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Inverse of Hermitian Adjacency Matrix of a Mixed Graph

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Abstract: A mixed graph D can be obtained from a graph by orienting some of its edges. Let α be a primitive n^{th} root of unity, then the $\alpha-$Hermitian adjacency matrix of a mixed graph is defined to be the matrix $H_\alpha = [h_{rs}]$ where $h_{rs} = \alpha$ if rs is an arc in D, $h_{rs} = \overline{\alpha}$ if sr is an arc in D, $h_{rs} = 1$ if sr is a digon in D and $h_{rs} = 0$ otherwise. Accordingly, in this paper we study the invertability of $\alpha-$hermitian adjacency matrix of a bipartite mixed graph with unique perfect matching. Additionally, we study the inverse of the $\alpha-$hermitian adjacency matrix of a tree mixed graph with perfect matching. Finally we restrict our study for $\alpha = \gamma$ the primitive third root of unity where we find that H_α^{-1} is $\{1,-1\}$ diagonally similar to $\gamma-$hermitian adjacency matrix of a bipartite graph.

Keywords: Adjacency matrix; Mixed graphs; Hermitian matrix; Inverse matrix; Spectrum; Bipartite graphs

1 Introduction

A mixed graph D is a digraph where both ways oriented edge is considered as an undirected edge; we call these edges digons. To be more formal, a mixed graph is a set of vertices $V(D)$ together with a set of undirected edges (digons) $E_0(D)$ and a set of directed edges (arcs) $E_1(D)$. For any two vertices $x, y \in V(D)$, if $xy \in E_1(D)$ then x(resp. y) is called initial (resp. terminal) vertex of the arc xy. The underlying graph of the mixed graph D, denoted by $\Gamma(D)$, is the graph obtained from D after stripping out the orientation of arcs of D.
A perfect matching of a mixed graph is just a perfect matching of its underlying graph, that is a collection of edges and arcs that are vertex disjoint and spans D. To be more precise $M \subset E_0(D) \cup E_1(D)$ is a perfect matching of D if no two elements in M have a common vertex and for every vertex v of D, v is an initial or a terminal vertex of an arc in D or an end vertex of a digon in D. If D has a unique perfect matching, then we denote it by $\mathscr{M}$. If D has a perfect matching M, then an arc (resp. digon) e in M is called a matching arc (resp. a matching digon) in D with respect to M. For a mixed subgraph X of a mixed graph D, the mixed graph $D\backslash X$ is defined to be the induced mixed graph over $V(D)\backslash V(X)$.
Algebraic graph theory includes the study of graphs and digraphs with respect to some graph matrix and its spectrum, where the adjacency matrix has been most intensively used/studied. For undirected graphs researchers focused on two kinds of adjacency matrices, the traditional adjacency matrix and the Laplacian adjacency matrix. On the other hand for directed graphs (digraphs) the traditional adjacency matrix was very challenging to deal with. Recently, many researchers have proposed other hermitian adjacency matrices of mixed graphs. For instance in [1], the author studied the singular values of the traditional adjacency matrix of digraphs, the author called them non-negative spectrum of digraphs. Alomari et al. in [2] proved that the non-negative spectrum is totally controlled by a vertex partition called common out neighbor partition. Around the same period of time Guo and Mohar in [3] proposed a new definition of adjacency matrix of mixed graphs as follows: For a mixed graph D, the $i-$hermitian adjacency matrix of D is a $|V| \times |V|$ matrix $H_i(D) = [h_{uv}]$, where

$$h_{uv} = \begin{cases} 1 & \text{if } uv \in E_0(D), \\ i & \text{if } uv \in E_1(D), \\ -i & \text{if } vu \in E_1(D), \\ 0 & \text{otherwise.} \end{cases}$$

Authors in [3] proved many interesting properties of H_i spectrum. Mohar in [4] extended the previously proposed adjacency matrix to a new kind of hermitian adjacency matrix called $\alpha-$hermitian adjacency matrix of mixed graphs D as follows: For a mixed graph D and the

