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Graph Based Approach for Error-Detecting and Correcting Codes

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Abstract: Combinatorial designs have properties that make them a significant tool for constructing good error detecting or correcting codes. In this paper, we use the fundamental properties of the incidence matrix of the graph designs (H,G,l) to construct some efficient error detecting and correcting codes. In this manner, we consider H a regular graph, G a subgraph of H and $l \geq 2$ to be an integer number. A (H,G,l) design is a collection of subgraphs $G_1, G_2, \cdots, G_b$ of H with each $G_i \simeq G; i \in \{1,2,\cdots b\}$, every edge from H appears exactly l times in that design and any two subgraphs G_i, G_j are orthogonal (have at most on edge common). We propose an approach that can generate an (H,G,l) design for some G and different H. Whenever building such a design, block graph binary codes are generated from the incidence matrix of such design. The resulting codes can be shown to be hamming codes with weights divisible by the cardinality of the edge set of G and the inner product of any two codewords ≤ 1. Using the minimum hamming distance of the constructed codes, one can efficiently detect and correct errors.

Keywords: Graph decomposition; Graph design; Orthogonal cover; Cayley graph ; Hamming codes

1 Introduction

Graphs are a vast class of combinatorial structures and are ubiquitous in that they are used to describe relationships. Graphs are used to model ecosystems, phylogenetic trees, and protein-protein interactions in biology; network flows, routing problems, and data structures in computer science and engineering; molecular structure in organic chemistry; countless problems from combinatorics, abstract algebra, matrix algebra, probability theory, and statistics. For more application of graph theory in applied mathematics and in applied science, we would refer the reader to [1] and [2]. A graph H is a pair of sets (V,E), where $V = V(H)$ is called the vertex set of H and $E = E(H) = \{\{x,y\} : x,y \in V(H)\}$ is called the edge set of H. Elements of V and E are called vertices and edges of H respectively. The cardinality of V is said to be the order of H, and the cardinality of E is said to be the size of H. If $\{x,y\}$ be an edge in H that is $\{x,y\} \in E$, we may write xy instead of $\{x,y\}$ whenever the context is clear. Many problems in combinatorics and related areas can be modeled as decomposition problems, where the goal is to decompose a whole structure into suitable smaller ones. Moreover, problems of combinatorial design may be efficiently modeled from viewpoint of graph theory. In graph theory, the problems of edge decomposition and graph covering have great attention as it is of immense importance for numerous applications in a wide range of areas [1,2,3]. In this paper, we consider the problem of designing orthogonal decompositions of the edge set of a regular graph. Throughout the paper we use $K_{m,n}$ for the complete bipartite graph with partition sets of sizes m and n, P_n for the path on n vertices, C_n for the cycle with length n, K_n for the complete graph on n vertices, $G \cup H$ for the disjoint union of G and H, and mG for m disjoint copies of G. Other terminology not defined here can be found in [4]. Orthogonal decompositions have been extensively studied for complete and complete bipartite graphs, see [5,6,7]. In [8], authors proved that:

1. There are orthogonal decompositions of H (H being a 2-regular graph except $H \in \{C_3,\ C_4,\ 2C_3\}$) by $2K_2$.
2. There are orthogonal decompositions of H (H being a 3-regular graph containing a 1-factor and without a component isomorphic to K_4) by P_4.

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3.There exist orthogonal decompositions of H (H being a 3-regular graph containing a 1-factor and $|V(H)| \geq 24$) by $P_3 \cup K_2$.

The problem of the existence of cyclic orthogonal decompositions of 4-regular circulant graphs has been studied in [9]. Here, we consider the problem of constructing orthogonal decompositions and the application of such decompositions in constructing error-detecting and correcting codes. In this manner, we are interested in graphs that are based on an algebraic group, that is the family of *Cayley graphs*. The theory of Cayley graphs provide a mathematical basis for the design of simple, undirected, uniform scalable families of interconnection networks that constitute the backbone of distributed memory parallel architectures.

