

Some Iterative Schemes for Solving Extended General Quasi Variational Inequalities

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Abstract: In this paper, we consider a new class of quasi variational inequalities involving three operators, which is called the *extended general quasi variational inequality*. It is shown that the extended general quasi variational inequalities are equivalent to the fixed problems. This equivalence is used to suggest and analyze some iterative methods for solving the extended general quasi variational inequalities. Convergence analysis is also considered. We have also shown that the extended general quasi variational inequalities are equivalent to the extended general implicit Wiener-Hopf equations. This alternative formulation is used to suggest and analyze some iterative methods. The convergence analysis of these new methods under some suitable conditions is investigated. Several special cases are discussed. Since the extended general quasi variational inequalities include general variational inequalities, quasi variational inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems. Results of this paper may stimulate further research in this fascinating area.

Keywords: Variational inclusions, fixed point problems, Wiener-Hopf equations, nonlinear operators, convergence criteria.

1 Introduction

Variational inequalities theory can be viewed as a natural generalization and extension of the variational principles, the origin of which can be traced back to Fermat, Newton, Euler and Lagrange. Variational inequalities contain a wealth of new and novel ideas with a wide class of applications in all areas of pure and applied sciences. Variational inequalities have played significant and fundamental part as a unifying influence and as a guide in the mathematical interpretation of many physical phenomena. It is well known that the optimality conditions of the minimum of a differentiable convex function can be characterized by the variational inequalities. This interplay between variational inequalities and optimization has been used to develop several numerical techniques for solving variational inequalities and nonlinear optimization problems. In recent years, considerable interest have been shown in developing various extensions and generalizations of variational inequalities, both for their own sake and for their applications. There are significant developments of these problems related to nonconvex optimization, iterative method and structural analysis. It is well-known

that, if the convex set depends upon the solution explicitly or implicitly, then the variational inequality is called the quasi variational inequality. Benssousan and Lions [2] have shown that a class of impulse control problems can be formulated as a quasi variational inequality problem. Variational and quasivariational inequalities theory with their applications to mathematical physics, pure and applied sciences provides us with a simple, natural, efficient and unified frame work to study a wide class of unrelated problems. This theory combines the theory of extremal problems and monotone operators under a unified view point For recent work on the quasi variational inequalities and their applications, see [1-42].

In recent years, Noor [19-23] has shown that the minimum of differentiable nonconvex functions on the nonconvex set can be characterized by a class of variational inequalities. This has motivated Noor [19-23] to introduce and consider a new class of variational inequalities, which is called the extended general variational inequalities. For the applications and numerical methods for solving the extended general variational inequalities, see [18-24, 27-2934] and the references therein. Noor et al [32] have introduced and studied a new class of variational inequalities, which is

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called the extended general quasi variational inequalities. It has been shown that the extended general quasi variational inequalities are equivalent to the fixed point problem. This fixed point formulation is used to study the existence of a solution of the extended general quasi variational inequalities. One of the most difficult and important problems in variational inequalities is the development of an efficient numerical methods. Several numerical methods have been developed for solving the variational inequalities and their variant forms. One of the technique is called the projection method and its variant forms. Projection methods represent an important tool for finding the approximate solution of various types of variational inequalities. The projection type methods were developed in 1970's. These methods have been extended and modified in various ways. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed point problem using the concept of the projection. This alternative equivalent formulation has played a significant role in the developments of various projection type methods for solving the variational inequalities and related optimization problems. Noor [8] proved that a class of quasi variational inequalities is equivalent to the fixed-point problem using the projection technique. This equivalent formulation has been used to develop iterative methods for solving the quasi variational inequality and its various variant forms, see [8,9-28] and the references therein. Using the projection method, it has been shown that the extended general quasi variational inequalities are equivalent to the implicit fixed point problem. We use this alternative equivalent formulation to suggest and analyze some iterative methods for solving the extended general quasi variational inequalities. We also discuss the convergence of these iterative methods.

