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The Approximate Solutions for Stiff Systems of Ordinary Differential Equations by Using Homotopy Analysis Method and Variational Iteration Method

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Abstract: In this study, the homotopy analysis method and the variational iteration method are used to find the approximate solutions for systems of initial value problems for ordinary differential equations for both linear and non-linear problems, by using size of a step with a sequence of subintervals. These methods are shown to work for a variety of systems with approximate-exact solutions. The illustrated numerical results demonstrate that the convergence of the approximate solutions with the exact solutions. The results achieved in the aforementioned systems demonstrate the method's consistency and efficiency.

Keywords: Homotopy analysis method; Variational iteration method; Lagrange multiplier; Stiff systems; He's polynomials

1 Introduction

In his Ph.D. dissertation in 1992, Shijun Liao [1,2] proposed a new and excellent method (so-called Homotopy Analysis Method (HAM)) for solving linear and non-linear (ordinary differential equations, partial differential equations, integral equations, and so on).

In 1978, suggested a general Lagrange multiplier method by Inokuti et. al. [3] to solve non-linear problems. In 1999, Ji-Huan He [4,5,6] proposed a new and excellent method (so-called Variational Iteration Method (VIM)) for solving linear and non-linear (ordinary differential, partial differential, integral, etc.) equations.

If an analytic solution exists, the HAM/VIM provides rapidly converging consecutive approximations; otherwise, a few approximations might be used for numerical results. The HAM/VIM generates a series solution that occasionally converges to the analytic solution to linear and non-linear deterministic. The VIM's concept is to use a general Lagrange multiplier [3] to create a correction functional, and the multiplier is set in such a way that the correction solution is optimal in relation to the initial approximation or experiment function.

The HAM/VIM has been utilized by many authors quite effectively for obtaining analytic and/or approximate solutions for a wide variety of scientific and engineering applications linear and non-linear, homogeneous and inhomogeneous as well, particularly in the field of ordinary differential, partial differential and integral equations [7,8,9,10,11,12] and [13,14,15,16]. The authors [17] have been used the VIM with Sumudu transform for solving Delay differential equations. Some comparisons with the efficiency of other methods in similar problems [18] have been performed, but a comprehensive analysis is still lacking. HAM/VIM has traditionally been used only at short intervals or with a fixed stepsize.

In the literature, Wu and Xia have been used numerical methods for stiff systems [19]. Mahmood et. al. utilized the Adomian decomposition method (ADM) for the same systems by using size of a step with a sequence of subintervals [20].

The HAM/VIM have been utilized only on small intervals or with a constant stepsize. This paper investigates the performance of HAM/VIM in the cases of linear/non-linear for stiff system IVPs of ODEs by using size of a step with a sequence of subintervals.

Consider the system of ordinary differential equations

$$y_i' = f_i(y_1, y_2, \ldots, y_n) + g_i(x), \; i = 1, \ldots, n \qquad (1)$$

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where f_i are linear/non-linear functions, x denotes the independent variable, $y_i(x)$ are unknown functions, $g_i(x)$ are known analytic functions, and we are searching the solution $y_i(x)$ satisfying (1). We assume that for every $g_i(x)$, the system (1) has one and only one solution. In what follows, we give a summarized description of HAM/VIM and we look at how the methods are used to three different problems with varying levels of difficulty.

2 Basic concept of the HAM

To describe the basic ideas of the HAM, the system (1) can be written as follows

$$N_i[y_i(x)] = g_i(x),\ i = 1,\dots,n \tag{2}$$

where N_i are non-linear operators. We'll ignore all initial/boundary conditions for the simplicity, but they may all be dealt in the same way. Shijun Liao [21] creates the zero-order deformation equation by generalizing the classical Homotopy method.

$$(1-p)L[\phi_i(x;p) - y_{i,0}(x)] = ph_i[N_i[\phi_i(x;p)] - g_i(x)], \tag{3}$$

where $p \in [0,1]$ is an embedding parameter, h_i are nonzero auxiliary functions, L is a linear auxiliary operator, $y_{i,0}(x)$ are initial estimates of $y_i(x)$ and the functions $\phi_i(x;p)$ are unknown. It is critical to note that in HAM, one has a large deal of flexibility in selecting auxiliary objects such as h_i and L. It is obvious that it holds true when $p=0$ and $p=1$

$$\phi_i(x;0) = y_{i,0}(x),\ \text{and}\ \phi_i(x;1) = y_i(x), \tag{4}$$

respectively. Thus, as p increases from 0 to 1, the solutions $\phi_i(x;p)$ change from the initial estimates $y_{i,0}(x)$ to the solutions $y_i(x)$. Expanding $\phi_i(x;p)$ in series of Taylor with respect to p, we have

$$\phi_i(x;p) = y_{i,0}(x) + \sum_{m=1}^{+\infty} y_{i,m}(x)p^m, \tag{5}$$

where

$$y_{i,m}(x) = \frac{1}{m!}\frac{\partial^m \phi_i(x;p)}{\partial p^m}\bigg|_{p=0}. \tag{6}$$

If the auxiliary linear operators, the initial guesses, the auxiliary parameters h_i, and the auxiliary functions are so properly chosen, the above series (5) converges at $p=1$, then we have

$$\phi_i(x;1) = y_{i,0}(x) + \sum_{m=1}^{+\infty} y_{i,m}(x), \tag{7}$$

which must be one of the solutions of the original non-linear equation, as proved by Shijun Liao [21]. If $h_i = -1$, Eq. (3) becomes

$$(1-p)L[\phi_i(x;p) - y_{i,0}(x)] + p[N_i[\phi_i(x;p)] - g_i(x)] = 0, \tag{8}$$

this is mostly employed in the Homotopy perturbation method [22].

The governing equation can be obtained from the zero-order deformation equation (3) using Eq. (6). We define the vectors

$$\overrightarrow{y}_{i,m}(x) = \{y_{i,0}(x), y_{i,1}(x), \dots, y_{i,m}(x)\}. \tag{9}$$

Differentiating Eq. (3) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[y_{i,m}(x) - \chi_m y_{i,m-1}(x)] = h_i R_{i,m}(\overrightarrow{y}_{i,m-1}), \tag{10}$$

where

$$R_{i,m}(\overrightarrow{y}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}\{N_i[\phi_i(x;p)] - g_i(x)\}_{p=0}, \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Substituting Eq. (5) into Eq. (11), we have

$$R_{i,m}(\overrightarrow{y}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}\left\{N_i\left(\sum_{m=0}^{+\infty} y_{i,m}(x)p^m\right) - g_i(x)\right\}_{p=0}. \tag{12}$$

It is important to note that $y_{i,m}(x)$ $(m \ge 1)$ is governed by Eq. (10) with the initial/boundary conditions derived from the original problem, which can be solved quickly using symbolic computation software like Maple and Mathematica.

