7970$ $\vartheta = 0.0069$	7.1991 0.0029
IK	$\delta = 14.6443$ $\vartheta = 4.8909$	1.2584 0.7972
TLIK	$\delta = 30.0526$ $\vartheta = 1.3699$ $\upsilon = 1.8741$	8.8631 0.4298 0.2976
NAPE	$\lambda = 135.5089$ $\xi = 13.2114$	79.5693 1.3148
NAPGT-II	$\lambda = 0.0063$ $\beta = 5.2431$ $\delta = 1.0064$	0.0043 0.4208 0.0044
EGT-II	$\xi = 6.4945$ $\beta = 2.4718$ $\kappa = 0.0238$	4.0462 0.5118 0.0282
Weibull	$\theta = 5.2152$ $\kappa = 0.1750$	0.6407 0.0060
GT-II	$\beta = 0.0025$ $\kappa = 3.1696$	0.0006 0.1391

Table (5): Goodness-of-fit measures and K-S statistic with P-value of the APPA distribution and other competing distributions for COVID-19 data in Italy.

Italy	K-S	W*	A*	P-value
APPA	0.05	0.0715	0.54148	0.96
ITL	0.1560	0.179	1.079	_
Weibull	0.1102	_	_	0.7480
KIT	0.0715	0.135	0.831	_
NAPE	0.2788	_	_	0.0068
IW	0.1907	1.324	7.127	_
NAPGT-II	0.0889	_	_	0.9217
IL	0.2922	0.986	5.442	_
EGT-II	0.9891	_	_	0.0000
IK	0.1202	0.398	2.330	_
GT-II	0.2537	_	_	0.0179
TLIK	0.0740	0.142	0.870	_



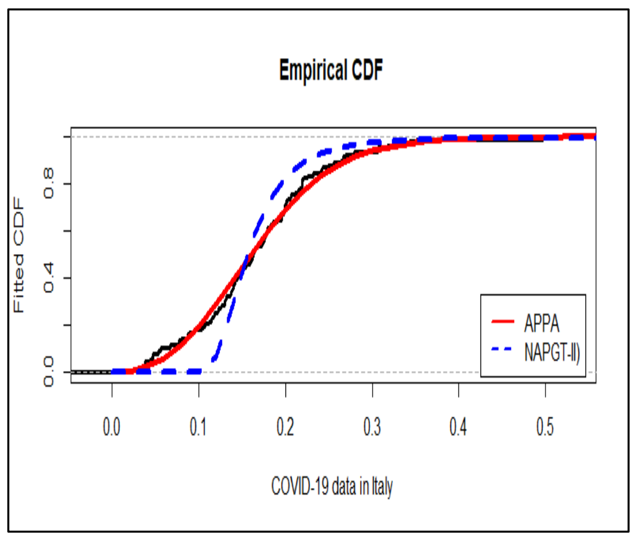


Fig. 8: Illustrate the estimated density and the empirical CDF of the APPA distribution for COVID-19 data in Italy.

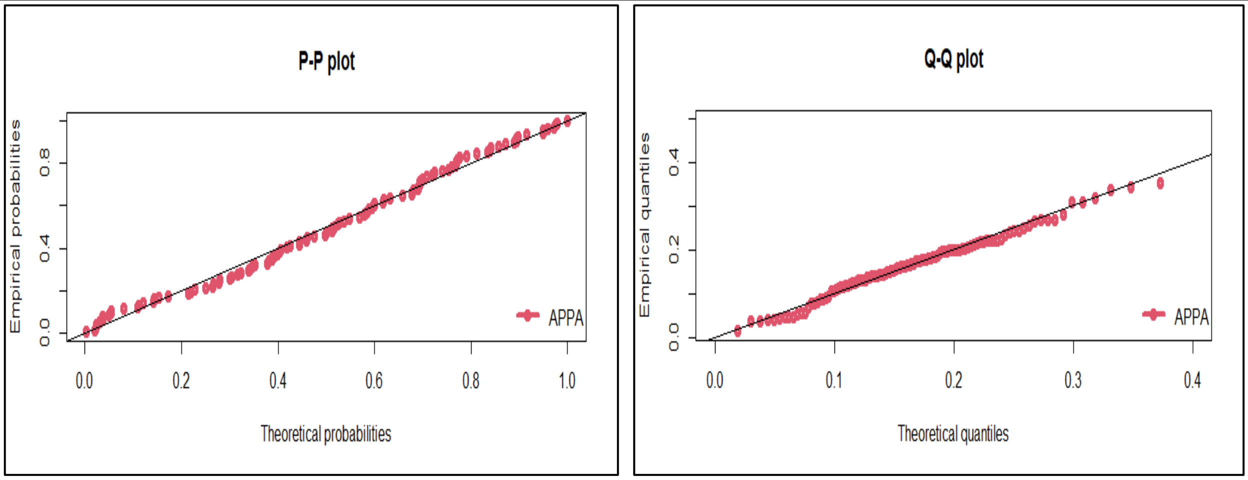


Fig. 9: Illustrate the P-P plot and the Q-Q plot of the APPA distribution' COVID-19 Italian data.