Projection iterative methods have been modified and generalized in several directions using various techniques. Shi [40] considered the problem of solving a system of nonlinear projections, which are called the Wiener-Hopf equations. It has been shown by Shi [40] that the Wiener-Hopf equations are equivalent to the variational inequalities. It turns out that this alternative formulation is more general and flexible. It has been shown that the Wiener-Hopf equations provide us a simple, natural, elegant and convenient device to develop some efficient numerical methods for solving variational and complementarity problems, see [6, 8,9 13,15,16-18] and the references therein. Essentially using the projection technique, we prove that the extended general quasi variational inequalities are equivalent to nonlinear implicit projection equations and the implicit Wiener-Hopf equations. We use these equivalent formulations to suggest and analyze some projection iterative methods for solving the extended general quasi variational inequalities under suitable conditions. Since the extended general variational inequalities include several classes of (quasi) variational inequalities and related optimization problems as special cases, results

proved in this paper continue to hold for these problems. Results proved in this paper may be starting point for a wide range of further new and novel applications.

2 Preliminaries and Basic Results

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Let $K(u)$ be a nonempty closed convex-valued set in H .

For given three operators $T, g, h : H \rightarrow H$, consider the problem of finding $u \in H, h(u) \in K(u)$ such that

$$\begin{aligned} \langle \rho Tu + h(u) - g(u), g(v) - h(u) \rangle &\geq 0, \\ \forall v \in H : g(v) \in K(u), \end{aligned} \quad (1)$$

where $\rho > 0$ is a constant. Inequality of type (1) is called the *extended general quasi variational inequality involving three operators, introduced by Noor and Noor [32]*.

We now list some special cases of the extended general quasi variational inequality (1).

I. If $K(u) \equiv K$, the convex set in H , then (1) is equivalent to finding $u \in H : h(u) \in K$ such that

$$\begin{aligned} \langle \rho Tu + h(u) - g(u), g(v) - h(u) \rangle &\geq 0, \\ \forall v \in H : g(v) \in K, \end{aligned} \quad (2)$$

which is called the extended general variational inequality, introduced and studied by Noor [20].

II. If $g = h$, then problem (1) is equivalent to finding $u \in H : g(u) \in K(u)$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K(u), \quad (3)$$

which is known as general quasi variational inequality and appears to be a new one. If $g = h$ and $K(u) \equiv K$, then problem (3) is called the general variational inequality involving two operator, which was introduced and studied by Noor [9] in 1988. It turned out that odd order and nonsymmetric obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via general variational inequality (3), see [11-28].

To convey an idea of the applications of the general quasi variational inequality (3), we consider the third-order implicit obstacle boundary value problem of finding u such that

$$\left. \begin{aligned} -u''' &\geq f(x) && \text{on } \Omega = [0, 1] \\ u &\geq M(x, u) && \text{on } \Omega = [0, 1] \\ [-u''' - f(x)][u - M(x, u)] &= 0 && \text{on } \Omega = [0, 1] \\ u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0. \end{aligned} \right\} (4)$$

where $f(x)$ is a continuous function and $M(x, u(x))$ is the obstacle function. We study the problem (4) in the framework of variational inequality approach. To do so, we first define the set $K(u)$ as

$$K(u) = \{u : u \in H_0^2(\Omega) : u \geq M(x, u) \text{ on } \Omega\},$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev (Hilbert) space, see [1,5]. One can easily show that the energy functional associated with the problem (2.2) is

$$\begin{aligned}
 I[v] &= - \int_0^1 \left(\frac{d^3v}{dx^3} \right) \left(\frac{dv}{dx} \right) dx - 2 \int_0^1 f(x) \left(\frac{dv}{dx} \right) dx, \\
 &\quad \text{for all } \frac{dv}{dx} \in K(u) \\
 &= \int_0^1 \left(\frac{d^2v}{dx^2} \right)^2 dx - 2 \int_0^1 f(x) \left(\frac{dv}{dx} \right) dx \\
 &= \langle Tv, g(v) \rangle - 2\langle f, g(v) \rangle \tag{5}
 \end{aligned}$$

where

$$\begin{aligned}
 \langle Tu, g(v) \rangle &= \int_0^1 \left(\frac{d^2u}{dx^2} \right) \left(\frac{d^2v}{dx^2} \right) dx \tag{6} \\
 \langle f, g(v) \rangle &= \int_0^1 f(x) \frac{dv}{dx} dx
 \end{aligned}$$

and $g = \frac{d}{dx}$ is the linear operator.