Let $(\Gamma, \circledast)$ be a finite group with I as its unit element, and $A \subseteq \Gamma \backslash \{I\}$ be a subset of non-identity elements of Γ such that $A = A^{-1}$.The set A is said to be the generating set of Γ. In the following, we refer to the group $(\Gamma, \circledast)$ simply as Γ. Given the pair (Γ, A), a Cayley graph, $Cay(\Gamma, A)$ is a graph whose vertex set V consists of elements of the group Γ. The set A is said to be the connection set (generating set) of $Cay(\Gamma, A)$, That is, the Cayley graph has $|V| = |\Gamma|$ number of vertices labelled by the elements of the group Γ. Further, any two vertices x and y are adjacent, i.e. $\{x,y\} \in E(Cay(\Gamma, A))$ if and only if $y = x \circledast a$ for some $a \in A$. Thus, $E(Cay(\Gamma, A)) = \{\{x,y\} : y \circledast x^{-1} \in A\}$. Consequently, a Cayley graph is a regular graph of degree $|A|$. As $I \notin A$, then there are no loops at any vertex. Furthermore, since all elements of A are distinct, i.e., $a_i \neq a_j$ for $1 \leq i < j \leq k$, there is at most one edge labeled by a_i between any two vertices. Moreover, Cayley graph defined above is a finite, simple, undirected, and regular graph. The rest of the paper is organized as follows. Section 2 describes the fundamentals and principals of our approach used in constructing orthogonal decompositions of Cayley graph. Section 3 introduces an application of this approach in designing graph designs ($G-$designs) of Cayley graphs where G is isomorphic to the union of cycle C_l and $K_{1,m}$ with a unique vertex belongs to that cycle and $K_{1,m}$. The construction of binary codes with efficient properties in detecting and correcting errors while data transmission is presented in Section 4. Section 5 summarizes the extracted results of the paper.

2 Orthogonal graph designs of Cayley graph

Let G be a finite simple graph. A $G-$design of $Cay(\Gamma, A)$, denoted by $GD(\Gamma, G, A)$ is a triple $(\Gamma, \mathscr{G}, 2)$, where $\mathscr{G}$ is a collection of subgraphs (called blocks) of $Cay(\Gamma, A)$, each isomorphic to the graph G, and any two blocks share exactly one edge. Thus the design $GD(\Gamma, G, A)$ (or the triple $(\Gamma, \mathscr{G}, 2)$) covers the edge set of $Cay(\Gamma, A)$ twice and we may refer to it as an orthogonal $G-$design of $Cay(\Gamma, A)$. Informally, we define G-design of $Cay(\Gamma, A)$ as a collection $\mathscr{G} = \{\mathscr{J}(x) : x \in \Gamma\}$ of subgraphs of $Cay(\Gamma, A)$, all isomorphic to G, such that

1. Every edge of $Cay(\Gamma, A)$ appears in exactly two blocks of $\mathscr{G}$.
2. $\mathscr{J}(x)$ and $\mathscr{J}(y)$ share an edge if and only if x and y are adjacent in $Cay(\Gamma, A)$.

For every $x \in \Gamma$, $\mathscr{J}(x)$ has exactly $|A|$ edges. Hereafter, we will introduce an effective approach to construct $GD(\Gamma, G, A)$ of $Cay(\Gamma, A)$. This approach based on translate a given subgraph of G by the group Γ. In this approach, we will use Multiplicative notation for groups as a default. Sometimes we will switch to additive notation when groups of residue classes are involved. In our study, an edge of a $Cay(\Gamma, A)$ will often be identified with one of its arcs, so we may write xy instead of $\{x,y\}$ whenever the context is clear.

Definition 1.*Let H be a cayley graph $Cay(\Gamma, A)$. An automorphism φ of $\mathscr{G}$ is a map from $V(H)$ to itself such that $\varphi(\mathscr{J}(x)) = \mathscr{J}(\varphi(x))$ for all $x \in \Gamma$. if a coloring is assigned to edges of H, the automorphism φ will be called colour-preserving if whenever $xy \in E(H)$ the edges xy and $\varphi(x)\varphi(y))$ have the same colour.*

Let Γ be a finite group and $A \subseteq \Gamma$ a subset of Γ, such that $A^{-1} = A$ and $1 \notin A$. Consider the Cayley graph $H = Cay(\Gamma, A)$ where $E(H) = \{\{x, ax\}) : x \in \Gamma, a \in A\}$. The colour a is assigned to each arc $\{x, ax\}$ of H. Sometimes a or its inverse will be mentioned as the colours of the corresponding edge.