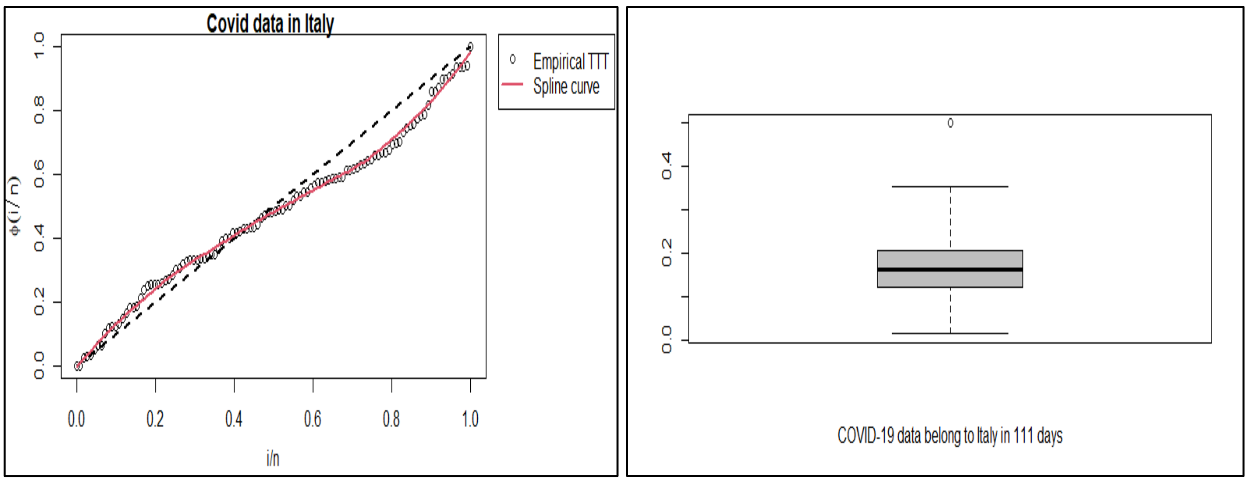


Fig. 10: Illustrate the empirical TTT and the boxplot of the APPA distribution for COVID-19 data in Italy.

For data set 2, Table (6) provides the APPA distribution comparison with other competitive distributions as, Transmuted Generalized Lomax (TGL), Burr-XII (KEBXII) with Kumaraswamy exponentiated [29], Weibull-Lomax (WL), Odds Exponential- Pareto IV (OEPIV) [30], Marshall–Olkin Alpha power Weibull (MOAPW) [31], Marshall–Olkin Alpha power extended Weibull (MOAPEW) [32], and Gompertz Lomax (GOLOM) distribution [33]. Fig. (11) shows the plots of the estimated density and the empirical CDF and Fig. (12) shows the P-P plot and the Q-Q plot of the APPA distribution for COVID-19 data of the United Kingdom. It is clear from Table (6), Fig. (11) and Fig. (12) that APPA model is the most effective model for fitting COVID-19 data of the United Kingdom where it has the greatest P-value and the lowest values of K-S, W* and A*. Furthermore, total time on test transform (TTT) is presented in Fig. (13), which takes convex shape followed by a concave shape. This corresponds to a bathtub shaped HF. Also, Fig. (13) displays the boxplot of the APPA model in the United Kingdom. Finally, APPA model is the most appropriate model for fitting the two real datasets.

Table 6: The numerical results of the APPA distribution and other rival distributions using United Kingdom COVID-19 data.

United Kingdom	K-S	P-value	W*	A*
APPA	0.044608	0.9968	0.012247	0.1080
TGL	0.0579	0.9313	0.0520	0.3666
KEBXII	0.0686	0.8093	0.0574	0.4058
WL	0.0589	0.9221	0.0573	0.4038
OEPIV	0.0600	0.9120	0.0557	0.3935
MOAPW	0.0620	0.8912	0.0614	0.4256
MOAPEW	0.0643	0.8646	0.0562	0.3930
GOLOM	0.1003	0.3574	0.0719	0.4967