It is clear that the operator T defined by (6) is linear, g -symmetric and g -positive. Using the technique of Noor [18,28], one can easily show that the minimum $u \in H : g(u) \in K(u)$ of the functional $I[v]$ defined by (5) associated with the problem (4) on the closed convex-valued set $K(u)$ can be characterized by the inequality of the type

$$\langle Tu, g(v) - g(u) \rangle \geq \langle f, g(v) - g(u) \rangle, \quad \forall g(v) \in K(u),$$

which is exactly the general quasi variational inequality (3).

III. For $g \equiv I$, the identity operator, the extended general quasi variational inequality (1) collapses to: find $u \in H : h(u) \in K(u)$ such that

$$\langle Tu, v - h(u) \rangle \geq 0, \quad \forall v \in K(u), \tag{7}$$

which is also called the general quasi variational inequality, see [27].

IV. For $h = I$, the identity operator, then problem (1) is equivalent to finding $u \in K(u)$ such that

$$\langle Tu, g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K(u), \tag{8}$$

which is also called the general quasi variational inequalities, introduced and studied by Noor [28].

V. For $g = h = I$, the identity operator, the extended general variational inequality (1) is equivalent to finding $u \in K(u)$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K(u), \tag{9}$$

which is known as the classical quasi variational inequality and was introduced by Benssousan and Lions [2].

VI. If $K^*(u) = \{u \in H; \langle u, v \rangle \geq 0, \quad \forall v \in K(u)\}$ is a polar(dual) cone of a closed convex-valued cone $K(u)$ in

H , then problem (1) is equivalent to finding $u \in H$ such that

$$g(u) \in K(u), \quad Tu \in K^*(u), \quad \langle g(u), Tu \rangle = 0, \tag{10}$$

which is known as the general quasi complementarity problem, see[3, 6,11, 17]. If $g = I$, the identity operator, then problem (10) is called the generalized quasi complementarity problem. For $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem (10) is called the quasi (implicit) complementarity problem, see [16,17] and the references therein.

From the above discussion, it is clear that the extended general quasi variational inequality (1) is most general and includes several previously known classes of variational inequalities and related optimization problems as special cases. These variational inequalities have important applications in mathematical programming and engineering sciences. For the recent applications, numerical methods, sensitivity analysis, dynamical systems and formulation of quasi variational inequalities and related fields, see [1-42] and the references therein.

We also need the following concepts and results.

Lemma 2.1. Let $K(u)$ be a closed convex set in H . Then, for a given $z \in H$, $u \in K(u)$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K(u),$$

if and only if

$$u = P_{K(u)}z,$$

where $P_{K(u)}$ is the projection of H onto the closed convex-valued set $K(u)$ in H .

Definition 2.3. An operator $T : H \rightarrow H$ is said to be:

(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

From (i) and (ii), it follows that $\alpha \leq \beta$.

We would like to point out that the implicit projection operator $P_{K(u)}$ is not nonexpansive. We shall assume that the implicit projection operator $P_{K(u)}$ satisfies the Lipschitz type continuity, which plays an important and fundamental role in the existence theory and in developing numerical methods for solving extended general quasi variational inequality (1) and its variant forms.

Assumption 2.1. The implicit projection operator $P_{K(u)}$ satisfies the condition

$$\|P_{K(u)}w - P_{K(v)}w\| \leq \nu \|u - v\|, \quad \forall u, v, w \in H, \tag{11}$$

where $\nu > 0$ is a positive constant.

In many important applications [1-6] the convex-valued set $K(u)$ can be written as

$$K(u) = m(u) + K, \quad (12)$$

where $m(u)$ is a point-point mapping and K is a convex set. In this case, we have

$$P_{K(u)}w = P_{m(u)+K}(w) = m(u) + P_K[w - m(u)], \quad \forall u, v \in H. \quad (13)$$

We note that if $K(u)$ is defined by (12) and $m(u)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then, using (13), we have

$$\begin{aligned} & \|P_{K(u)}w - P_{K(v)}w\| \\ &= \|m(u) - m(v) + P_K[w - m(u)] - P_K[w - m(v)]\| \\ &\leq 2\|m(u) - m(v)\| \leq 2\gamma\|u - v\|, \quad \forall u, v, w \in H. \end{aligned}$$

which shows that Assumption 2.1 holds with $\nu = 2\gamma$.

3 Projection Iterative Methods

In this section, we suggest and analyze some new approximation schemes for solving the extended general quasi variational inequality (1). One can show that the extended general quasi variational inequality (1) is equivalent to the fixed point problem by invoking Lemma 2.1.