3 Basic concept of the VIM

To illustrate the basic ideas of the VIM, the system (1) may be written as follows

$$Ly_i(x) + Ny_i(x) = g_i(x),\ i = 1,\dots,n \tag{13}$$

where L is the linear operator and N is the non-linear operator. The VIM was proposed by Ji-Huan He [4,5,6], where a correction functional for Eq. (13) can be rewritten as

$$y_{i,m+1}(x) = y_{i,m}(x) + \int_0^x \lambda_i(t)\left[Ly_{i,m}(t) + N\tilde{y}_{i,m}(t) - g_i(t)\right]dt, \tag{14}$$

where $\lambda_i(t)$ are general Lagrange multipliers [3,4,5,6] which can be identified optimally via variational theory, $y_{i,0}(x)$ are initial approximations, with possible unknowns. The functions $\tilde{y}_{i,m}(x)$ are considered as a restricted variation [23], which means $\delta\tilde{y}_{i,m}(t) = 0$. Therefore, we first determine the Lagrange multipliers λ_i

that will be identified optimally via integration by parts. The consecutive approximations $y_{i,m+1}(x)$, to the solutions $y_i(x)$ will be readily obtained upon using the Lagrange multipliers obtained and by using any selective functions $y_{i,0}(x)$, which can be easily solved by mathematical symbolic programs like Maple and Mathematica. Consequently, the approximate-analytic solutions may be obtained by using $y_i(x) = \lim_{m\to\infty} y_{i,m}(x)$.

4 Application of VIM

The following problems were chosen from the current literature have been studied in [19,20]. The environment has been used for symbolic computations is Maple 18.

Problem 1

Firstly, consider the non-linear stiff system of ordinary differential equations [19,20]:

$$Y'(x) = -BY(x) + UW,\ Y(0) = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \tag{15}$$

where

$$Y'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix},\ U = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

$$B = U\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}U,\ W = \begin{bmatrix} z_1^2 \\ z_2^2 \\ z_3^2 \\ z_4^2 \end{bmatrix},$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = UY,\ D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1000 \\ 800 \\ -10 \\ 0.001 \end{bmatrix}.$$

The exact solution of the system (15) is

$$Y(x) = UZ, \tag{16}$$

where

$$Y(x) = \begin{bmatrix} y_{1E}(x) \\ y_{2E}(x) \\ y_{3E}(x) \\ y_{4E}(x) \end{bmatrix}, \qquad Z = \begin{bmatrix} z_1(x) \\ z_2(x) \\ z_3(x) \\ z_4(x) \end{bmatrix},$$

$$z_i(x) = \frac{d_i}{1-(1+d_i)e^{d_i x}},\ i = 1,\ldots,4.$$

Operating and applying the matrix operations of the system (15), we get

$$\begin{aligned}
y_1'(x) &= -\alpha y_1 + \beta y_2 + \gamma y_3 + \mu y_4 \\
&\quad + \frac{1}{4}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right) \\
&\quad + \frac{1}{2}(y_1y_2 + y_1y_3 + y_1y_4) \\
&\quad - \frac{1}{2}(y_2y_3 + y_2y_4 + y_3y_4), \\
y_2'(x) &= \beta y_1 - \alpha y_2 - \mu y_3 - \gamma y_4 \\
&\quad + \frac{1}{4}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right) \\
&\quad + \frac{1}{2}(y_1y_2 - y_1y_3 - y_1y_4) \\
&\quad + \frac{1}{2}(y_2y_3 + y_2y_4 - y_3y_4), \\
y_3'(x) &= \gamma y_1 - \mu y_2 - \alpha y_3 - \beta y_4 \\
&\quad + \frac{1}{4}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right) \\
&\quad - \frac{1}{2}(y_1y_2 - y_1y_3 + y_1y_4) \\
&\quad + \frac{1}{2}(y_2y_3 - y_2y_4 + y_3y_4), \\
y_4'(x) &= \mu y_1 - \gamma y_2 - \beta y_3 - \alpha y_4 \\
&\quad + \frac{1}{4}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right) \\
&\quad - \frac{1}{2}(y_1y_2 + y_1y_3 - y_1y_4) \\
&\quad - \frac{1}{2}(y_2y_3 - y_2y_4 - y_3y_4),
\end{aligned} \tag{17}$$

where

$$\alpha = 447.50025,\ \beta = 452.49975,$$
$$\gamma = 47.49975, \quad \mu = 52.50025.$$

HAM: To solve the system (17) by means the standard of the HAM, we choose the initial approximations $y_{i,0}(x) = -1$, and the linear operators

$$L[\phi_i(x;p)] = \frac{\partial \phi_i(x;p)}{\partial x},$$

with the property $L[c_i] = 0$, where c_i are constants of integration. Furthermore, the system (17) suggest that a system of non-linear operators be defined as

$$\begin{aligned}
N_1[\phi_i(x;p)] &= \frac{\partial\phi_1(x;p)}{\partial x} + \alpha\phi_1(x;p) - \beta\phi_2(x;p) - \gamma\phi_3(x;p) \\
&-\mu\phi_4(x;p) - \frac{1}{4}\left[\phi_1^2(x;p) + \phi_2^2(x;p)\right. \\
&\left.+\phi_3^2(x;p) + \phi_4^2(x;p)\right] - \frac{1}{2}[\phi_1(x;p)\phi_2(x;p) \\
&+\phi_1(x;p)\phi_3(x;p) + \phi_1(x;p)\phi_4(x;p)] \\
&+\frac{1}{2}[\phi_2(x;p)\phi_3(x;p) + \phi_2(x;p)\phi_4(x;p) \\
&+\phi_3(x;p)\phi_4(x;p)], \\
N_2[\phi_i(x;p)] &= \frac{\partial\phi_2(x;p)}{\partial x} - \beta\phi_1(x;p) + \alpha\phi_2(x;p) + \mu\phi_3(x;p) \\
&+\gamma\phi_4(x;p) - \frac{1}{4}\left[\phi_1^2(x;p) + \phi_2^2(x;p)\right. \\
&\left.+\phi_3^2(x;p) + \phi_4^2(x;p)\right] - \frac{1}{2}[\phi_1(x;p)\phi_2(x;p) \\
&-\phi_1(x;p)\phi_3(x;p) - \phi_1(x;p)\phi_4(x;p)] \\
&-\frac{1}{2}[\phi_2(x;p)\phi_3(x;p) + \phi_2(x;p)\phi_4(x;p) \\
&-\phi_3(x;p)\phi_4(x;p)], \\
N_3[\phi_i(x;p)] &= \frac{\partial\phi_3(x;p)}{\partial x} - \gamma\phi_1(x;p) + \mu\phi_2(x;p) + \alpha\phi_3(x;p) \\
&+\beta\phi_4(x;p) - \frac{1}{4}\left[\phi_1^2(x;p) + \phi_2^2(x;p)\right. \\
&\left.+\phi_3^2(x;p) + \phi_4^2(x;p)\right] + \frac{1}{2}[\phi_1(x;p)\phi_2(x;p) \\
&-\phi_1(x;p)\phi_3(x;p) + \phi_1(x;p)\phi_4(x;p)] \\
&+\frac{1}{2}[\phi_2(x;p)\phi_3(x;p) - \phi_2(x;p)\phi_4(x;p) \\
&+\phi_3(x;p)\phi_4(x;p)], \\
N_4[\phi_i(x;p)] &= \frac{\partial\phi_4(x;p)}{\partial x} - \mu\phi_1(x;p) + \gamma\phi_2(x;p) + \beta\phi_3(x;p) \\
&+\alpha\phi_4(x;p) - \frac{1}{4}\left[\phi_1^2(x;p) + \phi_2^2(x;p)\right. \\
&\left.+\phi_3^2(x;p) + \phi_4^2(x;p)\right] + \frac{1}{2}[\phi_1(x;p)\phi_2(x;p) \\
&+\phi_1(x;p)\phi_3(x;p) - \phi_1(x;p)\phi_4(x;p)] \\
&+\frac{1}{2}[\phi_2(x;p)\phi_3(x;p) - \phi_2(x;p)\phi_4(x;p) \\
&-\phi_3(x;p)\phi_4(x;p)]. \qquad (18)
\end{aligned}$$