Definition 2.*Let Γ be a finite group and η be a permutation of Γ. The permutation η is said to be balanced if $\eta(yz)\eta(xz)^{-1} = \eta(y)\eta(x)^{-1}$ for all $x, y, z \in \Gamma$.*

All automorphisms η of Γ are balanced. Moreover, for fixed $a, b \in \Gamma$, the map $\eta(x) = axb$ is a balanced permutation.

Definition 3.*Let A be any non-empty subset of Γ and let η be a balanced permutation of Γ. For a map $f : A \to \Gamma$, the map $f^* : A \to A$ defined as $f^*(a) = f(a^{-1})^{-1} a f(a)$. We call f a starter map for (Γ, A, η) if f^* is injective and satisfies:*

$$yx^{-1} \in A \quad \text{if and only if} \; \exists \, a \in A \;\; f^*(a)\eta(x) = \eta(y). \quad (1)$$

When additive notation is used, a balanced map will be a permutation η satisfying $\eta(y+z) - \eta(x+z) = \eta(y) - \eta(x)$. Moreover, the map f^* will define as $f^*(a) = -f(-a) + a + f(a)$ and (1) will be:

$$y - x \in A \text{ if and only if } \exists \, a \in A \;\; f^*(a) + \eta(x) = \eta(y). \quad (2)$$

Let $H = Cay(\Gamma,\ A)$, η be a balanced permutation of Γ and f be a starter map for $(\Gamma,\ A,\ \eta)$. We define $\mathscr{B}(f)$ as the collection of graphs

$$\mathscr{J}(x) = \{(f(a)\eta(x),\ af(a)\eta(x)) : a \in A\}, \text{ where } x \in \Gamma.$$

Theorem 1. *Let $H = Cay(\Gamma,\ A)$. The collection $\mathscr{G}(f)$ is a $GD(\Gamma,G,A)$ and $\mathscr{J}(1)$ is the generator of such design. Moreover, the group of right translations $h \mapsto xh$ of Γ is a colour-preserving automorphism group of $\mathscr{G}(f)$.*

Proof. Firstly, we claim to check that each edge $(h,\ ah)$ of H occurs in exactly two blocks of $\mathscr{G}(f)$. If $(h,\ ah)$ is in $\mathscr{J}(x)$, then either $h = f(b)\eta(x)$ or $ah = f(b)\eta(x)$ for some $b \in A$. In the former case, $b = a$ and $\eta(x) = f(a)^{-1}h$. Conversely, with this choice of x we actually have $(h,\ ah) \in \mathscr{J}(x)$. In the latter case, $b = a^{-1}$ and $\eta(x) = f(a^{-1})^{-1}ah$. Again, we have $(h,\ ah) \in \mathscr{P}(x)$ for this choice of x. It remains to show that the elements of $\mathscr{G}$ are different. Now if $f(a)^{-1}h = f(a^{-1})^{-1}ah$ then $f(a^{-1})^{-1}af(a) = 1$, that is $f^*(a) = 1$. This is a contradiction because f is a starter map and $1 \notin A$. This proves the first claim.