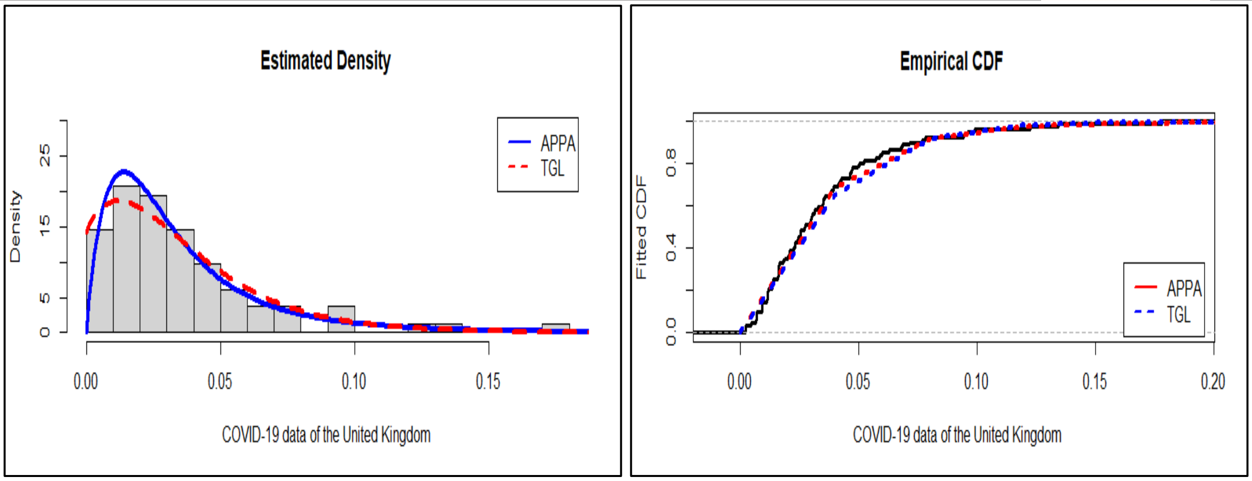


Fig. 11: Illustrate the estimated density and the empirical CDF of the APPA distribution for COVID-19 data of the United Kingdom.

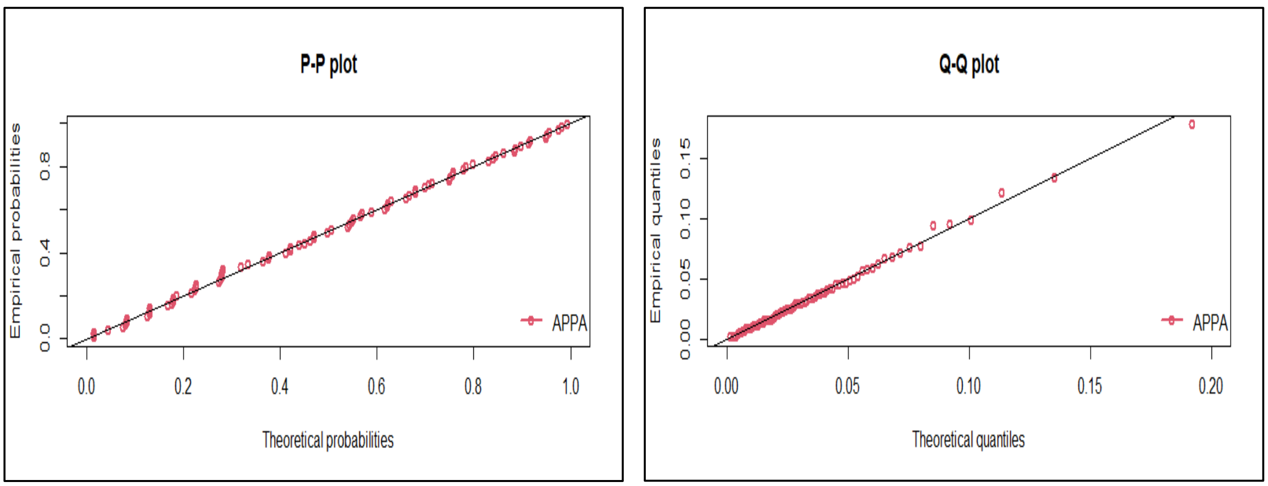


Fig. 12: Illustrate P-P plot and Q-Q plot of the APPA distribution for the United Kingdom's COVID-19 data.

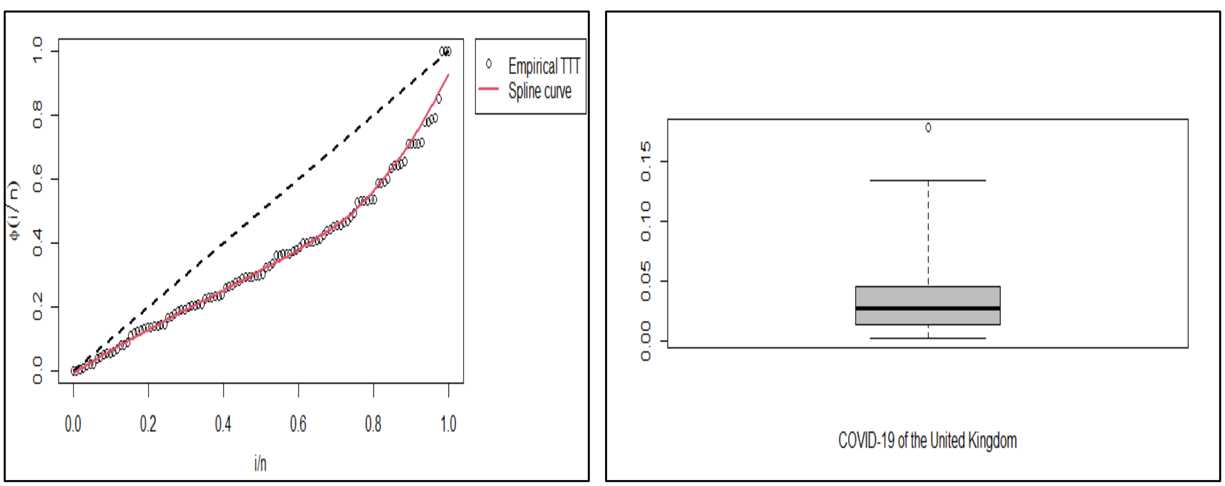


Fig. (13): Illustrate the empirical TTT and the boxplot of the APPA distribution for the United Kingdom's COVID-19 data.