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primitive n^{th} root of unity α, the $\alpha-$hermitian adjacency matrix of D is a $|V| \times |V|$ matrix $H_\alpha(D) = [h_{uv}]$, where

$$h_{uv} = \begin{cases} 1 \ \text{ if } uv \in E_0(D), \\ \alpha \ \text{ if } uv \in E_1(D), \\ \overline{\alpha} \ \text{ if } vu \in E_1(D), \\ 0 \ \text{ otherwise.} \end{cases}$$

The fact that these adjacency matrices (H_i and H_α) are hermitian, opens a hot research topic nowadays. For simplicity if we deal with one mixed graph D, then we write H_α instead of $H_\alpha(D)$.

Motivated by a chemistry problem, Godsil in [5] investigated invertibility of adjacency matrix of bipartite graphs. He proved that for a tree T, the inverse of the adjacency matrix of T is diagonally similar to adjacency matrix of another graph that contains a copy of T. More papers appeared after this paper that continued on Godsil's work, see for example, [6], [7] and [8].

In this paper we study the inverse of $\alpha-$hermitian adjacency matrix of a mixed graph. We examine the inverse of $\alpha-$hermitian adjacency matrix of bipartite mixed graphs and tree mixed graphs. In order to do that we need the following definitions and theorems.

Definition 1.[9] *Let D be a mixed graph and $H_\alpha = [h_{uv}]$ its α-hermitian adjacency matrix.*

- *D is called elementary mixed graph if for every component D' of D, $\Gamma(D')$ is either isomorphic to the complete graph K_2 or to a cycle C_k (for some $k \geq 3$).*
- *Let D be an elementary mixed graph. The rank of D is defined as $r(D) = n - c$, where $n = |V(D)|$ and c is the number of its components. The co-rank of D is defined as $s(D) = m - r(D)$, where $m = |E_0(D) \cup E_1(D)|$.*
- *The value $h_\alpha(W)$ of a mixed walk W with vertices $v_1, v_2, \ldots, v_k$ is defined as*

$$h_\alpha(W) = (h_{v_1v_2}h_{v_2v_3}h_{v_3v_4}\ldots h_{v_{k-1}v_k}) \in \{\alpha^r\}_{r\in\mathbb{Z}}$$

Recall that a permutation η of a set of n elements V, is just a bijective function from V to itself. The set of all permutations of V form a group under the functions composition. Let η be a permutation of a set of n elements V, then $sgn(\eta)$ is defined to be $(-1)^k$, where k is the number of transposition when η is decomposed as a product of transpositions. The following theorem is a well known result in linear algebra

Theorem 1.*If $A = [a_{ij}]$ is an $n \times n$ matrix, then*

$$det(A) = \sum_{\eta\in S_n} sgn(\eta)a_{1,\eta(1)}a_{2,\eta(2)}a_{3,\eta(3)}\ldots a_{n,\eta(n)}$$

2 Inverse of the α-Hermitian adjacency matrix of a Mixed Graph

A mixed graph D is called α non-singular mixed graph if its $\alpha-$hermitian adjacency matrix is non-singular. In this section we will study when a bipartite mixed graph is α non-singular, then we give a general description of the inverse $\alpha-$Hermitian adjacency matrix of mixed graphs. We start with the following elementary theorem which can be found in [9].

Theorem 2.*(Determinant expansion for H_α) [9] Let D be a mixed graph and H_α its α-hermitian adjacency matrix, then*

$$det(H_\alpha) = \sum_{D'}(-1)^{r(D')}2^{s(D')}Re(\prod_C h_\alpha(\mathbf{C}))$$

where the sum ranges over all spanning elementary mixed subgraphs D' of D, the product ranges over all mixed cycles C in D', and $\mathbf{C}$ is any mixed closed walk traversing C.