Secondly, at this time we claim to prove that two blocks $\mathscr{J}(x)$ and $\mathscr{J}(y)$ of $\mathscr{G}(f)$, with $x \neq y$, $|E(\mathscr{J}(x)) \cap E(\mathscr{J}(y))| = 0$, whenever x and y are not adjacent and $|E(\mathscr{J}(x)) \cap E(\mathscr{J}(y))| = 1$ otherwise. A common edge of $\mathscr{J}(x)$ and $\mathscr{J}(y)$ takes the two forms expressed as $\{f(a)\eta(x),\ af(a)\eta(x)\} = \{f(b)\eta(y),\ bf(b)\eta(y)\}$ for suitable $a,b \in A$. Note that if $f(a)\eta(x) = f(b)\eta(y)$ then $af(a)\eta(x) = bf(b)\eta(y)$, so that $a = b$ and also $x = y$, a contradiction. Therefore we must have $f(a)\eta(x) = bf(b)\eta(y)$ and $af(a)\eta(x) = f(b)\eta(y)$. As the colour of the arc is a, we must have $abf(b)\eta(y) = f(b)\eta(y)$ Then b is the inverse of a. The latter of these two equations can thus be rewritten as $f(a^{-1})^{-1}af(a)\eta(x) = \eta(y)$. That is $f^*(a)\eta(x) = \eta(y)$. Since f is a starter map, by (1) this is not possible if x and y are nonadjacent and so the intersection of the pages $\mathscr{J}(x)$ and $\mathscr{J}(y)$ is empty in this case. If x and y are adjacent, the equation is satisfied whenever $f^*(a) = x^{-1}y$. Since now $x^{-1}y \in A$ and f^* is injective, there is exactly one $a \in A$ satisfying this condition. This proves that $\mathscr{J}(x)$ and $\mathscr{J}(y)$ intersect in exactly one edge. These two claims prove that $\mathscr{G}(f)$ is a $GD(\Gamma,G,A)$ of H. For $h \in \Gamma$ the map: $x \mapsto xh$ takes the edge $(f(a)\eta(1),\ af(a)\eta(1))$ into $(f(a)\eta(1)g,\ af(a)\eta(1)g)$, hence $\mathscr{J}(1)$ into $\eta(g)$ where $\eta(g) = \eta(1)h$. Thus the group of right translations consists of colour-preserving automorphisms of $\mathscr{G}(f)$. In particular, $\mathscr{G}(f)$ is $GD(\Gamma,G,A)$ of H by $\mathscr{J}(1)$, so $\mathscr{J}(1)$ can be seen as a generator of $GD(\Gamma,G,A)$.

Throughout the next section we switch to additive notation and use $\Gamma = \mathbb{Z}_n$ for a finite (additive) abelian group where $\mathbb{Z}_n = \{0,1,2,\cdots,n-1\}$ is the group of all residual classes modulo n.

3 Construction of $C_l \cup^v K_{1,m}$ design of $Cay(\mathbb{Z}_n,\ A)$

Let m,l be positive integers such $m < n$ and $l < n$. We will apply the approach introduced in Section 2 to build some $GD(\mathbb{Z}_n,G,A)$ of $Cay(\mathbb{Z}_n,\ A)$ where G is isomorphic to $C_l \cup^v K_{1,m}$ (the union of cycle C_l and $K_{1,m}$ where the vertex v belongs to that cycle and $K_{1,m}$).

Theorem 2. *Let n be a positive integer such that $n \geq 5$ and let $v \in \mathbb{Z}_n$. Then there exists graph design $(\mathbb{Z}_n, C_3 \cup^v K_{1,n-4},\ A)$ of $Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_n \backslash \{0\})$.*

Proof. For $n \geq 5$, $A = \mathbb{Z}_n \backslash \{0\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_n$ by

$$f(a) = \begin{cases} 0 & \text{if} \quad a = 2 \\ 4 & \text{if} \quad a = n-2, n-1 \\ 2 & \quad otherwise. \end{cases}$$

From the definition of $f(a)$; $\mathscr{J}(0)$ is isomorphic to the graph $G = C_3 \cup^2 K_{1,n-4}$ and $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in \mathscr{G}(f)$. For $a \in \{1,2,n-2,n-1\}$; $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies (2) with $\eta = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying Theorem 1, proves the claim and moreover, $\mathscr{J}(0)$ is the generator of such a design.

Example 1. Let $n = 8$, $v \in \mathbb{Z}_8$ and $A = \mathbb{Z}_8 \backslash \{0\}$. According to Theorem 2, Figure 1, shows the graph design $(\mathbb{Z}_8, C_3 \cup^v K_{1,4},\ A)$ of $Cay(\mathbb{Z}_8,\ A)$.