Lemma 3.1 [32]. The function $u \in H : h(u) \in K(u)$ is a solution of the extended general quasi variational inequality (1) if and only if $u \in H : h(u) \in K(u)$ satisfies the relation

$$h(u) = P_{K(u)}[g(u) - \rho Tu], \quad (14)$$

where $P_{K(u)}$ is the projection operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the extended general quasi variational inequality (1) is equivalent to the implicit fixed point problem (14). This alternative equivalent formulation is very useful from the numerical and theoretical points of view. Zhao and Sun [35] used the concept of the exceptional family to study the existence of a solution of the nonlinear projection equations (14) for the case $K(u) = K$, the convex set. Liu and Cao [7] and Liu and Yang [8] have developed the recurrent neural network technique for solving the extended general variational inequalities. We hope this technique can be extended for solving the implicit fixed point problem (14), which is another direction for future research work.

Using the fixed point formulation (14), we suggest and analyze the following iterative method for solving the extended general quasi variational inequality (1).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n \\ &\quad + \alpha_n\{u_n - h(u_n) + P_{K(u_n)}[g(u_n) - \rho Tu_n]\}, \end{aligned} \quad n = 0, 1, \dots \quad (15)$$

which is known as the Mann iteration process for solving the extended general quasi variational inequalities (1).

Note that if $h = g$, then Algorithm 3.1 reduces to the following iterative method for solving the general quasi variational inequalities (3) and appears to be a new one.

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n\{u_n - g(u_n) \\ &\quad + P_{K(u_n)}[g(u_n) - \rho Tu_n]\}, \quad n = 0, 1, \dots \end{aligned}$$

For different and appropriate choice of the operators T, g, h , set K and the space H , one can obtain several known and new iterative methods for solving a wide class of variational inequalities and related complementarity p-problems. This clearly shows that the iterative methods considered in this paper are more general and unifying ones.

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result.

Theorem 3.1. Let the operators $T, g, h : H \rightarrow H$ be both strongly monotone with constants $\alpha > 0$, $\sigma > 0$, $\mu > 0$ and Lipschitz continuous with constants with $\beta > 0$, $\delta > 0$, $\eta > 0$ respectively. If Assumption 2.1 holds and

$$\begin{aligned} \left| \rho - \frac{\alpha}{\beta^2} \right| &< \frac{\sqrt{\alpha^2 - \beta^2 k(2 - k)}}{\beta^2}, \\ \alpha &> \beta \sqrt{k(2 - k)}, \quad k < 1, \end{aligned} \quad (16)$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\mu + \eta^2} + \nu, \quad (17)$$

and $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm 3.1 converge to a solution $u \in H : h(u) \in K(u)$ satisfying the extended general quasi variational inequality (1).

Proof. Let $u \in H : h(u) \in K(u)$ be a solution of the extended general quasi variational inequality (1). Then, using Lemma 3.1, we have

$$\begin{aligned} u &= (1 - \alpha_n)u \\ &\quad + \alpha_n\{u - h(u) + P_{K(u)}[g(u) - \rho Tu]\}, \end{aligned} \quad (18)$$

where $0 \leq \alpha_n \leq 1$ is a constant.

From Assumption 2.1, (15) and (18), we have

$$\begin{aligned} & \|u_{n+1} - u\| \\ &\leq \|(1 - \alpha_n)(u_n - u) + \alpha_n(u_n - u - (h(u_n) - h(u)))\| \end{aligned}$$

$$\begin{aligned}
 & +\alpha_n\| \{P_{K(u_n)}[g(u_n) - \rho Tu_n] - P_{K(u)}[g(u) - \rho Tu]\} \| \\
 & = \|(1 - \alpha_n)(u_n - u) + \alpha_n(u_n - u - (h(u_n) - h(u)))\| \\
 & + \|P_{K(u_n)}[g(u_n) - \rho Tu_n] - P_{K(u)}[g(u_n) - \rho Tu_n]\| \\
 & + \|P_{K(u)}[g(u_n) - \rho Tu_n] - P_{K(u)}[g(u) - \rho Tu]\| \\
 & \leq (1 - \alpha_n)\|u_n - u\| + \alpha_n\|u_n - u - (g(u_n) - g(u))\| \\
 & + \nu + \alpha_n\|u_n - u - (h(u_n) - h(u))\| \\
 & + \alpha_n\|u_n - u - \rho(Tu_n - Tu)\| + \nu\|u_n - u\|. \tag{19}
 \end{aligned}$$