Suppose that D is a bipartite mixed graph, then the first (obvious) thing we refer to is, if D has a perfect matching then the bipartition sets of D have the same cardinality. Therefore D is of even number of vertices. The second thing is that D contains no odd cycles, which means, if D has a unique perfect matching, then for any spanning elementary mixed subgraph D', D' should consist of K_2 components only. Thus D contains only one spanning elementary mixed subgraph. Now using these fact and Theorem 2 we get the following result:

Theorem 3.*Let D be a bipartite mixed graph, $|V(D)| = n$ and H_α its α-hermitian adjacency matrix. If D has a unique perfect matching then $det(H_\alpha) = (-1)^{\frac{n}{2}}$ and thus D is α-nonsingular.*

By Theorem 3, the determinant of the α-hermitian adjacency matrix of bipartite mixed graphs with unique perfect matching is either 1 or -1 and this is independent of the value of α. Therefore all entries of H_α^{-1} belong to the ring $\mathbb{Z}[\alpha]$.

In the following theorem we characterize the inverse of α-hermitian adjacency matrix of mixed graphs in terms of elementary mixed subgraphs.

Theorem 4.*Let D be a mixed graph, H_α be its α-hermitian adjacency matrix and for $i \neq j$, $\Im_{ij} = \{P : P$ is a mixed path from the vertex i to the vertex $j\}$. If $\det(H_\alpha) \neq 0$, then*

$$[H_\alpha^{-1}]_{ij} = \frac{1}{\det(H\alpha)} \sum_{P_{i\to j}\in\Im_{ij}} \left[(-1)^{|E(P_{i\to j})|} h\alpha(P_{i\to j}) \left(\sum_{D'}(-1)^{r(D')}2^{s(D')}Re\left(\prod_C h\alpha(\mathbf{C})\right)\right)\right]$$

where the second sum ranges over all spanning elementary mixed subgraphs D' of $D\backslash P$, the product is being taken over all mixed cycles C in D' and $\mathbf{C}$ is any mixed closed walk traversing C.

*Proof.*Suppose that $i \neq j$, then

$$[H_\alpha^{-1}]_{ij} = \frac{m_{ji}}{\det(H_\alpha)},$$

where

$$m_{ji} = (-1)^{i+j}\det((H_\alpha)_{(j,i)}),$$

and $(H_\alpha)_{(j,i)}$ is the matrix obtained from $H_\alpha(D)$ after removing the j^{th} row and i^{th} column.

Now let M_{ji} be the matrix obtained from H_α by replacing the (ji)-entry by 1 and all other entries of j^{th} row and i^{th} column by 0, then

$$m_{ji} = \overline{\det(M_{ji})} \quad (1)$$

On the other hand, using Theorem 1 we have,

$$\det(M_{ji}) = \sum_{\eta \in S_n} sgn(\eta) h_{1\eta(1)} h_{2\eta(2)} \dots h_{n\eta(n)}$$

Now for any $\eta \in S_n$, since (j,k)-entries of M_{ji} are zeros and the (j,i)-entry is one, if η does not take j to i then η contributes zero in the expansion of $\det(M_{ji})$. Let

$$\psi_{j\to i} = \{\phi \in S_n : \phi \text{ is a permutation that takes } j \text{ to } i\}.$$

For each $\phi \in \psi_{j\to i}$ let δ_ϕ be the cycle in ϕ that permutes j to i and δ_ϕ^c be all other cycles in ϕ, then

$$\begin{aligned}\det(M_{ji}) &= \sum_{\phi \in \psi_{j\to i}} sgn(\phi) \prod_{k\in V(G)\backslash\{j\}} h_{k\phi(k)} \\ &= \sum_{\phi \in \psi_{j\to i}} sgn(\delta_\phi^c) sgn(\delta_\phi) \prod_{k\in\delta_\phi} h_{k\delta_\phi(k)} \prod_{k\in\delta_\phi^c} h_{k\delta_\phi^c(k)} \\ &= \sum (-1)^{|E(P_{j\to i})|} h_\alpha(P_{j\to i}) \det(H_\alpha(X)) \\ &= \sum (-1)^{|E(P_{j\to i})|} \overline{h_\alpha(P_{i\to j})} \det(H_\alpha(X))\end{aligned}$$