Theorem 3. *Let n be a positive integer such that $n \geq 6$, and let $v \in \mathbb{Z}_n$. Then there exists graph design $(\mathbb{Z}_n, C_4 \cup^v K_{1,n-7},\ A)$ of $Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_n \backslash \{0,3,n-3\})$.*

Proof. For $n \geq 5$, $A = \mathbb{Z}_n \backslash \{0,3,n-3\}$ and for each $a \in A$, $f : A \to \mathbb{Z}_n$ is defined as

$$f(a) = \begin{cases} n-1 & \text{if} \quad a = 1, n-1 \\ 2 & \quad otherwise \end{cases}$$

From the definition of $f(a)$; $\mathscr{J}(0)$ is isomorphic to the graph $G = C_4 \cup^2 K_{1,n-7}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in \mathscr{G}(f)$. For $a \in A$; $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies (2) with $\eta = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying Theorem 1, proves the claim and moreover, $\mathscr{J}(0)$ is the generator of such a design.

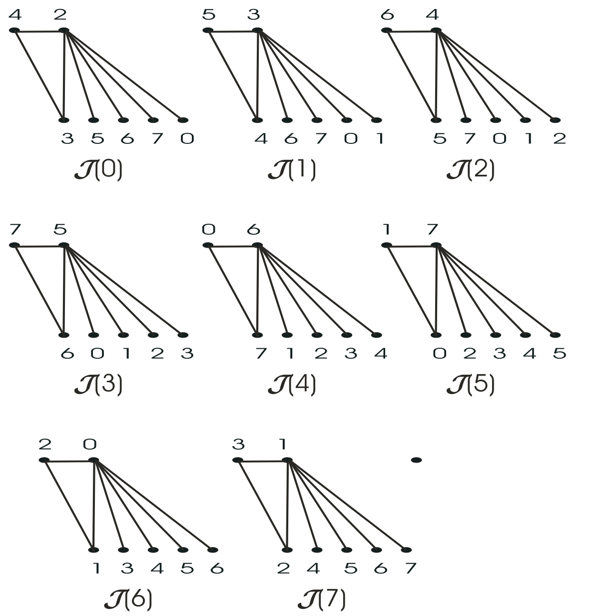

Fig. 1: $C_3 \cup^v K_{1,4}$-design for $Cay(\mathbb{Z}_8, A = \mathbb{Z}_8\backslash\{0\})$ where $v \in \mathbb{Z}_8$.

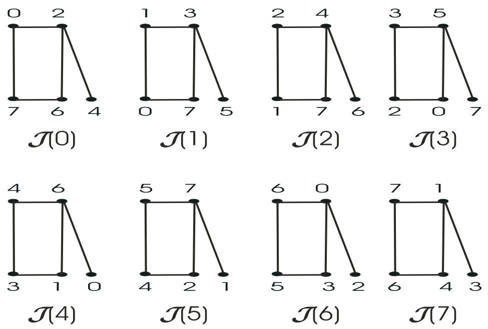

Fig. 2: $C_4 \cup^v K_{1,1}$-design for $Cay(\mathbb{Z}_8,\ A = \mathbb{Z}_8\backslash\{0,3,n-3\})$ where $v \in \mathbb{Z}_8$.

*Example 2.*Let $n = 8$, $v \in \mathbb{Z}_8$ and $A = \mathbb{Z}_8\backslash\{0,3,n-3\}$. According to Theorem 3, Figure 2, shows the graph design $(\mathbb{Z}_8, C_4 \cup^v K_{1,1},\ A)$ of $Cay(\mathbb{Z}_8,\ A)$.