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned}
 & \|u_n - u - \rho(Tu_n - Tu)\|^2 \\
 & \leq \|u_n - u\|^2 - 2\rho\langle Tu_n - Tu, u_n - u \rangle \\
 & \quad + \rho^2\|Tu_n - Tu\|^2 \\
 & \leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_n - u\|^2. \tag{20}
 \end{aligned}$$

In a similar way, we have

$$\begin{aligned}
 & \|u_n - u - (g(u_n) - g(u))\|^2 \\
 & \leq (1 - 2\sigma + \delta^2)\|u_n - u\|^2, \tag{21}
 \end{aligned}$$

$$\begin{aligned}
 & \|u_n - u - (h(u_n) - h(u))\|^2 \\
 & \leq (1 - 2\mu + \eta^2)\|u_n - u\|^2, \tag{22}
 \end{aligned}$$

using the strongly monotonicity and Lipschitz continuity of the operators g and h .

From (17), (19), (20), (21) and (22), we have

$$\begin{aligned}
 & \|u_{n+1} - u\| \\
 & \leq (1 - \alpha_n)\|u_n - u\| + \{\sqrt{1 - 2\sigma + \delta^2} \\
 & + \sqrt{1 - 2\mu^2 + \eta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}\|u_n - u\| \\
 & = (1 - \alpha_n)\|u_n - u\| + (k + t(\rho))\|u_n - u\|, \\
 & = (1 - \alpha_n)\|u_n - u\| + \theta\|u_n - u\|,
 \end{aligned}$$

where

$$t(\rho) = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}. \tag{23}$$

and

$$\theta = k + t(\rho) < 1.$$

Thus, we have

$$\begin{aligned}
 \|u_{n+1} - u\| & \leq (1 - \alpha_n)\|u_n - u\| + \alpha_n\theta\|u_n - u\| \\
 & = [1 - (1 - \theta)\alpha_n]\|u_n - u\| \\
 & \leq \prod_{i=0}^n [1 - (1 - \theta)\alpha_i]\|u_0 - u\|.
 \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\lim_{n \rightarrow \infty} \{\prod_{i=0}^n [1 - (1 - \theta)\alpha_i]\} = 0$. Consequently the sequence $\{u_n\}$ converges strongly to $u \in H : h(u) \in K(u)$ satisfying (1). This completes the proof. \square

We again use the fixed point formulation (14) to suggest the following implicit method for solving the extended general quasi variational inequality (1).

Algorithm 3.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$h(u_{n+1}) = P_{K(u_n)}[g(u_n) - \rho Tu_{n+1}], \quad n = 0, 1, 2, \dots$$

To implement this implicit method, one usually uses the predictor-corrector technique. Consequently, Algorithm 3.3 can be rewritten in the following form.

Algorithm 3.4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned}
 h(w_n) & = P_{K(u_n)}[g(u_n) - \rho Tu_n] \\
 h(u_{n+1}) & = P_{K(u_n)}[g(u_n) - \rho Tw_n], \quad n = 0, 1, 2, \dots
 \end{aligned}$$

Algorithm 3.4 is called the extragradient method. The implementation and comparison of Algorithm 3.4 is an open problem.

We can use the fixed point formulation (14) to suggest the following two-step iterative method for solving (1).

Algorithm 3.5. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned}
 h(w_n) & = P_{K(u_n)}[g(u_n) - \rho Tu_n] \\
 h(u_{n+1}) & = P_{K(u_n)}[g(w_n) - \rho Tw_n], \quad n = 0, 1, 2, \dots
 \end{aligned}$$

Algorithm 3.5 is also known as the modified extragradient method. Such type of the modified projection methods for solving the variational inequalities and their variant forms are due to Noor [17]. We remark that Algorithm 3.4 and Algorithm 3.5 are quite different. The problem of comparing these methods is an open problem.