where X is the induced mixed graph over $V(D)\backslash V(P_{i\to j})$ and $P_{i\to j} \in \Im_{ij}$. Therefore using Equation 1 together with Theorem 2 we have,

$$\det(M_{ij}) = \sum_{P_{i\to j}\in\Im_{ij}} \left[(-1)^{|E(P_{i\to j})|} h_\alpha(P_{i\to j}) \sum_{D'} (-1)^{r(D')} 2^{s(D')} Re\left(\prod_C h_\alpha(\mathbf{C})\right)\right]$$

where the second sum is taken over all spanning elementary mixed subgraphs of $D\backslash P$, the product ranges over all mixed cycles C in D', and $\mathbf{C}$ is any mixed closed walk traversing C

Theorem 4 is related to the non-diagonal entries of H_α^{-1}. Based on Theorem 2, and since a bipartite mixed graph with unique perfect matching have only one elementary mixed subgraph that consist of the matching edges of D, we have the following observation which takes care of the diagonal entries of H_α^{-1}.

Theorem 5. *Let D be a bipartite mixed graph with unique perfect matching and H_α its α−hermitian adjacency matrix. Then*

1. *$[H_\alpha^{-1}]_{ii} = 0$ if and only if $D\backslash\{i\}$ is not α invertible mixed graph.*
2. *Suppose that T is a tree mixed graph that has a perfect matching, then $[H_\alpha^{-1}]_{ij} = 0$ if and only if $T\backslash P_{i\to j}$ does not have a perfect matching.*

Proof. 1. Since D has a unique perfect matching, then D is an α invertible mixed graph, Further

$$[H_\alpha^{-1}]_{ii} = \frac{det((H_\alpha)_{(i,i)})}{det(H_\alpha)},$$

where $(H_\alpha)_{(i,i)}$ is the matrix obtained from H_α by deleting the i^{th} row and the i^{th} column.
Therefore $[H_\alpha^{-1}]_{ii} = 0$ if and only if $det((H_\alpha)_{(i,i)}) = 0$.
The fact that $(H_\alpha)_{(i,i)}$ is the α adjacency matrix of $D\backslash\{i\}$ ends the proof.

2. Obvious.

Example 1. Consider the mixed graph D shown in Figure 1 and let $\alpha = e^{\frac{\pi i}{7}}$.

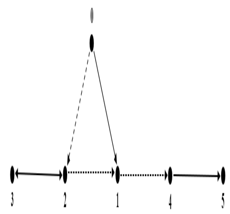

Fig. 1: The mixed graph D of Example 1

Obviously, D has a unique perfect matching. Thus

$$\det(H_\alpha) = (-1)^3 2^0 = -1.$$

Now for $i \neq 2$ and $i \neq 3$, $D\backslash P_{2\to i}$ will leave the vertex 3 as an isolated vertex, which means $D\backslash P_{2\to i}$ does not have a spanning elementary mixed subgraph. Also $D\backslash\{2\}$ leaves the vertex 3 as an isolated vertex which means $[H_\alpha^{-1}]_{22} = 0$. So the row corresponds to 2 in H_α^{-1} will be all zeros except $[H_\alpha^{-1}]_{23}$. We have

$$[H_\alpha^{-1}]_{23} = \frac{-1}{-1} h_\alpha(P_{2\to 3})(-1)^{r(D')} 2^{s(D')}.$$