Applying the above definition, the zeroth-order deformation equation is constructed as (3) and (4), and the deformation equation of mth-order for $m \geq 1$ is constructed as

$$L[y_{i,m}(x) - \chi_m y_{i,m-1}(x)] = h_i R_{i,m}(\overrightarrow{y}_{i,m-1}), \qquad (19)$$

with the initial conditions $y_{i,m}(0) = 0$ where

$$\begin{aligned}
R_{1,m}(\overrightarrow{y}_{i,m-1}) &= y'_{1,m-1} + \alpha y_{1,m-1} - \beta y_{2,m-1} - \gamma y_{3,m-1} \\
&-\mu y_{4,m-1} - \frac{1}{4}[A_{1,m-1} + A_{2,m-1} \\
&+A_{3,m-1} + A_{4,m-1}] - \frac{1}{2}[A_{1,2,m-1} \\
&+A_{1,3,m-1} + A_{1,4,m-1}] + \frac{1}{2}[A_{2,3,m-1} \\
&+A_{2,4,m-1} + A_{3,4,m-1}], \\
R_{2,m}(\overrightarrow{y}_{i,m-1}) &= y'_{2,m-1} - \beta y_{1,m-1} + \alpha y_{2,m-1} + \mu y_{3,m-1} \\
&+\gamma y_{4,m-1} - \frac{1}{4}[A_{1,m-1} + A_{2,m-1} \\
&+A_{3,m-1} + A_{4,m-1}] - \frac{1}{2}[A_{1,2,m-1} \\
&-A_{1,3,m-1} - A_{1,4,m-1}] - \frac{1}{2}[A_{2,3,m-1} \\
&+A_{2,4,m-1} - A_{3,4,m-1}], \\
R_{3,m}(\overrightarrow{y}_{i,m-1}) &= y'_{3,m-1} - \gamma y_{1,m-1} + \mu y_{2,m-1} + \alpha y_{3,m-1} \\
&+\beta y_{4,m-1} - \frac{1}{4}[A_{1,m-1} + A_{2,m-1} \\
&+A_{3,m-1} + A_{4,m-1}] + \frac{1}{2}[A_{1,2,m-1} \\
&-A_{1,3,m-1} + A_{1,4,m-1}] + \frac{1}{2}[A_{2,3,m-1} \\
&-A_{2,4,m-1} + A_{3,4,m-1}], \\
R_{4,m}(\overrightarrow{y}_{i,m-1}) &= y'_{4,m-1} - \mu y_{1,m-1} + \gamma y_{2,m-1} + \beta y_{3,m-1} \\
&+\alpha y_{4,m-1} - \frac{1}{4}[A_{1,m-1} + A_{2,m-1} \\
&+A_{3,m-1} + A_{4,m-1}] + \frac{1}{2}[A_{1,2,m-1} \\
&+A_{1,3,m-1} - A_{1,4,m-1}] + \frac{1}{2}[A_{2,3,m-1} \\
&-A_{2,4,m-1} - A_{3,4,m-1}].
\end{aligned}$$

For the non-linear term $y_i^2 = \sum_{n=0}^{\infty} A_{i,n}$ $(i = 1,\ldots,4)$.the corresponding He's polynomials [24] are:

$$A_{i,n} = \sum_{m=0}^{n} y_{i,m} y_{i,n-m},\ n \geq m,\ \ n = 0,1,\ldots\ .$$

In the same way, for $y_i y_j = \sum_{n=0}^{\infty} A_{i,j,n}$ $(i \neq j, i, j = 1,\ldots,4)$ the corresponding He's polynomials [24] are:

$$A_{i,j,n} = \sum_{m=0}^{n} y_{i,m} y_{j,n-m},\ n \geq m,\ \ n = 0,1,\ldots\ .$$

Now, for $m \geq 1$ The solution of the mth-order deformation Eq. (19) is

$$y_{i,m}(x) = \chi_m y_{i,m-1}(x) + h_i \int_0^x R_{i,m}(\overrightarrow{y}_{i,m-1})\, d\tau + c_i, \quad (20)$$

where the constants of integration c_i are determined by the given initial conditions in the system (15). We now successively obtain the iterations $y_{i,m}(x)$. Thus, the approximate solutions in a series form given by HAM is

$$y_i(x) = y_{i,0}(x) + \sum_{m=1}^{4} y_{i,m}(x), \quad (21)$$

therefore, the series solutions obtained when $h = -1$.

VIM: A correction functional of the system (17) is an iteratively described VIM

$$\begin{aligned} y_{1,m+1}(x) = y_{1,m}(x) + \int_0^x \lambda_1(t) \Big[& y'_{1,m} + \alpha \widetilde{y}_{1,m} \\ & - \beta \widetilde{y}_{2,m} - \gamma \widetilde{y}_{3,m} - \mu \widetilde{y}_{4,m} \\ & - \frac{1}{4}\left(\widetilde{y}_{1,m}^2 + \widetilde{y}_{2,m}^2 + \widetilde{y}_{3,m}^2 + \widetilde{y}_{4,m}^2\right) \\ & - \frac{1}{2}\left(\widetilde{y}_{1,m}\widetilde{y}_{2,m} + \widetilde{y}_{1,m}\widetilde{y}_{3,m} + \widetilde{y}_{1,m}\widetilde{y}_{4,m}\right) \\ & + \frac{1}{2}\left(\widetilde{y}_{2,m}\widetilde{y}_{3,m} + \widetilde{y}_{2,m}\widetilde{y}_{4,m} + \widetilde{y}_{3,m}\widetilde{y}_{4,m}\right) \Big] dt, \end{aligned}$$