Data Set 3: According to [34, 35], the following data includes 34 observations of vinyl chloride data (in mg/L) acquired from clean-up gradient groundwater monitoring wells.

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3,1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

The MLEs of the parameters with standard errors, K-S, P-value, W*, A*, AIC and BIC are computed and displayed in Tables (7) and (8). The APPA distribution is compared with other some competitive models as, generalized Burr XII (GBXII), BXII, odd exponential logarithmic Weibull (OELW), Beta BXII (BBXII), log-logistic Weibull (LLoGW) distributions. Fig. (14) shows the plots of the estimated density and the empirical CDF and Fig. (15) shows the P-P plot and the Q-Q plot of the APPA distribution for vinyl chloride data. It is clear from Table (8), Fig. (14) and Fig. (15) that APPA model is the best for modeling vinyl chloride data where it has the greatest P-value and the least K-S, W*, A*, AIC and BIC values. Furthermore, total time on test transform (TTT) is presented in Fig. (16), which takes convex shape followed by a concave shape. This corresponds to a bathtub shaped HF. Also, the boxplot of the APPA model is displayed in Fig. (16).

Data Set 4: Next data indicate 30 observations of March precipitation in Minneapolis/St Paul (measured in inches), as described in [35, 36].

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Tables (9) and (10) display the MLEs of the parameters with standard errors, K-S, P-value, W*, A*, AIC, and BIC. The APPA distribution is compared with other models as, generalized Burr XII (GBXII), BXII, Nakagami Weibull (NW), Beta BXII (BBXII), Exponentiated Weibull Weibull (EWW) distributions. Fig. (17) shows the plots of the estimated density and the empirical CDF and Fig. (18) shows the P-P plot and the Q-Q plot of the APPA distribution for the March precipitation in Minneapolis/St Paul. It is clear from Table (10), Fig. (17) and Fig. (18) that APPA model is the best for modeling fourth data where it has the greatest P-value and the least K-S, W*, A*, AIC and BIC values. Furthermore, total time on test transform (TTT) is presented in Fig. (19), which takes convex shape followed by a concave shape. This corresponds to a bathtub shaped HF. Also, the boxplot of the APPA model is displayed in Fig. (19).

Table 7: MLEs of parameters and standard errors for the vinyl chloride data.

Models	MLEs	Standard Errors
APPA	$\alpha = 0.58971$ $\beta = 0.71987$ $\lambda = 0.40942$	0.27493 0.10893 0.82015
GBXII	$\alpha = 1.3204$ $c = 0.8542$ $k = 2.7298$	1.9609 0.6688 2.4948
BXII	$c = 1.5621$ $k = 0.9305$	0.2479 0.1791
OELW	$a = 1.9409$ $b = 8.7259$ $p = 0.0023$ $\beta = 2.0977$	_ _ _ _
BBXII	$a = 3.5874$ $b = 13.211$ $c = 0.5429$ $k = 1.4257$ $s = 30.417$	7.0388 72.677 0.5453 7.4133 140.35
LLoGW	$s = 9.7787$ $c = 5.0155$ $\beta = 0.9910$ $\alpha = 0.5270$	_ _ _ _

Table (8): Goodness-of-fit measures and K-S statistic with P-value of the APPA distribution and other competing distributions for the vinyl chloride data.

Models	K-S	P-value	W*	A*	AIC	BIC
APPA	0.077733	0.9864	0.028745	0.18981	115.8297	120.4087
GBXII	0.089	_	0.034	0.248	117.11	121.69
BXII	0.104	_	0.045	0.302	118.15	122.20
OELW	0.0785	0.9849	0.0289	0.2002	116.4218	122.5273
BBXII	0.083	_	0.037	0.250	119.88	127.51
LLoGW	0.0931	0.9301	0.0438	0.2881	118.708	124.8134