Theorem 4.*Let n be a positive integer such that $n > 10$. Then there exists a graph-design $(\mathbb{Z}_n, C_5 \cup^v K_{1,n-8},\ A)$ of $Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_n\backslash\{0,4,n-4\})$.*

*Proof.*For $n > 10, A = \mathbb{Z}_n\backslash\{0,4,n-4\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_n$ by

$$f(a) = \begin{cases} 2 & \text{if} \quad a = 2, n-1 \\ 6 & \text{if} \quad a = n-2 \\ 0 & \quad otherwise. \end{cases}$$

From the definition of $f(a)$; $\mathscr{J}(0)$ is isomorphic to the graph $G = C_5 \cup^0 K_{1,n-8}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in \mathscr{G}(f)$. For $a \in \{1,2,n-2,n-1\}$; $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies (2) with $\eta = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying Theorem 1, proves the claim and moreover, $\mathscr{J}(0)$ is the generator of such a design.

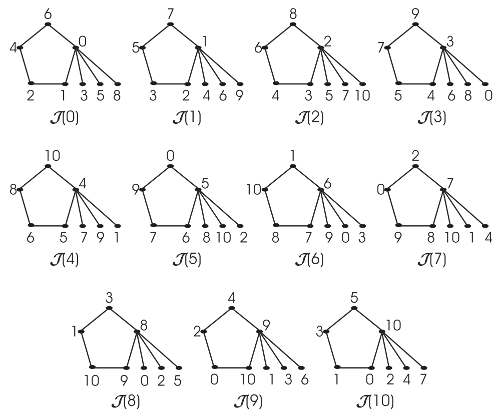

Fig. 3: $C_5 \cup^v K_{1,3}$-design for $Cay(\mathbb{Z}_{11},\ A = \mathbb{Z}_{11}\backslash\{0,4,n-4\})$ where $v \in \mathbb{Z}_{11}$.

*Example 3.*Let $n = 11, v \in \mathbb{Z}_{11}$ and $A = \mathbb{Z}_{11}\backslash\{0,4,n-4\}$. According to Theorem 4, Figure 3, shows the graph design $(\mathbb{Z}_{11}, C_5 \cup^v K_{1,3},\ A)$ of $Cay(\mathbb{Z}_{11},\ A)$.

Theorem 5.*Let n be a positive integer such that $n > 7$ and let $v \in \mathbb{Z}_n$. Then there exists a graph-design $(\mathbb{Z}_n, C_6 \cup^v K_{1,n-7},\ A)$ of $Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_n\backslash\{0\})$.*

*Proof.*For $n > 7, A = \mathbb{Z}_n\backslash\{0\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_n$ by

$$f(a) = \begin{cases} 4 & \text{if} \quad a = 1, n-2 \\ 0 & \text{if} \quad a = 2, 3 \\ 6 & \text{if} \quad a = n-3, n-1. \\ 3 & \quad otherwise. \end{cases}$$

From the definition of $f(a)$, $\mathscr{J}(0)$ is isomorphic to the graph $G = C_6 \cup^3 K_{1,n-7}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in \mathscr{G}(f)$. For $a \in \{1,2,3,n-3,n-2,n-1\}$; $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies (2) with $\eta = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying Theorem 1, proves the claim and moreover, $\mathscr{J}(0)$ is the generator of such a design.

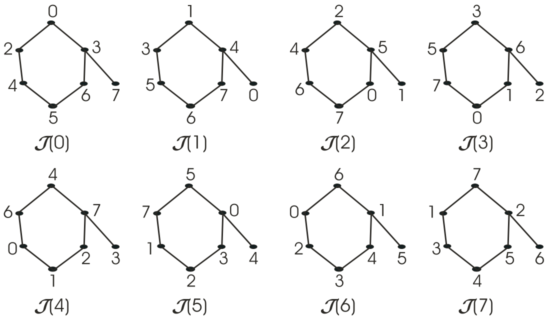

Fig. 4: $C_6 \cup^{v} K_{1,1}$-design for $Cay(\mathbb{Z}_8,\ A = \mathbb{Z}_8 \backslash \{0\})$ where $v \in \mathbb{Z}_8$.

*Example 4.*Let $n = 8, v \in \mathbb{Z}_8$ and $A = \mathbb{Z}_8 \backslash \{0\}$. According to Theorem 5, Figure 4, shows the graph design $(\mathbb{Z}_8, C_6 \cup^{v} K_{1,1},\ A)$ of $Cay(\mathbb{Z}_8,\ A)$.

