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Solution of an Initial Value Problem of Cauchy Type for One Equation

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Abstract: In this work, using pre-constructed partial solutions, a representation of the solution to the Cauchy problem for an equation with two variables, in which both derivatives are of fractional order, is obtained.

Keywords: RiemannLiouville fractional derivative, self-similar solution, Cauchy problem, explicit solution.

1 Introduction and Finding Particular Solutions

Consider the equation

$$L[u] \equiv D_{0y}^{\alpha} u(x,y) - dD_{0x}^{\beta} u(x,y) = 0, \tag{1}$$

where

$$x,y > 0, 0 < \alpha < 1; 1 < \beta < 2; \alpha + \beta < 2; d < 0,$$

and D_{0y}^{α}, D_{0x}^{β} are derivatives of fractional order in the sense of Riemann-Liouville, respectively, orders α, β (see [1]) :

$$D_{st}^{\nu}\varphi(t) = \begin{cases} \frac{sign(t-s)}{\Gamma(-\nu)} \int\limits_{s}^{t} \frac{\varphi(\tau)d\tau}{|t-\tau|^{\nu+1}}, \nu < 0, \\ \varphi(t), \nu = 0, \\ sign^{n}(t-s)\frac{d^n}{dt^n} D_{st}^{\nu-n}\varphi(t), n-1 < \nu \le n. \end{cases}$$

Recently, specialists have increasingly intensively studied equations that have a fractional order. This is due to the wide application of these equations in the natural sciences and in life (see, for example, [1]-[6] and others).

When solving initial and boundary problems, it is important to know the fundamental solution. One of the methods for finding a fundamental solution is the method of preliminary construction of a self-similar solution. For example, this method was used to construct a fundamental solution of the heat equation [7]. The construction of self-similar solutions themselves is also important from both theoretical and practical points of view. In the articles [8]-[9] particular solutions of the type of self-similar solutions for model equations of high integer order were found. In works [10]-[15] self-similar solutions were constructed for equations with fractional derivatives using special integro-differential operators.

In this part of the article we will construct self-similar solutions to equation (1) in the form of the following series:

$$u(x,y) = y^b \sum_{n=0}^{\infty} c_n (x^a y^c)^{n+\gamma} = \sum_{n=0}^{\infty} c_n x^{an+a\gamma} y^{cn+c\gamma+b}, \tag{2}$$

where the parameters a, b, c, γ are unknown yet. Let be

$$\overline{(a)}_s = a(a-1)\ldots(a-(s-1)), \ \overline{(a)}_0 = 1, \ \overline{(a)}_1 = a.$$

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Taking (2) into account, we formally have that

$$D_{0y}^{\alpha}u(x,y)=\frac{1}{\Gamma(1-\alpha)}\sum_{n=0}^{\infty}c_n x^{an+a\gamma}\frac{d}{dy}\int_0^y\frac{\tau^{cn+c\gamma+b}d\tau}{(y-\tau)^{\alpha}}=$$

$$=\sum_{n=0}^{\infty}c_n(x^a y^c)^{n+\gamma}y^{b-\alpha}\frac{\Gamma(cn+c\gamma+b+1)}{\Gamma(cn+c\gamma+b-\alpha+1)},\tag{3}$$

the same way

$$D_{0x}^{\beta}u(x,y)=\sum_{n=0}^{\infty}\overline{(an+a\gamma+2-\beta)_2}c_n(x^a y^c)^{n+\gamma}x^{-\beta}y^b\frac{\Gamma(an+a\gamma+1)}{\Gamma(an+a\gamma+3-\beta)}.\tag{4}$$

Let now $a=\beta$, $c=-\alpha$, then taking into account (3),(4) from equation (1) we get

$$\sum_{n=0}^{\infty}c_n(x^a y^c)^n x^{\beta}y^{-\alpha}\frac{\Gamma(cn+c\gamma+b+1)}{\Gamma(cn+c\gamma+b-\alpha+1)}=$$

$$=d\sum_{n=0}^{\infty}\overline{(an+a\gamma+2-\beta)_2}c_n(x^a y^c)^n\frac{\Gamma(an+a\gamma+1)}{\Gamma(an+a\gamma+3-\beta)},$$

or

$$\sum_{n=0}^{\infty}c_n\left(x^{\beta}y^{-\alpha}\right)^{n+1}\frac{\Gamma(-\alpha(n+\gamma)+b+1)}{\Gamma(-\alpha(n+\gamma+1)+b+1)}=$$