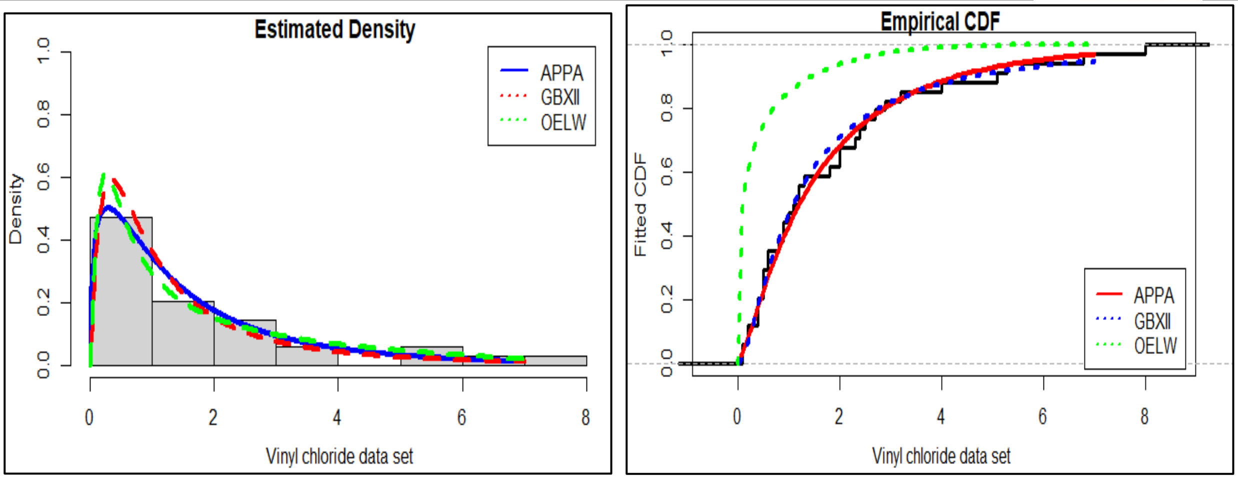


Fig. 14: Illustrate the estimated density and the empirical CDF of the APPA distribution for the vinyl chloride data.

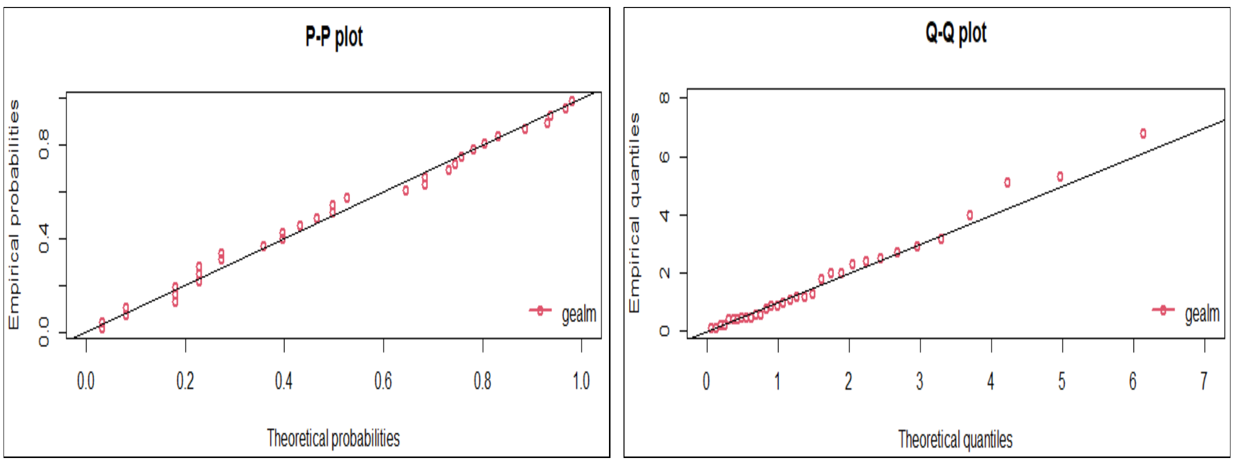


Fig. 15: Illustrate the P-P plot and the Q-Q plot of the APPA distribution' vinyl chloride data.

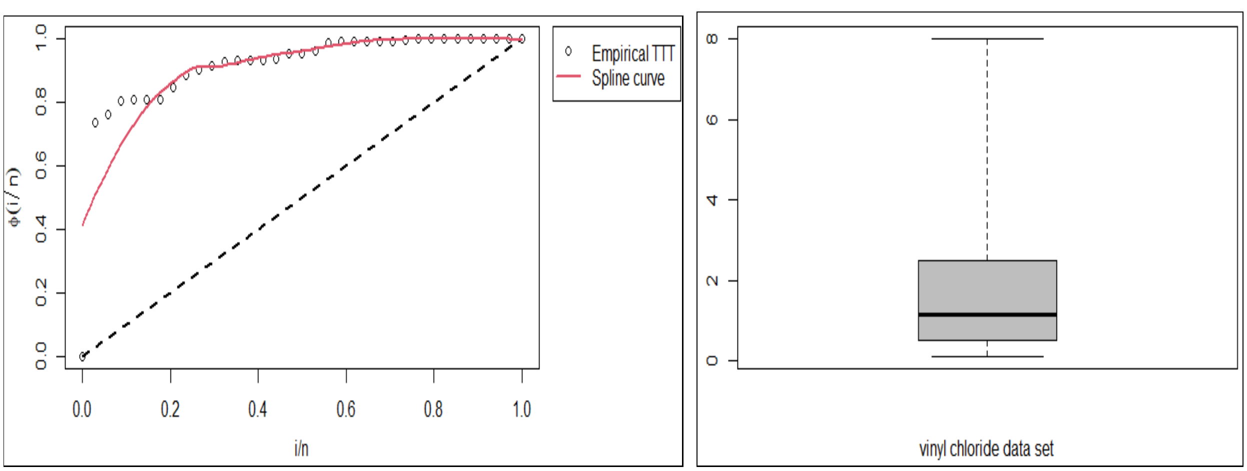


Fig. 16: Illustrate the empirical TTT and the boxplot of the APPA distribution for vinyl chloride data.