$$\begin{aligned} y_{2,m+1}(x) = y_{2,m}(x) + \int_0^x \lambda_2(t) \Big[& y'_{2,m} - \beta \widetilde{y}_{1,m} \\ & + \alpha \widetilde{y}_{2,m} + \mu \widetilde{y}_{3,m} + \gamma \widetilde{y}_{4,m} \\ & - \frac{1}{4}\left(\widetilde{y}_{1,m}^2 + \widetilde{y}_{2,m}^2 + \widetilde{y}_{3,m}^2 + \widetilde{y}_{4,m}^2\right) \\ & - \frac{1}{2}\left(\widetilde{y}_{1,m}\widetilde{y}_{2,m} - \widetilde{y}_{1,m}\widetilde{y}_{3,m} - \widetilde{y}_{1,m}\widetilde{y}_{4,m}\right) \\ & - \frac{1}{2}\left(\widetilde{y}_{2,m}\widetilde{y}_{3,m} + \widetilde{y}_{2,m}\widetilde{y}_{4,m} - \widetilde{y}_{3,m}\widetilde{y}_{4,m}\right) \Big] dt, \end{aligned}$$

$$\begin{aligned} y_{3,m+1}(x) = y_{3,m}(x) + \int_0^x \lambda_3(t) \Big[& y'_{3,m} - \gamma \widetilde{y}_{1,m} \\ & + \mu \widetilde{y}_{2,m} + \alpha \widetilde{y}_{3,m} + \beta \widetilde{y}_{4,m} \\ & - \frac{1}{4}\left(\widetilde{y}_{1,m}^2 + \widetilde{y}_{2,m}^2 + \widetilde{y}_{3,m}^2 + \widetilde{y}_{4,m}^2\right) \\ & + \frac{1}{2}\left(\widetilde{y}_{1,m}\widetilde{y}_{2,m} - \widetilde{y}_{1,m}\widetilde{y}_{3,m} + \widetilde{y}_{1,m}\widetilde{y}_{4,m}\right) \\ & - \frac{1}{2}\left(\widetilde{y}_{2,m}\widetilde{y}_{3,m} - \widetilde{y}_{2,m}\widetilde{y}_{4,m} + \widetilde{y}_{3,m}\widetilde{y}_{4,m}\right) \Big] dt, \end{aligned}$$

$$\begin{aligned} y_{4,m+1}(x) = y_{4,m}(x) + \int_0^x \lambda_4(t) \Big[& y'_{4,m} - \mu \widetilde{y}_{1,m} \\ & + \gamma \widetilde{y}_{2,m} + \beta \widetilde{y}_{3,m} + \alpha \widetilde{y}_{4,m} \\ & - \frac{1}{4}\left(\widetilde{y}_{1,m}^2 + \widetilde{y}_{2,m}^2 + \widetilde{y}_{3,m}^2 + \widetilde{y}_{4,m}^2\right) \\ & + \frac{1}{2}\left(\widetilde{y}_{1,m}\widetilde{y}_{2,m} + \widetilde{y}_{1,m}\widetilde{y}_{3,m} - \widetilde{y}_{1,m}\widetilde{y}_{4,m}\right) \\ & + \frac{1}{2}\left(\widetilde{y}_{2,m}\widetilde{y}_{3,m} - \widetilde{y}_{2,m}\widetilde{y}_{4,m} - \widetilde{y}_{3,m}\widetilde{y}_{4,m}\right) \Big] dt, \quad (22) \end{aligned}$$

where $\lambda_i(t)$ $(i = 1,\ldots,4)$ are general Lagrange multipliers and $\widetilde{y}_{i,m}$ $(i = 1,\ldots,4)$ denote restricted variations. Then, we have

$$\begin{aligned} \delta y_{1,m+1}(x) = \delta y_{1,m}(x) + \delta \int_0^x \lambda_1(t) \Big[& y'_{1,m} + \alpha \widetilde{y}_{1,m} \\ & - \beta \widetilde{y}_{2,m} - \gamma \widetilde{y}_{3,m} - \mu \widetilde{y}_{4,m} - \frac{1}{4}\big(\widetilde{y}_{1,m}^2 \\ & + \widetilde{y}_{2,m}^2 + \widetilde{y}_{3,m}^2 + \widetilde{y}_{4,m}^2\big) - \frac{1}{2}\big(\widetilde{y}_{1,m}\widetilde{y}_{2,m} \\ & + \widetilde{y}_{1,m}\widetilde{y}_{3,m} + \widetilde{y}_{1,m}\widetilde{y}_{4,m}\big) + \frac{1}{2}\big(\widetilde{y}_{2,m}\widetilde{y}_{3,m} \\ & + \widetilde{y}_{2,m}\widetilde{y}_{4,m} + \widetilde{y}_{3,m}\widetilde{y}_{4,m}\big)\Big] dt = 0, \end{aligned}$$

$$\begin{aligned} \delta y_{2,m+1}(x) = \delta y_{2,m}(x) + \delta \int_0^x \lambda_2(t) \Big[& y'_{2,m} - \beta \widetilde{y}_{1,m} \\ & + \alpha \widetilde{y}_{2,m} + \mu \widetilde{y}_{3,m} + \gamma \widetilde{y}_{4,m} - \frac{1}{4}\big(\widetilde{y}_{1,m}^2 \\ & + \widetilde{y}_{2,m}^2 + \widetilde{y}_{3,m}^2 + \widetilde{y}_{4,m}^2\big) - \frac{1}{2}\big(\widetilde{y}_{1,m}\widetilde{y}_{2,m} \\ & - \widetilde{y}_{1,m}\widetilde{y}_{3,m} - \widetilde{y}_{1,m}\widetilde{y}_{4,m}\big) - \frac{1}{2}\big(\widetilde{y}_{2,m}\widetilde{y}_{3,m} \\ & + \widetilde{y}_{2,m}\widetilde{y}_{4,m} - \widetilde{y}_{3,m}\widetilde{y}_{4,m}\big)\Big] dt = 0, \end{aligned}$$

$$\delta y_{3,m+1}(x) = \delta y_{3,m}(x) + \delta \int_0^x \lambda_3(t)\,[y'_{3,m} - \gamma \tilde{y}_{1,m} + \mu \tilde{y}_{2,m} + \alpha \tilde{y}_{3,m} + \beta \tilde{y}_{4,m} - \frac{1}{4}(\tilde{y}_{1,m}^2 + \tilde{y}_{2,m}^2 + \tilde{y}_{3,m}^2 + \tilde{y}_{4,m}^2) + \frac{1}{2}(\tilde{y}_{1,m}\tilde{y}_{2,m} - \tilde{y}_{1,m}\tilde{y}_{3,m} + \tilde{y}_{1,m}\tilde{y}_{4,m}) - \frac{1}{2}(\tilde{y}_{2,m}\tilde{y}_{3,m} - \tilde{y}_{2,m}\tilde{y}_{4,m} + \tilde{y}_{3,m}\tilde{y}_{4,m})]\,dt = 0,$$

$$\delta y_{4,m+1}(x) = \delta y_{4,m}(x) + \delta \int_0^x \lambda_4(t)\,[y'_{4,m} - \mu \tilde{y}_{1,m} + \gamma \tilde{y}_{2,m} + \beta \tilde{y}_{3,m} + \alpha \tilde{y}_{4,m} - \frac{1}{4}(\tilde{y}_{1,m}^2 + \tilde{y}_{2,m}^2 + \tilde{y}_{3,m}^2 + \tilde{y}_{4,m}^2) + \frac{1}{2}(\tilde{y}_{1,m}\tilde{y}_{2,m} + \tilde{y}_{1,m}\tilde{y}_{3,m} - \tilde{y}_{1,m}\tilde{y}_{4,m}) + \frac{1}{2}(\tilde{y}_{2,m}\tilde{y}_{3,m} - \tilde{y}_{2,m}\tilde{y}_{4,m} - \tilde{y}_{3,m}\tilde{y}_{4,m})]\,dt = 0. \quad (23)$$