$$=d\sum_{n=0}^{\infty}\overline{(\beta n+\beta\gamma+2-\beta)_2}c_n(x^a y^c)^n\frac{\Gamma(an+a\gamma+1)}{\Gamma(\beta n+\beta\gamma+3-\beta)},$$

hence the equality must be satisfied that

$$\gamma_1=1-\frac{1}{\beta},\gamma_2=1-\frac{2}{\beta}.$$

Taking this into account, after some calculations and transformations we obtain a formula for finding the coefficients c_n in the form:

$$c_0=\frac{1}{\Gamma(-\alpha\gamma+b+1)\Gamma(\beta\gamma+1)},j=1,2;$$

$$c_n=\frac{1}{d^n\Gamma\left(-\alpha n-\alpha+\frac{\alpha}{\beta}j+b+1\right)\Gamma(\beta n+\beta-j+1)},j=1,2;n=1,2,....$$

So, we have obtained the following formal partial solutions of equation (1), such as self-similar solutions, in the following form:

$$u_j(x,y)=x^{\beta-j}y^{-\alpha+j\frac{\alpha}{\beta}+b}e_{\beta,\alpha}^{\beta-j+1,-\alpha+j\frac{\alpha}{\beta}+b+1}\left(\frac{1}{d}x^{\beta}y^{-\alpha}\right),j=1,2,\tag{5}$$

where

$$e_{\lambda,\nu}^{\mu,\delta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\mu)\Gamma(\delta-\nu n)},\lambda>\nu,\lambda>0,z\in\mathbb{C}$$

is a Wright type function [1]. All the above calculations were of a formal nature. Let us give expression (5) a legitimate meaning. The following theorem is true.

Theorem 1. If $d<0$; $x>0,y>0$; $0<\alpha<1;1<\beta<2;\alpha+\beta<2$; $j\frac{\alpha}{\beta}+b+1>0$, $j=1,2$; then the expressions (5) are partial solutions of the equation (1).

Proof. By direct calculation, we have

$$D_{0x}^{\beta}u_j(x,y)=\frac{1}{d}x^{\beta-j}y^{b-2\alpha+j\frac{\alpha}{\beta}}e_{\beta,\alpha}^{-j+1+\beta,-2\alpha+\frac{\alpha}{\beta}j+b+1}\left(\frac{1}{d}x^{\beta}y^{-\alpha}\right).$$

Using formula (2.2.12) from [1], we obtain

$$x^{\beta-j}D_{0y}^{\beta}\left(y^{-\alpha+j\frac{\alpha}{\beta}+b}e_{\beta,\alpha}^{\beta-j+1,-\alpha+j\frac{\alpha}{\beta}+b+1}\left(\frac{1}{d}x^{\beta}y^{-\alpha}\right)\right)=$$

$$=x^{\beta-j}y^{-2\alpha+j\frac{\alpha}{\beta}+b}e_{\beta,\alpha}^{\beta-j+1,-2\alpha+j\frac{\alpha}{\beta}+b+1}\left(\frac{1}{d}x^{\beta}y^{-\alpha}\right).$$

Substituting the found fractional derivatives into the equation (1), we obtain an identity.

Theorem 1 is proved.

Solutions of the form (5) coincide with the solutions obtained in [11]. Let now

$$b=\alpha-1-\frac{\alpha}{\beta},d=-1,$$

consider expressions

$$V_1(x,\xi,y,\eta)=|x-\xi|^{\beta-1}(y-\eta)^{-1}e_{\beta,\alpha}^{\beta,0}\left(-|x-\xi|^{\beta}(y-\eta)^{-\alpha}\right),$$

$$V_2=|x-\xi|^{\beta-2}(y-\eta)^{\frac{\alpha}{\beta}-1}e_{\beta,\alpha}^{\beta-1,\frac{\alpha}{\beta}}\left(-|x-\xi|^{\beta}(y-\eta)^{-\alpha}\right),$$

known estimate [1]

$$|V_1(x,\xi,y,\eta)|\leqslant C|x-\xi|^{\beta-1-\beta\theta}|y-\eta|^{-1+\alpha\theta},C-const,0\leqslant\theta\leqslant 2,$$

$$|V_2(x,\xi,y,\eta)|\leqslant C|x-\xi|^{\beta-2-\beta\theta}(y-\eta)^{\frac{\alpha}{\beta}-1+\alpha\theta}.$$

True lemma.