Where D' is the spanning elementary mixed subgraph of $D\backslash P_{2\to 3}$. Observing that

$$h_\alpha(P_{2\to 3}) = 1, r(D') = 2 \text{ and } s(D') = 0,$$

we have, $[H_\alpha^{-1}]_{23} = 1$. Let's turn to the value of $[H_\alpha^{-1}]_{05}$. In this case there are two paths between the vertices 0 and 5,

1. $P_{0\to 5}$ is the path passing through 2. This contributes 0 because the vertex 3 will be an isolated vertex in $D\backslash P_{0\to 5}$.
2. $Q_{0\to 5}$ is the path that doesn't pass through the vertex 2. This contributes

$$[H_\alpha^{-1}]_{05} = \frac{1}{-1}(-1)^3 e^{\frac{3\pi i}{7}} (-1)^1 2^0 = -e^{\frac{3\pi i}{7}}.$$

Finally, we notice here $D\backslash\{3\}$ is a mixed graph with unique elementary mixed subgraph, according to Theorem 2

$$det(H_\alpha(D\backslash\{3\})) = -e^{\frac{i\pi}{7}} - e^{-\frac{i\pi}{7}}.$$

One can continue this process to find all the entries of H_α^{-1}. Below we give the $\alpha-$hermitian adjacency matrix of D and its inverse,

$$H_\alpha = \begin{pmatrix} 0 & e^{\frac{i\pi}{7}} & e^{\frac{i\pi}{7}} & 0 & 0 & 0 \\ e^{-\frac{i\pi}{7}} & 0 & e^{-\frac{i\pi}{7}} & 0 & e^{\frac{i\pi}{7}} & 0 \\ e^{-\frac{i\pi}{7}} & e^{\frac{i\pi}{7}} & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & e^{-\frac{i\pi}{7}} & 0 & 0 & 0 & e^{\frac{i\pi}{7}} \\ 0 & 0 & 0 & 0 & e^{-\frac{i\pi}{7}} & 0 \end{pmatrix}$$

$$H_\alpha^{-1} = \begin{pmatrix} 0 & e^{\frac{i\pi}{7}} & 0 & -1 & 0 & -e^{\frac{3i\pi}{7}} \\ e^{-\frac{i\pi}{7}} & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & e^{-\frac{i\pi}{7}}+e^{\frac{i\pi}{7}} & 0 & e^{\frac{2i\pi}{7}} \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{i\pi}{7}} \\ -e^{-\frac{3i\pi}{7}} & 0 & 0 & e^{-\frac{2i\pi}{7}} & e^{-\frac{i\pi}{7}} & 0 \end{pmatrix}$$

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3 The inverse of α-hermitian adjacency matrix of a Tree Mixed Graph

In this section, we will introduce a combinatorial representation of the entries of the inverse of the α-hermitian adjacency matrix of a tree mixed graph. Suppose that D is a mixed graph with unique perfect matching. A path P between two vertices i and j is called co-augmenting path if the edges of the underlying path of P alternates between matching edges and non-matching edges where both first and last edge are matched. In [6] authors proved that if T is a tree graph and A is its adjacency matrix then whenever there is a co-augmenting path between two vertices in T, the corresponding entry of the matrix A^{-1} is either 1 or -1. In the following theorem we show that the co-augmenting path part is still applicable with the tree mixed graph and its α-hermitian adjacency matrix. Note that if a tree has a perfect matching M, then this matching is unique.

Theorem 6. *Let T be a tree mixed graph with perfect matching $\mathscr{M}$, $|V(T)| = n$ and H_α its α-hermitian adjacency matrix. Then*

1. $\det(H_\alpha) = (-1)^{\frac{n}{2}}$.
2. $[H_\alpha^{-1}]_{ij} \neq 0$ *if and only if the path from i to j is a co-augmenting path.*
3. $[H_\alpha^{-1}]_{ij} = \begin{cases} (-1)^{\frac{k}{2}-1} h_\alpha(P_{i\to j}) & P_{i\to j} \text{ is a co-augmenting path and } k = |V(P_{i\to j})|, \\ 0 & \text{otherwise}. \end{cases}$

Proof. 1 Using Theorem 3, bipartite mixed graphs of order n with unique perfect matching are non singular and its determinant equals to $(-1)^{\frac{n}{2}}$. Obviously tree mixed graphs that have a perfect matching will satisfy the above conditions.