Calculus of variations and integration by parts for the system (23), and noting that $\delta \tilde{y}_{i,m}(0) = 0$ $(i = 1,\dots,4)$, we get the following system

$$\begin{cases} \lambda_i'(t) = 0, \\ 1 + \lambda_i(t)|_{t=x} = 0. \end{cases} \quad (24)$$

Solving the systems (24) for $\lambda_i(t)$ yields the Lagrange multipliers $\lambda_i(t) = -1$, and the formula of variational iteration can be obtained

$$y_{1,m+1}(x) = y_{1,m}(x) - \int_0^x \Big[y'_{1,m} + \alpha y_{1,m} - \beta y_{2,m} - \gamma y_{3,m} - \mu y_{4,m} - \frac{1}{4}(y_{1,m}^2 + y_{2,m}^2 + y_{3,m}^2 + y_{4,m}^2) - \frac{1}{2}(y_{1,m}y_{2,m} + y_{1,m}y_{3,m} + y_{1,m}y_{4,m}) + \frac{1}{2}(y_{2,m}y_{3,m} + y_{2,m}y_{4,m} + y_{3,m}y_{4,m})\Big]\,dt,$$

$$y_{2,m+1}(x) = y_{2,m}(x) - \int_0^x \Big[y'_{2,m} - \beta y_{1,m} + \alpha y_{2,m} + \mu y_{3,m} + \gamma y_{4,m} - \frac{1}{4}(y_{1,m}^2 + y_{2,m}^2 + y_{3,m}^2 + y_{4,m}^2) - \frac{1}{2}(y_{1,m}y_{2,m} - y_{1,m}y_{3,m} - y_{1,m}y_{4,m}) - \frac{1}{2}(y_{2,m}y_{3,m} + y_{2,m}y_{4,m} - y_{3,m}y_{4,m})\Big]\,dt,$$

$$y_{3,m+1}(x) = y_{3,m}(x) - \int_0^x \Big[y'_{3,m} - \gamma y_{1,m} + \mu y_{2,m} + \alpha y_{3,m} + \beta y_{4,m} - \frac{1}{4}(y_{1,m}^2 + y_{2,m}^2 + y_{3,m}^2 + y_{4,m}^2) + \frac{1}{2}(y_{1,m}y_{2,m} - y_{1,m}y_{3,m} + y_{1,m}y_{4,m}) - \frac{1}{2}(y_{2,m}y_{3,m} - y_{2,m}y_{4,m} + y_{3,m}y_{4,m})\Big]\,dt,$$

$$y_{4,m+1}(x) = y_{4,m}(x) - \int_0^x \Big[y'_{4,m} - \mu y_{1,m} + \gamma y_{2,m} + \beta y_{3,m} + \alpha y_{4,m} - \frac{1}{4}(y_{1,m}^2 + y_{2,m}^2 + y_{3,m}^2 + y_{4,m}^2) + \frac{1}{2}(y_{1,m}y_{2,m} + y_{1,m}y_{3,m} - y_{1,m}y_{4,m}) + \frac{1}{2}(y_{2,m}y_{3,m} - y_{2,m}y_{4,m} - y_{3,m}y_{4,m})\Big]\,dt. \quad (25)$$

We start with the initial approximations $y_{i,0}(x) = -1$ $(i = 1,\dots,4)$ and using the formulas (25), the rest of the iterations can be obtained.

According to our testing requirements, we utilized the HAM and VIM in $0 \le x \le 50$ using size of a step $h \in [h_0, 13h_0]$, where $h_0 = 0.0002$ with the total number of subintervals (nodes) used is $N = 19292$.

In Table 1 shows the errors of our approximations HAM with the exact solutions $(\|y_{iE}(x) - y_i(x)\|_\infty)$, the errors of our approximations VIM with the exact solutions $(\|y_{iE}(x) - y_{i,4}(x)\|_\infty)$, the errors of the approximations ADM with the exact solutions $(\|y_{iE}(x) - \phi_{i,5}(x)\|_\infty)$ in [20] and those of the numerical solutions $(\|y_{iE}(x) - \psi_i(x)\|_\infty)$ with $h = 0.002$ and 25000 nodes mentioned in [19].

Problem 2

Consider the non-linear stiff system of ordinary differential equations [19,20]:

$$y_1'(x) = -1002 y_1(x) + 1000 y_2^2(x), \quad y_1(0) = 1,$$

$$y_2'(x) = y_1(x) - y_2(x) - y_2^2(x), \quad y_2(0) = 1. \quad (26)$$

The exact solution of the system (26) is

$$y_{1E}(x) = e^{-2x},$$

$$y_{2E}(x) = e^{-x}.$$

HAM: By means the standard of the HAM solving the system (26), the initial approximations $y_{i,0}(x) = 1$ and the linear operators are chosen

$$L[\phi_i(x;p)] = \frac{\partial \phi_i(x;p)}{\partial x}, \; i = 1,2$$

Table 1: Errors for problem 1 in $x = 50$

i	HAM $(n=5)$	VIM $(n=4)$	[20]	[19]
1	$2.1606E-15$	$2.8736E-16$	$2.1607E-15$	$5.8853E-07$
2	$2.1606E-15$	$2.8736E-16$	$2.1607E-15$	$5.8880E-07$
3	$2.1606E-15$	$2.8736E-16$	$2.1607E-15$	$5.8880E-07$
4	$2.1606E-15$	$2.8736E-16$	$2.1607E-15$	$5.8853E-07$

with the property $L[c_i] = 0$, where c_i are constants of integration. Furthermore, the system (26) suggest that a system of non-linear operators be defined as

$$N_1[\phi_i(x;p)] = \frac{\partial\phi_1(x;p)}{\partial x} + 1002\phi_1(x;p) - 1002\phi_2^2(x;p),$$
$$N_2[\phi_i(x;p)] = \frac{\partial\phi_2(x;p)}{\partial x} - \phi_1(x;p) + \phi_2(x;p) + \phi_2^2(x;p). \quad (27)$$