Table 9: MLEs of parameters and standard errors using fourth data.

Models	MLEs	Standard Errors
APPA	$\alpha = 0.39057$ $\beta = 1.30558$ $\lambda = 0.35102$	0.27685 0.22248 1.01011
GBXII	$\alpha = 1.8505$ $c = 1.4092$ $k = 2.5352$	2.5782 1.0352 2.2729
BXII	$c = 3.2555$ $k = 0.5769$	0.6455 0.1371
NW	$\lambda = 1.5347926$ $\beta = 0.5178431$ $\delta = 0.3999663$ $\alpha = 0.5250054$	_ _ _ _
BBXII	$a = 3.2965$ $b = 7.9098$ $c = 0.9976$ $k = 3.3293$ $s = 14.163$	6.0155 49.794 1.0339 20.744 69.095
EWW	$\lambda = 1.5264454$ $\beta = 0.6836198$ $\delta = 1.3895067$ $\alpha = 0.6091976$ $\theta = 0.6701214$	_ _ _ _ _

Table 10: Goodness-of-fit measures and K-S statistic with P-value of the APPA distribution and other competing distributions based on fourth data

Models	K-S	P-value	W*	A*	AIC	BIC
APPA	0.05861	1	0.014073	0.10542	82.25	86.454
GBXII	0.103	_	0.044	0.273	84.11	88.31
BXII	0.138	_	0.102	0.531	84.52	88.34
NW	0.066	0.999	_	0.170	85.493	91.098
BBXII	0.102	_	0.043	0.271	86.19	93.19
EWW	0.1023	0.911	_	0.235	88.412	95.418

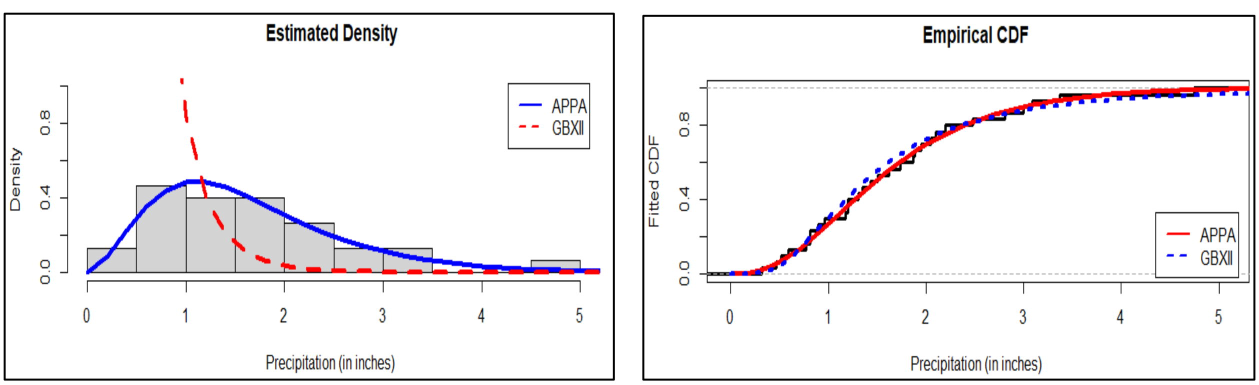


Fig. 17: Illustrate the estimated density and the empirical CDF of the APPA distribution for fourth data.

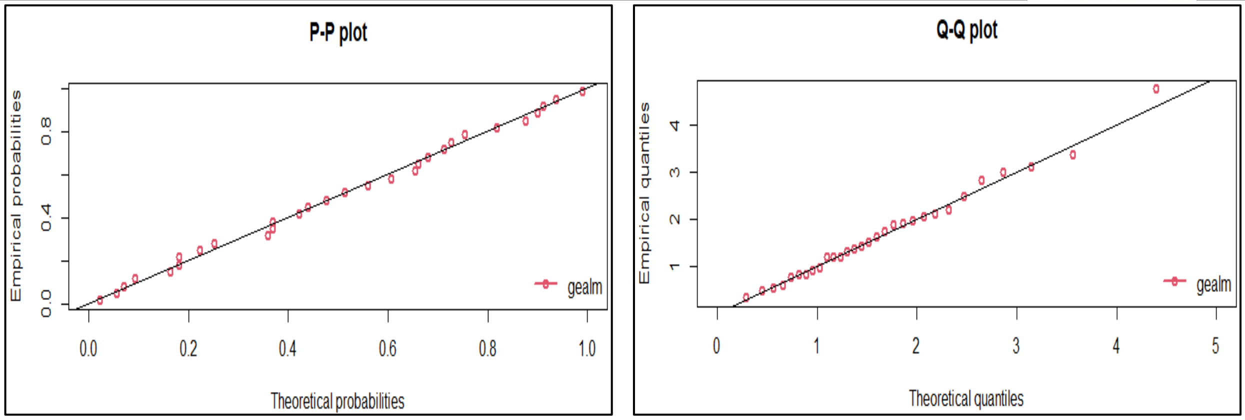


Fig. 18: Illustrate the P-P plot and the Q-Q plot of the APPA distribution' fourth data.