2 Suppose that $i \neq j$ are two vertices of T and $P_{i\to j}$ is the path from i to j, then

$$[H\alpha(T)]_{ij}^{-1} = (-1)^{\frac{n}{2}} \begin{cases} (-1)^{|E(P_{i\to j})|} h\alpha(P_{i\to j})(-1)^{r(D')} 2^{s(D')} & \text{if } T\backslash P_{i\to j} \text{ has a perfect matching,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $[H_\alpha(T)]_{ij}^{-1} \neq 0$, then $T\backslash P_{i\to j}$ has a perfect matching M. Now suppose that $P_{i\to j}$ is not a co-augmenting path, $M_1 = E(P_{i\to j}) \cap \mathscr{M}$ and $M_2 = \mathscr{M}\backslash M_1$. Since $P_{i\to j}$ is not a co-augmenting path, there is $v \in V(P)$ such that v is matched by M_2, say (vv'), this means v' is unmatched vertex in $T\backslash P_{i\to j}$, and so, the component of $T\backslash P_{i\to j}$ that contains v' together with v form a tree with perfect matching which contradicts the uniqueness of the perfect matching in a tree.
The other direction is obvious and we left it for the reader.

3 Simple calculation we leave it to the reader.

The above theorem characterizes the entries of the inverse of the $\alpha-$hermitian adjacency matrix of a tree mixed graph with a perfect matching. Note here that H_α^{-1} nonzero entries depends firstly on the existence of a co-augmenting path between the two vertices that corresponds to the location of the entry and secondly on the $\pm h_\alpha(P)$. Therefore, if T is a tree mixed graph, H_α its $\alpha-$hermitian adjacency matrix and α is the primitive n^{th} root of unity, then whenever T contains two vertices i and j with $1 < |E(P_{ij}| < n$, H_α^{-1} nonzero entries may not belong to $\{\pm\alpha, \pm\overline{\alpha}, \pm 1\}$. This is not like the inverse of tree graph adjacency matrix case, that is described in the following theorem.

Theorem 7. *[5] If T is a tree graph with perfect matching and A is its adjacency matrix, then*

1. $A^{-1} = \begin{cases} (-1)^{u(i,j)} & \text{if there is a co-augmenting path between } i \text{ and } j, \\ 0 & \text{otherwise} \end{cases}$
   where $u(i,j) = \frac{|V(P)|}{2} - 1$ is the number of unmatched edges along the i,j path.
2. *There is a bipartite graph G such that A^{-1} is $\{1,-1\}$ diagonally similar to $A(G)$, where $A(G)$ is the adjacency matrix of G.*

The above theorem is not true in general for $\alpha-$hermitian adjacency matrix for tree mixed graphs. We get similar result to previous theorem for the $\alpha-$hermitian adjacency matrix of a tree mixed graph when $\alpha = \gamma$, where γ is the third root of unity. We state this result in the following theorem.

Theorem 8. *Let T be a tree mixed graph with perfect matching, γ is the primitive third root of unity and H_γ is the $\gamma-$hermitian adjacency matrix of T. Then, H_γ^{-1} is $\{+1,-1\}$-diagonally similar to $\gamma-$hermitian matrix of a mixed graph.*