Lemma 1. Let $\varphi(x)\in C(R)$ and $|\varphi(x)|\leq S,\forall x\in R,0<S-const$, then

$$\lim_{y\to+0}\left(y^{1-\alpha}\int_{-\infty}^{+\infty}V_1(x,\xi,y,0)\varphi(\xi)d\xi\right)=2\frac{\varphi(x)}{\Gamma(\alpha)}.$$

Proof. We have

$$y^{1-\alpha}\int_{-\infty}^{+\infty}V_1(x,\xi,y,0)\varphi(\xi)d\xi=I_1(x,y)+I_2(x,y),$$

where

$$I_1=y^{1-\alpha}\int_{-\infty}^{x}V_1(x,\xi,y,0)\varphi(\xi)d\xi,$$

$$I_2=y^{1-\alpha}\int_{x}^{+\infty}V_1(x,\xi,y,0)\varphi(\xi)d\xi.$$

Let's make a change of variables

$$z=(x-\xi)^{\beta}y^{-\alpha},\xi=x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}},$$

then we will have

$$I_1=\frac{\varphi(x)}{\beta}\int_0^{+\infty}e_{\beta,\alpha}^{\beta,0}(-z)dz+\frac{1}{\beta}\int_0^{+\infty}e_{\beta,\alpha}^{\beta,0}(-z)\left(\varphi\left(x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}}\right)-\varphi(x)\right)dz.$$

Let's consider each term separately. Considering formula (2.2.7) from [1], we have

$$\frac{\varphi(x)}{\beta}\int_0^{+\infty}e_{\beta,\alpha}^{\beta,0}(-z)dz=\frac{\varphi(x)}{\beta}\lim_{A\to+\infty}\sum_{n=0}^{\infty}\int_0^{A}\frac{(-1)^n z^n dz}{\Gamma(\beta n+\beta)\Gamma(-\alpha n)}=$$

$$=\varphi(x)\lim_{A\to+\infty}\sum_{n=0}^{\infty}\frac{(-1)^n A^{n+1}}{(\beta n+\beta)\Gamma(\beta n+\beta)\Gamma(-\alpha n)}=$$

$$=\varphi(x)\lim_{A\to+\infty}Ae_{\beta,\alpha}^{\beta+1,0}(-A)=\frac{\varphi(x)}{\Gamma(\alpha)}. \quad (6)$$

Further

$$\int_0^{+\infty} e_{\beta,\alpha}^{\beta,0}(-z)\left(\varphi\left(x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}}\right)-\varphi(x)\right)dz=\int_0^{M} e_{\beta,\alpha}^{\beta,0}(-z)\left(\varphi\left(x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}}\right)-\varphi(x)\right)dz+$$

$$+\int_M^{+\infty} e_{\beta,\alpha}^{\beta,0}(-z)\left(\varphi\left(x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}}\right)-\varphi(x)\right)dz,$$

$$\forall\varepsilon>0,\exists M>0,0<y<\delta\Rightarrow$$

$$\int_0^{M}\left|e_{\beta,\alpha}^{\beta,0}(-z)\right|\left|\varphi\left(x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}}\right)-\varphi(x)\right|dz\leq$$

$$\leq\sup_{0<z<M,0<y<\delta}\left|\varphi\left(x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}}\right)-\varphi(x)\right|\int_0^{M}\left|e_{\beta,\alpha}^{\beta,0}(-z)\right|dz\leq\frac{\varepsilon}{2}, \tag{7}$$

$$\int_M^{+\infty}\left|e_{\beta,\alpha}^{\beta,0}(-z)\right|\left|\varphi\left(x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}}\right)-\varphi(x)\right|dz\leq$$

$$\leq 2S\int_M^{+\infty}\left|e_{\beta,\alpha}^{\beta,0}(-z)\right|dz=2S\int_M^{+\infty}z^{-2}\left|z^2e_{\beta,\alpha}^{\beta,0}(-z)\right|dz,$$

taking into account relation (2.2.8) from [1], we have

$$\lim_{z\to+\infty}\left(z^2e_{\beta,\alpha}^{\beta,0}(-z)\right)=\frac{1}{\Gamma(-\beta)\Gamma(2\alpha)},$$

from here

$$\int_M^{+\infty}\left|e_{\beta,\alpha}^{\beta,0}(-z)\right|\left|\varphi\left(x-z^{\frac{1}{\beta}}y^{\frac{\alpha}{\beta}}\right)-\varphi(x)\right|dz\leq\frac{\varepsilon}{2}. \tag{8}$$

From (6),(7) and (8) we get

$$\lim_{y\to+0}I_1(x,y)=\frac{\varphi(x)}{\Gamma(\alpha)}.$$

Similarly

$$\lim_{y\to+0}I_2(x,y)=\frac{\varphi(x)}{\Gamma(\alpha)}.$$

Lemma 1 is proved.