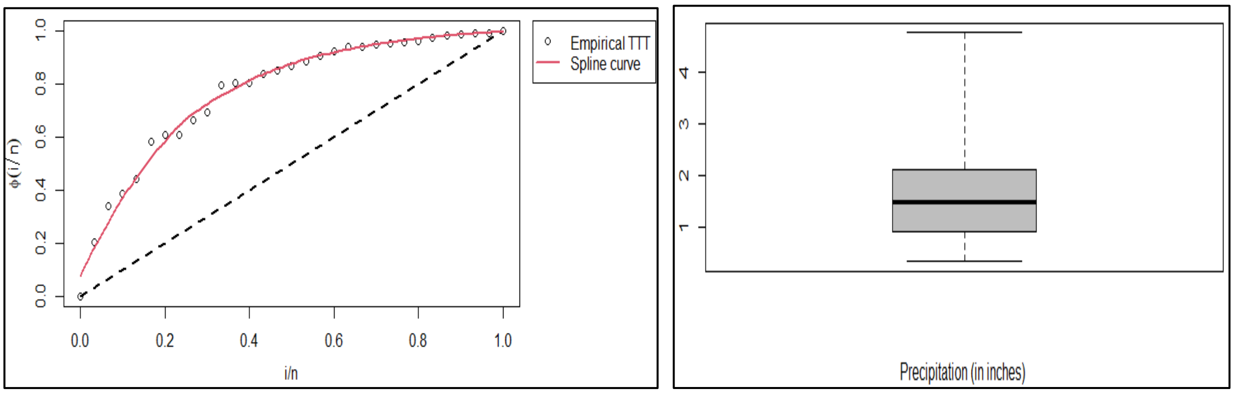


Fig. 19: Illustrate the empirical TTT and the boxplot of the APPA distribution for fourth data.

7 Conclusions

In this paper, a new three- parameters distribution, Alpha Power of the Power Ailamujia (APPA) distribution, is introduced using alpha power transformation. The log-concavity, moments, quantile function, moment generating function, median, mode, characteristic function, Shanon's entropy, order statistics, and other reliability measures are among the statistical qualities of the distribution that are derived and discussed. The method of maximum likelihood estimation is used for determining the unknown parameters of the newly proposed distribution. The performance of the APPA distribution is determined by fitting it to four real-life datasets using several criteria. In comparison to other well-known distributions, the APPA distribution offers a best fitting to the data, yielding the least K-S, W*, A*, AIC and BIC values and the greatest P-value of all fitted models. Finally, the proposed distribution is suitable for most of the lifetime data.

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

References

[1] S. Dey, V. K. Sharma and M. Mesfioui, A new extension of Weibull distribution with application to lifetime data, Annals of Data Science,**4(1)**, 31–61 (2017).

[2] H. Q. Lv, L. H. Gao and C. L. Chen, Эрланга distribution and its application in supportability data analysis, Journal of Academy of Armored force Engineering,**16(3)**, 48-52 (2002).

[3] G. S. Mudholkar and D. K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, IEEE transactions on reliability,**42(2)**, 299–302 (1993).

[4] A. W. Marshall and I. Olkin, A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, Biometrika,**84(3)**, 641–52 (1997).

[5] A. A. Rather, C. Subramanian, S. Shafi, k. A. Malik, P. J. Ahmad, B. A. Para and T. R. Jan, A new size biased distribution with applications in engineering and medical science, IJSRMSS,**5(4)**, 66-76 (2018).

[6] C. Lee, F. Famoye and A. Y. Alzaatreh, Methods for generating families of univariate continuous distributions in the recent decades, Wiley Interdisciplinary Reviews: Computational Statistics,**5(3)**, 219–38 (2013).

[7] A. Aijaz, A. Ahmad and R. Tripathi, Inverse analogue of Ailamujia distribution with statistical properties and applications, Asian Research Journal of Mathematics,**16(9)**, 36-46 (2020).

[8] A. Aijaz, S. Qurat ul Ain, A. Ahmad and R. Tripathi, Bayesian estimation of inverse Ailamujia distribution using different loss functions, Journal of Xi'an University of Architecture & Technology, 226-235 (2020).

[9] A. Alzaatreh, C. Lee and F. Famoye, A new method for generating families of continuous distributions, Metron,**71(1)**, 63–79 (2013).

[10] M. Mohiuddin and R. Kannan, Alpha power transformed Aradhana distributions, its properties and applications, Indian Journal of Science and Technology,**14(30)**, 2483–2493 (2021).

[11] N. J. Barinaandaa, E. D. Isaac and A. Emeka, Nwikpe probability distribution, statistical properties and goodness of fit, Asian Journal of Probability and Statistics, (2021).

[12] A. Mahdavi and D. Kundu, A new method for generating distributions with an application to exponential distribution, Communications in Statistics-Theory and Methods,**46(13)**, 6543–57 (2017).