Proof. We observe that in tree mixed graphs $h_\alpha(P) \in \{\alpha^r\}_{r\in\mathbb{Z}}$, restricting α to the primitive third root of unity, we have $\{\alpha^r\}_{r\in\mathbb{Z}} = \{\alpha, \overline{\alpha}, 1\}$. Moreover there is no integer $r \in \mathbb{Z}$ such that $\alpha^r = -\alpha$ or $\alpha^r = -\overline{\alpha}$. This means the negative sign only comes from the part $(-1)^{u(i,j)}$ which is the same in both inverse of $\gamma-$hermitian adjacency matrix and inverse of its underlying adjacency matrix. Therefore, if A is the adjacency matrix of $\Gamma(T)$ and S is a $\{1,-1\}$ diagonal matrix with $SA^{-1}S$ is the adjacency matrix of a graph Y then $SH_\gamma S$ is $\gamma-$hermitian adjacency matrix of a mixed graph X with $\Gamma(X) \simeq Y$.

Definition 2. *Let T be a tree mixed graph and that has a perfect matching. Define T_γ^{-1} to be the mixed graph which has γ-hermitian adjacency matrix $H_\gamma = [h_{ij}]$ where*

$$h_{ij} = \begin{cases} h(P) & \text{if } P \text{ is a co-augmenting path from } i \text{ to } j, \\ 0 & \text{otherwise} \end{cases}$$

Corollary 1. *If T is tree mixed graph with perfect matching, then*

$$\Gamma(T_\gamma^{-1}) = (\Gamma(T))^{-1}.$$

Example 2. In Figure 4 a tree mixed graph T and γ–inverse T_γ^{-1} are depicted. The bold arcs and digons correspond to the positions where $|E(P_{ij})|$ are odd. In this case the $\{1,-1\}$ diagonal matrix described in [5] is

$$D = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&-1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&-1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&-1 \end{pmatrix}$$

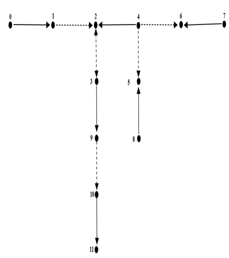
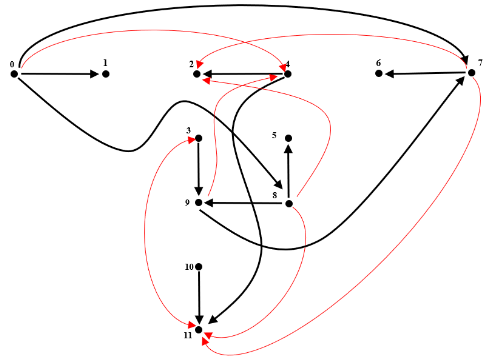


Fig. 2: The tree mixed graph T



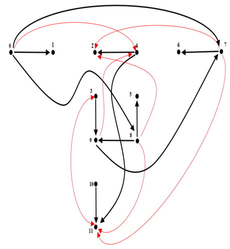


Fig. 3: The mixed graph T_γ^{-1}

Fig. 4: The non-bold arcs and digon in the mixed graph T_γ^{-1} are corresponding to the entries $\{-\gamma, -\overline{\gamma}, -1\}$.

A question is raised here which bipartite graphs have a tree γ inverse. This is an important open question where we solve it partially in the following theorem:

Theorem 9. *If D is a bipartite mixed graph and $D \simeq T_\gamma^{-1}$ of some tree mixed graph T, then*

$$\sigma_\gamma(D) = \sigma(\Gamma(D)),$$

where $\sigma_\gamma(D)$ is the spectrum of $H_\gamma(D)$ and $\sigma(\Gamma(D))$ is the traditional spectrum of the underlying graph of D, $\Gamma(D)$.

Proof. Suppose that $D \simeq T_\gamma^{-1}$, then there is $\{1,-1\}$ diagonal matrix S such that

$$SH_\gamma^{-1}(T)S = H_\gamma(D).$$

Therefore, $\gamma-$spectrum of D is totally controlled by $\gamma-$spectrum of T, but it is known (see [9]) that $\sigma_\gamma(T) = \sigma(\Gamma(T))$. Corollary 1 finishes the proof.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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