Applying the above definition, the zeroth-order deformation equation is constructed as (3) and (4), and the deformation equation of mth-order for $m \geq 1$ is constructed as

$$L[y_{i,m}(x) - \chi_m y_{i,m-1}(x)] = h_i R_{i,m}(\overrightarrow{y}_{i,m-1}), \quad (28)$$

with the initial conditions $y_{i,m}(0) = 0$, where

$$R_{1,m}(\overrightarrow{y}_{i,m-1}) = y'_{1,m-1} + 1002y_{1,m-1} - 1002A_{2,m-1},$$
$$R_{2,m}(\overrightarrow{y}_{i,m-1}) = y'_{2,m-1} - y_{1,m-1} + y_{2,m-1} + A_{2,m-1}.$$

For the non-linear term $y_2^2 = \sum_{n=0}^{\infty} A_{2,n}$, the He's polynomials as given before.
Now, for $m \geq 1$ The solution of the mth-order deformation Eq. (28) is

$$y_{i,m}(x) = \chi_m y_{i,m-1}(x) + h_i \int_0^x R_{i,m}(\overrightarrow{y}_{i,m-1})\, d\tau + c_i, \quad (29)$$

swhere the constants of integration c_i are determined by the given initial conditions in the system (26). We now successively obtain the iterations $y_{i,m}(x)$. Thus, the approximate solutions in a series form given by HAM is

$$y_i(x) = y_{i,0}(x) + \sum_{m=1}^{3} y_{i,m}(x), \quad (30)$$

therefore, the series solutions obtained when $h = -1$.

VIM: A correction functional of the system (26) is an iteratively described VIM

$$y_{1,m+1}(x) = y_{1,m}(x) + \int_0^x \lambda_1(t)\left(y'_{1,m} + 1002\tilde{y}_{1,m} - 1000\tilde{y}_{2,m}^2\right) dt,$$
$$y_{2,m+1}(x) = y_{2,m}(x) + \int_0^x \lambda_2(t)\left(y'_{2,m} - \tilde{y}_{1,m} + \tilde{y}_{2,m} + \tilde{y}_{2,m}^2\right) dt, \quad (31)$$

where $\lambda_i(t)$ are general Lagrange multipliers and $\tilde{y}_{i,m}$ denote restricted variations. Then, we have

$$\delta y_{1,m+1}(x) = \delta y_{1,m}(x) + \delta \int_0^x \lambda_1(t)\left(y'_{1,m} + 1002\tilde{y}_{1,m} - 1000\tilde{y}_{2,m}^2\right) dt = 0,$$
$$\delta y_{2,m+1}(x) = \delta y_{2,m}(x) + \delta \int_0^x \lambda_2(t)\left(y'_{2,m} - \tilde{y}_{1,m} + \tilde{y}_{2,m} + \tilde{y}_{2,m}^2\right) dt = 0. \quad (32)$$

Calculus of variations and integration by parts for the system (32), and noting that $\delta\tilde{y}_{i,m}(0) = 0$, we get the following system

$$\begin{cases} \lambda'_i(t) = 0, \\ 1 + \lambda_i(t)|_{t=x} = 0. \end{cases} \quad (33)$$

Solving the systems (33) for $\lambda_i(t)$ yields the Lagrange multipliers $\lambda_i(t) = -1$, and the formula of variational iteration can be obtained

$$y_{1,m+1}(x) = y_{1,m}(x) - \int_0^x \left(y'_{1,m} + 1002y_{1,m} - 1000y_{2,m}^2\right) dt,$$
$$y_{2,m+1}(x) = y_{2,m}(x) - \int_0^x \left(y'_{2,m} - y_{1,m} + y_{2,m} + y_{2,m}^2\right) dt. \quad (34)$$

We start with the initial approximatations $y_{1,0}(x) = y_{2,0}(x) = 1$ and using the formulas (34), we can get the rest of components.
In Table 2 reproduces the errors of our approximations HAM with the exact solutions

$(\|y_{iE}(x)-y_i(x)\|_\infty)$, the errors of our approximations VIM with the exact solutions $\left(\|y_{iE}(x)-y_{i,4}(x)\|_\infty\right)$, the errors of the approximations ADM with the exact solutions $\left(\|y_{iE}(x)-\phi_{i,4}(x)\|_\infty\right)$ in [20] and those of the numerical solutions $(\|y_{iE}(x)-\psi_i(x)\|_\infty)$ in [19].

Problem 3

Finally, consider the linear stiff system of ordinary differential equations [19,20]:

$$\begin{aligned} y_1'(x) &= -20y_1(x)-0.25y_2(x)-19.75y_3(x),\ y_1(0)=1,\\ y_2'(x) &= 20y_1(x)-20.25y_2(x)+0.25y_3(x),\ y_2(0)=0,\\ y_3'(x) &= 20y_1(x)-19.75y_2(x)-0.25y_3(x),\ y_3(0)=-1. \end{aligned} \tag{35}$$

The exact solution of the system (35) is

$$\begin{aligned} y_{1E}(x) &= \frac{1}{2}\left[e^{-0.5x}+e^{-20x}\{\cos(20x)+\sin(20x)\}\right],\\ y_{2E}(x) &= \frac{1}{2}\left[e^{-0.5x}-e^{-20x}\{\cos(20x)-\sin(20x)\}\right],\\ y_{3E}(x) &= -\frac{1}{2}\left[e^{-0.5x}+e^{-20x}\{\cos(20x)-\sin(20x)\}\right]. \end{aligned}$$

HAM: By means the standard of the HAM solving the system (35), we choose the initial approximations

$$y_{1,0}(x)=1,\ y_{2,0}(x)=0,\ y_{3,0}(x)=-1,$$

and the linear operators

$$L[\phi_i(x;p)]=\frac{\partial\phi_i(x;p)}{\partial x},\ i=1,2,3$$

with the property $L[c_i]=0$, where c_i are constants of integration. Furthermore, the system (35) suggest that a system of non-linear operators be defined as

$$\begin{aligned} N_1[\phi_i(x;p)] &= \frac{\partial\phi_1(x;p)}{\partial x}+20\phi_1(x;p)+0.25\phi_2(x;p)\\ &\quad +19.75\phi_3(x;p),\\ N_2[\phi_i(x;p)] &= \frac{\partial\phi_2(x;p)}{\partial x}-20\phi_1(x;p)+20.25\phi_2(x;p)\\ &\quad -0.25\phi_3(x;p),\\ N_3[\phi_i(x;p)] &= \frac{\partial\phi_3(x;p)}{\partial x}-20\phi_1(x;p)+19.75\phi_2(x;p)\\ &\quad +0.25\phi_3(x;p). \end{aligned} \tag{36}$$