4 Wiener-Hopf Equations Technique

In this Section, we first consider the problem of solving the extended general implicit Wiener-Hopf equations. These problems are related with the extended general quasi variational inequality (1). To be more precise, let $Q_{K(u)} = I - gh^{-1}P_{K(u)}$, where I is the identity operator and h^{-1} inverse exist. For given nonlinear operators T, g, h , we consider the problem of finding $z \in H$ such that

$$Th^{-1}P_{K(u)}z + \rho^{-1}Q_{K(u)}z = 0, \tag{24}$$

which is called the extended general implicit Wiener-Hopf equation. We note that if $gh^{-1} = I$, that is, $g = h$, then the Extended general implicit Wiener-Hopf equations (24) are exactly the general implicit Wiener-Hopf equations introduced and studied by Noor [13,21]. In addition if $g = h = I$ and $K(u) \equiv K$, then one can obtain the original Wiener-Hopf equations, which are mainly due to Shi [40]. It has been shown that the Wiener-Hopf equations have played an important and significant role in developing several numerical techniques for solving variational inequalities and related optimization problems. One can obtain a wide class of implicit Wiener-Hopf equations as special cases of the extended general quasi Wiener-Hopf equations, see [11-33].

It has been shown in [32] that the problems (1) and (24) are equivalent. This equivalence has been used to study the sensitivity analysis of the extended general quasi variational inequality (1). For the sake of completeness and to convey an idea, we include its proof.

Lemma 4.1 [32]. The solution $u \in H : h(u) \in K(u)$ satisfies the extended general quasi variational inequality (1), if and only if, $z \in H$ is a solution of the extended general implicit Wiener-Hopf equation (24), where

$$h(u) = P_{K(u)}z \tag{25}$$

$$z = g(u) - \rho Tu, \tag{26}$$

where $\rho > 0$ is a positive constant.

Proof. Let $u \in H : h(u) \in K(u)$ be a solution of (1). Then, from Lemma 3.1, we have

$$h(u) = P_{K(u)}[g(u) - \rho Tu]. \tag{27}$$

Let

$$z = g(u) - \rho Tu. \tag{28}$$

Then

$$h(u) = P_{K(u)}z. \tag{29}$$

Combining (27) and (28), and using the fact that h^{-1} exists, we have

$$z = g(u) - \rho Tu = g(h^{-1}(P_{K(u)}z)) - \rho T(h^{-1}(P_{K(u)}z)),$$

from which it follows that $z \in H$ is a solution of the extended general implicit Wiener-Hopf equation (24), the required result. \square

Lemma 4.1 implies that the extended general quasi variational inequality (1) and the Wiener-Hopf equation (4) are equivalent. We use this equivalent formulation to suggest a number of iterative methods for solving the extended general quasi variational inequalities.

I. Using (25), the Wiener-Hopf equation (24) can be rewritten in the form as:

$$Q_{K(u)}z = -\rho Th^{-1}P_{K(u)}z,$$

which implies that

$$z = gh^{-1}P_{K(u)}z - \rho Th^{-1}P_{K(u)}z = g(u) - \rho Tu,$$

This fixed point formulation enables to suggest the following iterative method for solving problem (24).

Algorithm 4.1. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$h(u_n) = P_{K(u_n)}z_n, \tag{30}$$

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{g(u_n) - \rho Tu_n\}, \tag{31}$$

$$n = 0, 1, \dots,$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

II. By an appropriate and suitable rearrangement of the terms and using (25), the Wiener-Hopf equations (24) can be written as:

$$z = gh^{-1}P_{K(u)} - \rho Th^{-1}P_{K(u)}z + (1 - \rho^{-1})Q_{K(u)}z$$

$$= g(u) - \rho Tu + (1 - \rho^{-1})Q_{K(u)}z,$$

which is another fixed point formulation. Using this fixed point formulation, we can suggest the following iterative method.

Algorithm 4.2. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$h(u_n) = P_{K(u_n)}z_n$$

$$z_{n+1} = g(u_n) - \rho Tu_n + (1 - \rho^{-1})Q_{K(u_n)}z_n,$$

$$n = 0, 1, \dots$$

III. If T is linear and T^{-1} exists, then the implicit Wiener-Hopf equation (2.11) can be written as:

$$z = (I - \rho hT^{-1}) Q_{K(u)}z.$$

This fixed point formulation allows us to suggest the following iterative method for solving the extended general quasi variational inequality (1).