Theorem 6.*Let n,m be positive integers such that $n = 2m+1$, and $m > 5$. Assume that $v \in \mathbb{Z}_n$, then there exists a graph-design $(\mathbb{Z}_n, C_7 \cup^{v} K_{1,2m-11},\ A)$ of $Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_{2m+1} \backslash \{0,4,6,2m-5,2m-3\})$.*

*Proof.*For $n = 2m+1$, $m > 5$, $A = \mathbb{Z}_{2m+1} \backslash \{0,4,6,2m-5,2m-3\}$, and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m+1}$ by

$$f(a) = \begin{cases} 1 & \text{if} \quad a = 1,\ 2m \\ 2 & \text{if} \quad a = 2 \\ 6 & \text{if} \quad a = 2m-1. \\ 2m-1 & \quad a = 8,\ 2m-7 \\ 0 & \quad otherwise. \end{cases}$$

From the definition of $f(a)$; $\mathscr{J}(0)$ is isomorphic to the graph $G = C_7 \cup^{0} K_{1,2m-11}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in \mathscr{G}(f)$. For $a \in \{2, 2m-1\}$; $f^{*}(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^{*}(a) = f(a) - f(-a) + a = a$. And hence f^{*} is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$.Therefore f^{*} satisfies (2) with $\eta = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying Theorem 1, proves the claim and moreover, $\mathscr{J}(0)$ is the generator of such a design.

Theorem 7.*Let n be a positive integer such that $n \geq 14$ and let $v \in \mathbb{Z}_n$. Then there exists a graph-design $(\mathbb{Z}_n, C_8 \cup^{v} K_{1,n-13},\ A)$ of $Cay(\mathbb{Z}_n, A = \mathbb{Z}_n \backslash \{0,2,4,n-2,n-4\})$.*

*Proof.*For $n \geq 14$, $A = \mathbb{Z}_n \backslash \{0,2,4,n-2,n-4\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_n$ by

$$f(a) = \begin{cases} 1 & \text{if} \quad a = 1,\ n-1 \\ 5 & \text{if} \quad a = 3,\ n-3 \\ 3 & \text{if} \quad a = 5,\ n-5 \\ n+6-a & \quad otherwise. \end{cases}$$

From the definition of $f(a)$; $\mathscr{J}(0)$ is isomorphic to the graph $G = C_8 \cup^{6} K_{1,n-13}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in \mathscr{G}(f)$. For $a \in \{1,3,5,n-5,n-3,n-1\}$; $f^{*}(a) = f(a) - f(-a) + a = a$; for otherwise, $f^{*}(a) = f(a) - f(-a) + a = -a$. And hence f^{*} is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^{*} satisfies (2) with $\eta = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying Theorem 1, proves the claim and moreover, $\mathscr{J}(0)$ is the generator of such a design.

Theorem 8.*Let n,m be positive integers such that $n = 2m+1$, and $m \geq 7$. Assume that $v \in \mathbb{Z}_n$, then there exists a graph-design $(\mathbb{Z}_n, C_9 \cup^{v} K_{1,2m-13},\ A)$ of $Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_{2m+1} \backslash \{0,4,8,2m-7,2m-3\})$.*

*Proof.*For $n = 2m+1$, $m \geq 7$, $A = \mathbb{Z}_{2m+1} \backslash \{0,4,8,2m-7,2m-3\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m+1}$ by

$$f(a) = \begin{cases} 1 & \text{if} \quad a = 1,\ 2m \\ 2m-3 & \text{if} \quad a = 2,\ 2m-1 \\ 2 & \text{if} \quad a = 3 \\ 3 & \text{if} \quad 5,\ 2m-4 \\ 8 & \text{if} \quad 2m-2 \\ 0 & \quad otherwise. \end{cases}$$

From the definition of $f(a)$; $\mathscr{J}(0)$ is isomorphic to the graph $G = C_9 \cup^{0} K_{1,2m-13}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in \mathscr{G}(f)$. For $a \in \{3, 2m-2\}$; $f^{*}(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^{*}(a) = f(a) - f(-a) + a = a$. And hence f^{*} is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^{*} satisfies (2) with $\eta = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying Theorem 1, proves the claim and moreover, $\mathscr{J}(0)$ is the generator of such a design.