Applying the above definition, the zeroth-order deformation equation is constructed as (3) and (4), and the deformation equation of mth-order for $m\geq 1$ is constructed as

$$L[y_{i,m}(x)-\chi_m y_{i,m-1}(x)]=h_i R_{i,m}(\overrightarrow{y}_{i,m-1}), \tag{37}$$

with the initial conditions $y_{i,m}(0)=0$, where

$$\begin{aligned} R_{1,m}(\overrightarrow{y}_{i,m-1}) &= y_{1,m-1}'+20y_{1,m-1}+0.25y_{2,m-1}\\ &\quad +19.75y_{3,m-1},\\ R_{2,m}(\overrightarrow{y}_{i,m-1}) &= y_{2,m-1}'-20y_{1,m-1}+20.25y_{2,m-1}\\ &\quad -0.25y_{3,m-1},\\ R_{3,m}(\overrightarrow{y}_{i,m-1}) &= y_{3,m-1}'-20y_{1,m-1}+19.75y_{2,m-1}\\ &\quad +0.25y_{3,m-1}. \end{aligned}$$

Now, for $m\geq 1$ The solution of the mth-order deformation Eq. (37) is

$$y_{i,m}(x)=\chi_m y_{i,m-1}(x)+h_i\int_0^x R_{i,m}(\overrightarrow{y}_{i,m-1})\,d\tau+c_i, \tag{38}$$

where the constants of integration c_i are determined by the given initial conditions in the system (38). We now successively obtain the iterations $y_{i,m}(x)$. Thus, the approximate solutions in a series form given by HAM is

$$y_i(x)=y_{i,0}(x)+\sum_{m=1}^{3}y_{i,m}(x), \tag{39}$$

therefore, the series solutions obtained when $h=-1$.

VIM: A correction functional of the system (35) is an iteratively described VIM

$$\begin{aligned} y_{1,m+1}(x) &= y_{1,m}(x)+\int_0^x\lambda_1(t)\,(y_{1,m}'+20\widetilde{y}_{1,m}\\ &\quad +0.25\widetilde{y}_{2,m}+19.75\widetilde{y}_{3,m})\,dt,\\ y_{2,m+1}(x) &= y_{2,m}(x)+\int_0^x\lambda_2(t)\,(y_{2,m}'-20\widetilde{y}_{1,m}\\ &\quad +20.25\widetilde{y}_{2,m}-0.25\widetilde{y}_{3,m})\,dt,\\ y_{3,m+1}(x) &= y_{3,m}(x)+\int_0^x\lambda_2(t)\,(y_{2,m}'-20\widetilde{y}_{1,m}\\ &\quad +19.75\widetilde{y}_{2,m}+0.25\widetilde{y}_{3,m})\,dt, \end{aligned} \tag{40}$$

where $\lambda_i(t)$ are general Lagrange multipliers and $\widetilde{y}_{i,m}$ denote restricted variations. Then, we have

$$\begin{aligned} \delta y_{1,m+1}(x) &= \delta y_{1,m}(x)+\delta\textstyle\int_0^x\lambda_1(t)\Big(y_{1,m}'+20\widetilde{y}_{1,m}\\ &\quad +0.25\widetilde{y}_{2,m}+19.75\widetilde{y}_{3,m})\,dt=0,\\ \delta y_{2,m+1}(x) &= \delta y_{2,m}(x)+\delta\textstyle\int_0^x\lambda_2(t)\Big(y_{2,m}'-20\widetilde{y}_{1,m}\\ &\quad +20.25\widetilde{y}_{2,m}-0.25\widetilde{y}_{3,m})\,dt=0,\\ \delta y_{3,m+1}(x) &= \delta y_{3,m}(x)+\delta\textstyle\int_0^x\lambda_2(t)\Big(y_{2,m}'-20\widetilde{y}_{1,m}\\ &\quad +19.75\widetilde{y}_{2,m}+0.25\widetilde{y}_{3,m})\,dt=0. \end{aligned} \tag{41}$$

Table 2: Errors for problem 2

x	h	N	i	HAM $(n=4)$	VIM $(n=4)$	[20]	[19]
1	0.002	500	1	$9.2110E-11$	$1.1015E-07$	$9.2111E-11$	$2.5606E-07$
			2	$1.2389E-10$	$1.0999E-10$	$1.2390E-10$	$8.0150E-08$
10	0.001	10000	1	$1.7231E-18$	$5.5468E-17$	$1.7232E-18$	$5.5468E-16$
			2	$1.8957E-14$	$3.7332E-18$	$1.8958E-14$	$6.0936E-12$

Calculus of variations and integration by parts for the system (41), and noting that $\delta\widetilde{y}_{i,m}(0)=0$, we get the following systems

$$\begin{cases} \lambda_i'(t)=0, \\ 1+\lambda_i(t)|_{t=x}=0. \end{cases} \tag{42}$$

Solving the systems (42) for $\lambda_i(t)$ yields the Lagrange multipliers $\lambda_i(t)=-1$, and the formula of variational iteration can be obtained

$$y_{1,m+1}(x) = y_{1,m}(x) - \int_0^x (y_{1,m}' + 20y_{1,m} + 0.25y_{2,m} + 19.75y_{3,m})\,dt,$$

$$y_{2,m+1}(x) = y_{2,m}(x) - \int_0^x (y_{2,m}' - 20y_{1,m} + 20.25y_{2,m} - 0.25y_{3,m})\,dt,$$

$$y_{3,m+1}(x) = y_{3,m}(x) - \int_0^x (y_{2,m}' - 20y_{1,m} + 19.75y_{2,m} + 0.25y_{3,m})\,dt. \tag{43}$$

We start with the initial approximatations

$$y_{1,0}(x)=1,\ y_{2,0}(x)=0,\ y_{3,0}(x)=-1,$$

and using the formulas (43), we can obtain the rest of components.

In Table 3 reproduces the errors of our approximations HAM with the exact solutions $(\|y_{iE}(x)-y_i(x)\|_\infty)$, the errors of our approximations VIM with the exact solutions $(\|y_{iE}(x)-y_{i,4}(x)\|_\infty)$, the errors of the approximations ADM with the exact solutions $(\|y_{iE}(x)-\phi_{i,4}(x)\|_\infty)$ in [20] and those of the numerical solutions $(\|y_{iE}(x)-\psi_i(x)\|_\infty)$ in [19].

In the following figures 1, 3, 5 for problem 3, we show a very good agreement between the exact solutions $(y_{iE}(x),\ i=1,2,3)$ and 3-terms of approximate solutions HAM $\left(y_i(x)=\sum_{m=0}^{2} y_{i,m}(x), i=1,2,3\right)$ with errors 6.886×10^{-7} at $x=1.2$. In the following figures 2, 4, 6 for problem 3, we show a very good agreement between the exact solutions $(y_{iE}(x),\ i=1,2,3)$ and 3-terms of approximate solutions VIM $(y_{i,3}(x),\ i=1,2,3)$ with errors 8.61×10^{-11} at $x=1.2$. We represent the approximate solutions with a continuous lines and the exact solutions with the symbol $\circ$.