[13] M. Nassar, A. Alzaatreh, M. Mead and O. Abo-Kasem, Alpha power Weibull distribution: properties and applications, Communications in Statistics-Theory and Methods,**46(20)**, 10236–52 (2017).

[14] S. Dey, A. Alzaatreh, C. Zhang and D. Kumar, A new extension of generalized exponential distribution with application to Ozone data, Ozone: Science & Engineering,**39(4)**, 273–85 (2017).

[15] A. S. Hassan, R. E. Mohamd, M. Elgarhy and A. Fayomi, Alpha power transformed extended exponential distribution: properties and applications, Journal of Nonlinear Sciences and Applications,**12(4)**, 62– 67 (2018).

[16] S. Dey, I. Ghosh and D. Kumar, Alpha-power transformed Lindley distribution: properties and associated inference with application to earthquake data, Annals of Data Science, 1–28 (2018).

[17] S. Dey, M. Nassar and D. Kumar, Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function, Journal of Computational and Applied Mathematics,**348**, 130–45 (2019).

[18] S. Qurat Ul Ain, A. Aijaz and R. Tripathi, A new two parameter Ailamujia distribution with applications in bio-medicine, Journal of Xi'an University of Architecture & Technology,**XII(XI)**, 592-604 (2020).

[19] F. Jamal, C. Chesneau, k. Aidi and A. Ali, Theory and application of the power Ailamujia distribution, Journal of Mathematical Modeling,**9(3)**, 391-413 (2021).

[20] G. Casella and R. L. Berger, Statistical Inference, Brooks/Cole Publishing Company, California, (1990).

[21] S. Abbas, M. Mohsin and J. Pilz, A new life time distribution with applications in reliability and environmental science, J. Stat. Manag. Syst.,**24(3)**, 453- 479 (2021).

[22] A. S. Hassan, E. M. Almetwally and G. M. Ibrahim, Kumaraswamy inverted Topp–Leone distribution with applications to COVID-19 data, J. Computers, Materials & Continua,**68(1)**, 337- 358 (2021).

[23] S. A. Lone, T. N. Sindhu, A. Shafiq and F. Jarad, A novel extended Gumbel type II model with statistical inference and Covid-19 applications, J. Results in Physics,**35**, 1- 12 (2022).

[24] W. S. Abu El Azm, E. M. Almetwally, S. N. AL-Aziz, A. A. H. El-Bagoury, R. Alharbi and O. E. Abo-Kasem, A New transmuted generalized Lomax distribution: properties and applications to COVID-19 data, Computational Intelligence and Neuroscience, 14 pages, (2021).

[25] H. Reyad, F. Jamal, S. Othman and N. Yahia, The Topp Leone generalized inverted Kumaraswamy distribution: properties and applications, Asian Research Journal of Mathematics,**13(1)**, 1–15 (2019).

[26] M. Ijaz, W. K. Mashwani, A. Göktaş and Y. A. Unvan, A novel alpha power transformed exponential distribution with real-life applications, J. Appl. Stat.,**48(11)**, I–XVI (2021).

[27] I. E. Okorie, A. C. Akpanta and J. Ohakwe, The exponentiated Gumbel type-2 distribution: properties and application,

Int. J. Math. Sci., 10 pages, (2016).

[28] A. A. Ogunde, S. T. Fayose, B. Ajayi and D. O. Omosigho, Extended Gumbel type-2 distribution: properties and applications, J. Appl. Math., 11 pages, (2020).

[29] M. E. Mead and A. Z. Afify, On five-parameter Burr XII distribution: properties and applications, South African Statistical Journal,**15 (1)**, 67–81 (2017).

[30] L. A. Baharith, K. M. AL-Beladi and H. S. Klakattawi, The odds exponential-Pareto IV distribution: regression model and application, Entropy,**22(5)**, p. 497, (2020).

[31] E. M. Almetwally, R. Alharbi, D. Alnagar and E. H. Hafez, A new inverted Topp-Leone distribution: applications to the COVID- 19 mortality rate in two different countries, Axioms,**10(1)**, p. 25, (2021).

[32] E. M. Almetwally, M. A. Sabry, R. Alharbi, D. Alnagar, S. A. Mubarak and E. H. Hafez, Marshall–Olkin alpha power Weibull distribution: different methods of estimation based on type-I and type-II censoring, Complexity, 18 pages, (2021).

[33] P. E. Oguntunde, M. A. Khaleel, M. T. Ahmed, A. O. Adejumo and O. A. Odetunmibi, A new generalization of the Lomax distribution with increasing, decreasing, and constant failure rate, Modelling and Simulation in Engineering, 6 pages, (2017).

[34] C. Chesneau, L. Tomy, M. Jose and K. V. Jayamol, Odd exponential-logarithmic family of distributions: features and modeling, Math. Comput. Appl., 24 pages, (2022).

[35] M. Aslam, R. M. Usman and M. Z. Raqab, A new generalized Burr XII distribution with real life applications, researchgate, 23 pages, (2021).

[36] I. Abdullahi and O. Job, The Nakagami–Weibull distribution in modeling real-life data, Cumhuriyet Sci. J.,**42(2)**, 422-433 (2021).