Lemma 2. Let be $\varphi(x)\in C(R)$ and $|\varphi(x)|\leq M,\forall x\in R,0<M-const$, then, function

$$u(x,y)=\int_{-\infty}^{+\infty}V_1(x,\xi,y,0)\varphi(\xi)d\xi, \tag{9}$$

is a solution to equation (1).

Proof. According to the estimate of function V_1, the improper integral (9) converges. Further, we have

$$D_{0x}^{\alpha}\left\{\int_{-\infty}^{+\infty}V_1(x,\xi,y,0)\varphi(\xi)d\xi\right\}=$$

$$=\frac{1}{y}\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_0^x\left(\int_{-\infty}^{+\infty}\frac{|t-\xi|^{\beta-1}e_{\beta,\alpha}^{\beta,0}\left(-|t-\xi|^{\beta}y^{-\alpha}\right)\varphi(\xi)}{|x-t|^{\alpha-1}}d\xi\right)dt=$$

$$= \frac{1}{y}\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty}\varphi(\xi)\,d\xi\left(\int_0^x \frac{|t-\xi|^{\beta-1}e_{\beta,\alpha}^{\beta,0}\left(-|t-\xi|^{\beta}y^{-\alpha}\right)}{|x-t|^{\alpha-1}}dt\right) =$$

$$= \int_{-\infty}^{+\infty}\varphi(\xi)\,D_{0x}^{\alpha}V_1(x,\xi,y,0)\,d\xi.$$

The convergence of the last improper integral is proved in the same way as in Lemma 1. Similarly,

$$D_{0y}^{\beta}\left\{\int_{-\infty}^{+\infty}V_1(x,\xi,y,0)\,\varphi(\xi)\,d\xi\right\} = \int_{-\infty}^{+\infty}\varphi(\xi)\,D_{0y}^{\beta}V_1(x,\xi,y,0)\,d\xi.$$

From here

$$D_{0y}^{\alpha}u(x,y) + D_{0x}^{\beta}u(x,y) = 0.$$

Lemma 2 is proved.

2 Representation of the Solution to the Initial Problem

In this part of the article, we will apply the constructed self-similar solutions to solve the following initial value problem, such as the Cauchy problem.

Cauchy problem. In the region $\Omega = \{(x,y) : -\infty < x < +\infty, 0 < y \le T\}, 0 < T < +\infty, T - const$, find a solution to the problem :

$$\begin{cases} D_{0y}^{\alpha}u(x,y) + D_{0x}^{\beta}u(x,y) = 0, \\ \lim\limits_{y\to+0}\left(y^{\alpha-1}u(x,y)\right) = \varphi(x), \end{cases}$$

where

$$y^{\alpha-1}u(x,y) \in C\left(\overline{\Omega}\right), D_{0y}^{\alpha}u(x,y), D_{0x}^{\beta}u(x,y) \in C(\Omega),$$

$$0 < \alpha < 1; 1 < \beta < 2; \alpha + \beta < 2,$$

$$\varphi(x) \in C(R), |\varphi(x)| < M < \infty, \forall x \in R, 0 < M - const.$$

The results of Lemma 1 and Lemma 2 imply the validity of the following theorem.

Theorem 2. Function having the form:

$$u(x,y) = \frac{\Gamma(\alpha)}{2}\int_{-\infty}^{+\infty}\frac{|x-\xi|^{\beta-1}}{y}e_{\beta,\alpha}^{\beta,0}\left(-\frac{|x-\xi|^{\beta}}{y^{\alpha}}\right)\varphi(\xi)\,d\xi$$

is a solution to the Cauchy problem.

Note. Note that the constructed self-similar solutions can be used to obtain a fundamental solution of the heat equation. Indeed, the heat equation

$$u_y(x,y) - u_{xx}(x,y) = 0,$$

can be written in equation form (1) (limiting case $\alpha = 1, \beta = 2$) as

$$u_y(x,y) + u_{xx}(ix,y) = 0,$$

further, having made some transformations, we will have

$$V_2 = \frac{1}{\sqrt{y-\eta}}e_{2,1}^{1,\frac{1}{2}}\left(-i^2|x-\xi|^2(y-\eta)^{-1}\right) =$$

$$= \frac{1}{\sqrt{y-\eta}}\sum_{n=0}^{\infty}\frac{\left(\frac{(x-\xi)^2}{y-\eta}\right)^n}{\Gamma(2n+1)\Gamma\left(-n+\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi(y-\eta)}}e^{-\frac{(x-\xi)^2}{4(y-\eta)}}.$$

That is, have obtained a fundamental solution of the heat equation.

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