Algorithm 4.3. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$z_{n+1} = (I - \rho hT^{-1}) Q_{K(u)}z_n, \quad n = 0, 1, \dots$$

For $g = h, K(u) \equiv K$, Algorithm 4.1- Algorithm 4.3 are due to Noor [13]. In brief, by an appropriate and suitable rearrangements of the terms of the extended general Wiener-Hopf equations (24), one can suggest and analyze a number of iterative methods for solving the extended general variational inequality (1) and related optimization problems. The investigation of such type of projection iterative methods and the verification of their numerical efficiency, further research efforts are needed.

We now consider the convergence analysis of Algorithm 4.1. In a similar way, one can study the convergence analysis of Algorithm 4.2 and Algorithm 4.3.

Theorem 4.1. Let the operators T, g, h satisfy all the assumptions of Theorem 3.1. If the condition (16) and Assumption 2.1 hold, then the approximate solution $\{z_n\}$ obtained from Algorithm 4.1 converges to a solution $z \in H$ satisfying the Wiener-Hopf equation (24) strongly in H .

Proof. Let $u \in H$ be a solution of (1). Then, using Lemma 4.1, we have

$$z = (1 - \alpha_n)z + \alpha_n\{g(u) - \rho Tu\}, \tag{32}$$

where $0 \leq \alpha_n \leq 1$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$. From(20), (21), (31) and (32), we have

$$\|z_{n+1} - z\|$$

$$\leq (1 - \alpha_n)\|z_n - z\|$$

$$+ \alpha_n\|g(u_n) - g(u) - \rho(Tu_n - Tu)\|$$

$$\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n\|u_n - u - (g(u_n) - g(u))\|$$

$$+ \alpha_n\|u_n - u - \rho(Tu_n - Tu)\|$$

$$\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n \left\{ \sqrt{1 - 2\sigma + \delta^2} \right.$$

$$\left. + \sqrt{1 - 2\rho\alpha + \beta^2} \right\} \|u_n - u\|, \tag{33}$$

Also from (17), (25), (30), (22) and Assumption 2.1, we have

$$\begin{aligned} & \|u_n - u\| \\ & \leq \|u_n - u - (h(u_n) - h(u))\| \\ & + \|P_{K(u_n)}z_n - P_{K(u)}z\|, \\ & = \|u_n - u - (h(u_n) - h(u))\| \\ & + \|P_{K(u_n)}z_n - P_{K(u_n)}z\| + \|P_{K(u_n)}z - P_{K(u)}z\| \\ & \leq \{\sqrt{1 - 2\mu + \eta^2}\|u_n - u\| + \|z_n - z\| + \nu\|u_n - u\| \end{aligned}$$

which implies that

$$\|u_n - u\| \leq \frac{1}{1 - (\nu + \sqrt{1 - 2\mu + \eta^2})} \|z_n - z\|. \quad (34)$$

Combining (33) and (34), we have

$$\|z_{n+1} - z\| \leq (1 - \alpha_n)\|z_n - z\| + \alpha_n\theta_1\|z_n - z\|, \quad (35)$$

where

$$\theta_1 = \frac{\sqrt{1 - 2\sigma + \delta^2} + t(\rho)}{1 - (\nu + \sqrt{1 - 2\mu + \eta^2})}. \quad (36)$$

Using (16),(17) and (36), we see that $\theta_1 < 1$. Consequently, from (35), we have

$$\begin{aligned} & \|z_{n+1} - z\| \\ & \leq (1 - \alpha_n)\|z_n - z\| + \alpha_n\theta_1\|z_n - z\| \\ & = [1 - (1 - \theta_1)\alpha_n]\|z_n - z\| \\ & \leq \prod_{i=0}^n [1 - (1 - \theta_1)\alpha_i]\|z_0 - z\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta_1 > 0$, we have $\lim_{n \rightarrow \infty} \{\prod_{i=0}^n [1 - (1 - \theta_1)\alpha_i]\} = 0$. Consequently the sequence $\{z_n\}$ converges strongly to z in H satisfying (24), the required result. \square

5 Conclusion

In this paper, we have introduced and considered a class of quasi variational inequalities involving three operators. We have shown that the third order obstacle boundary value problems can be studied in the general framework of the new quasi variational inequalities. It have been shown that the extended general quasi variational inequalities are equi-valent to the fixed point and implicit Wiener-Hopf equations. These equivalent formulations have been used to suggest and analyze several iterative methods for solving the quasi variational inequalities. We expect that the ideas and techniques of this paper will motivate and inspire the interested readers to explore its applications in various fiel-ds.

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