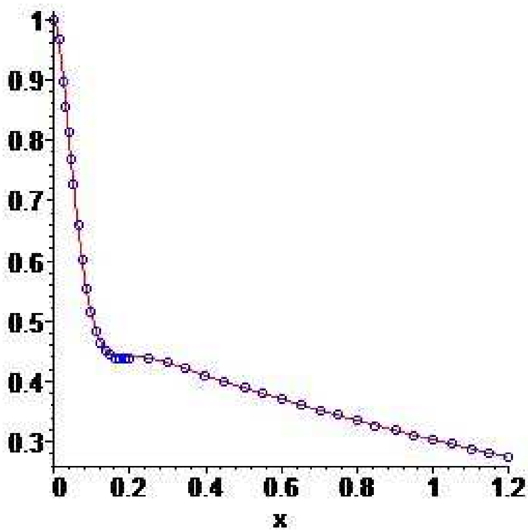

Fig. 1: Exact: $y_{1E}(x)$, HAM: $y_1(x)$

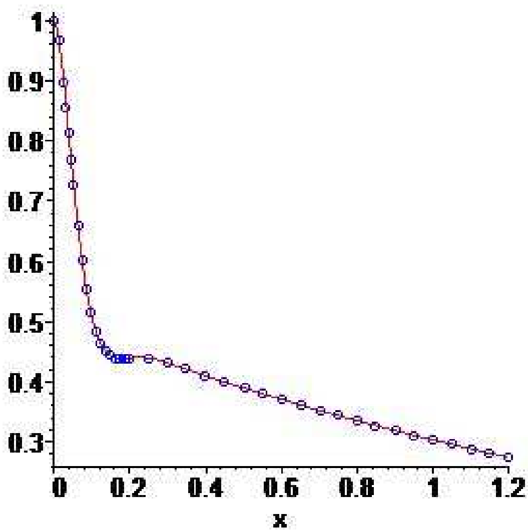

Fig. 2: Exact: $y_{1E}(x)$, VIM: $y_{1,3}(x)$

Table 3: Errors for problem 3

x	h	N	i	HAM $(n=4)$	VIM $(n=4)$	[20]	[19]
1	0.004	250	1	$4.8089E-11$	$3.2913E-14$	$4.8090E-11$	$7.2921E-05$
			2	$4.9837E-11$	$4.8413E-14$	$4.9838E-11$	$7.2921E-05$
			3	$5.1410E-11$	$7.9101E-15$	$5.1411E-11$	$7.2921E-05$
1.2	0.01	120	1	$8.6144E-10$	$9.1606E-13$	$8.6145E-10$	$3.9360E-04$
			2	$8.6183E-10$	$8.7639E-13$	$8.6184E-10$	$3.9360E-04$
			3	$8.6005E-10$	$8.4580E-13$	$8.6006E-10$	$3.9360E-04$

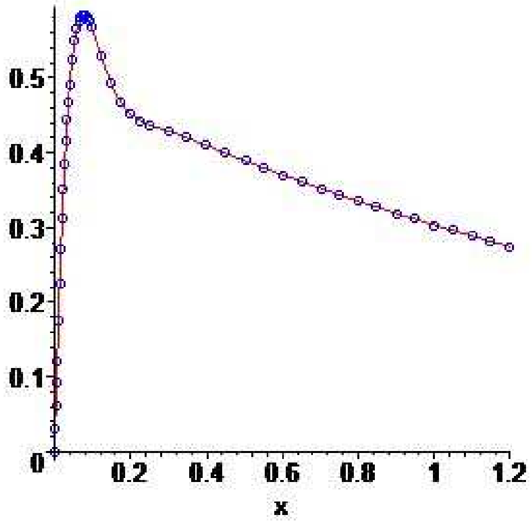

Fig. 3: Exact: $y_{2E}(x)$, HAM: $y_2(x)$

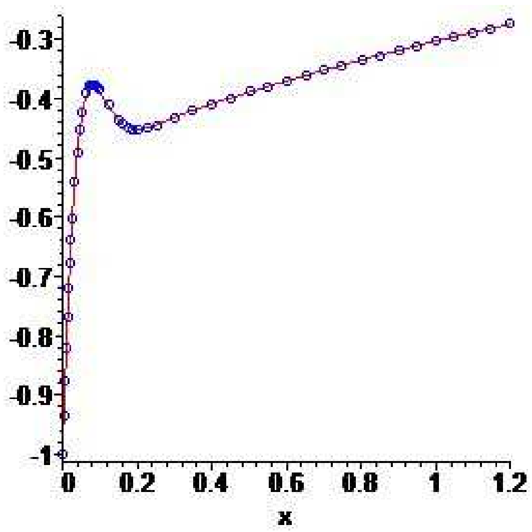

Fig. 5: Exact: $y_{3E}(x)$, HAM: $y_3(x)$

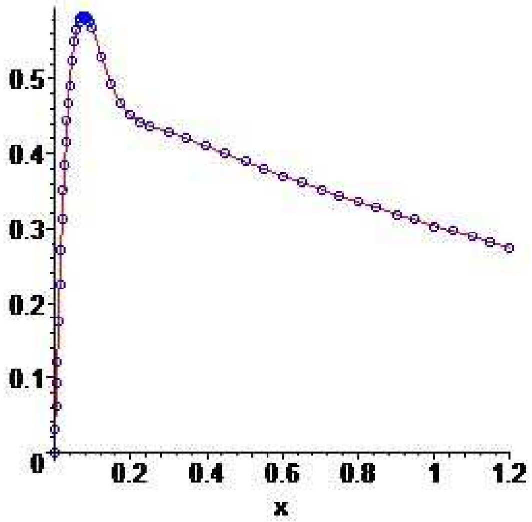

Fig. 4: Exact: $y_{2E}(x)$, VIM: $y_{2,3}(x)$

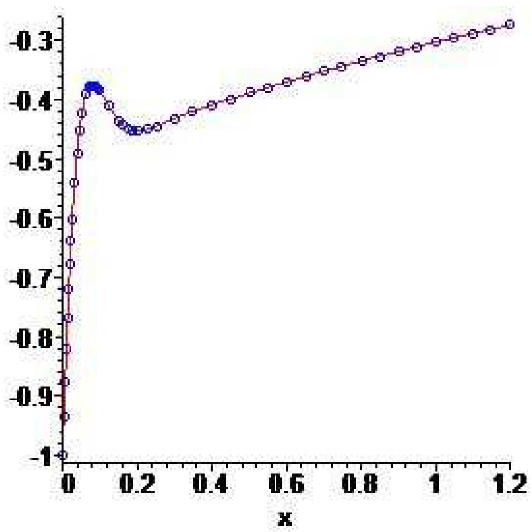

Fig. 6: Exact: $y_{3E}(x)$, VIM: $y_{3,3}(x)$

5 Conclusions

The homotopy analysis method and the variational iteration method have been put to the test by using the methods to obtain the approximate solution for three problems of stiff systems initial value problem of ordinary differential equations for both linear and non-linear systems by using size of a step with a sequence of subintervals and a suitable test of convergence. In all cases, the results obtained demonstrate that these methods are reliable and effective. It has been shown that the errors are monotically reduced with the increment of the integer n, where the errors by these methods are less than the errors presented by the Adomian decomposition method and the numerical methods.

Conflict of Interest The authors declare that they have no conflict of interest.

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