4 Codes and $GD(\mathbb{Z}_n, G, A)$ of $Cay(\mathbb{Z}_n,\ A)$

For a graph design $GD(\mathbb{Z}_n, G, A)$ of $Cay(\mathbb{Z}_n,\ A)$, we may associate an incidence matrix $\mathscr{I}$ .The incidence matrix $\mathscr{I}$ is a $n \times \frac{n\,|A|}{2}$ binary matrix. The rows of $\mathscr{I}$ correspond to the blocks of the collection $\mathscr{G}$ while the columns of $\mathscr{I}$ correspond to the edges of $Cay(\mathbb{Z}_n,\ A)$.The entry $w_{i,j}$ of $\mathscr{I}$ has value equals 1 if and only if the edge corresponds to the column, j belongs to the block corresponding to the row i, and $w_{i,j} = 0$ otherwise. The rows and columns of $\mathscr{I}$ can be used to construct binary hamming codes. Here, we consider the rows binary codes (C_r) constructed from the rows of $\mathscr{I}$. We refer to the code generated from the

Fig. 5: A flow chart for the process of building codes from the graph design.

rows of $\mathscr{I}$ as $\mathscr{R}$. Each row in $\mathscr{I}$ is a codeword in $\mathscr{R}$.Thus $\mathscr{R}$ has n codewords each of length $\frac{n\,|A|}{2}$.

$$C_r = \mathscr{R} = \{\underbrace{w_{1,1}w_{1,2}\ldots w_{1,j}}_{1^{st}\text{ codeword}}, \underbrace{w_{2,1}w_{2,2}\ldots w_{2.j}}_{2^{nd}\text{ codeword}}, \ldots, \underbrace{w_{n,1}w_{n,2}\ldots w_{n,j}}_{n^{th}\text{ codewords}}\}$$

A flow chart for the process of building codes from the graph design is illustrated in Figure 5.

Since every block $\mathscr{J}(x)$ where $x \in \mathbb{Z}_n$ has the same number of edges that is $|A|$, every row in $\mathscr{I}$ has the same number of ones. Moreover, any two rows have exactly one position of ones common, then the minimum distance $d(\mathscr{R})$ of the code $\mathscr{R}$ is $2(|A|-1)$. Following [10] and [11], $\mathscr{R}$ can detect up to $d(\mathscr{R})-1 = 2|A|-3$ errors, and correct up to $\left\lfloor \frac{d(\mathscr{R})-1}{2} \right\rfloor = \left\lfloor \frac{2|A|-3}{2} \right\rfloor$. Let S,T be two distinct codewords in $\mathscr{R}$. The distance $d(S,T)$ between S,T defines the number of positions where S and T differ. The minimum distance of the code $\mathscr{R}$ is $d(\mathscr{R}) = \min\{d(S,T) : S,T \in R,\ S \neq T\}$. For binary codes, the minimum distance is a significant notation. Such distance plays an important role in checking whether the binary code can detect or correct errors [10, 11, 12, 13].

For more illustration, the construction of the code $\mathscr{R}$ from an incidence matrix $\mathscr{I}$ of a design $GD(\mathbb{Z}_n, G, A)$ of $Cay(\mathbb{Z}_n, A)$ is described in the following example.

*Example 5.*Let $n = 6$ and $A = \mathbb{Z}_n \backslash \{0\}$ then the edge set of $Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_n \backslash \{0\})$ is the set $E(Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_n \backslash \{0\})) = \{01, 02, 03, 04, 05, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45\}$

Following Theorem 2 and Theorem 1, we can build a $GD(\mathbb{Z}_6, C_3 \cup^v K_{1,n-4},\ A)$ of $Cay(\mathbb{Z}_6,\ A = \mathbb{Z}_6 \backslash \{0\})$. In such a design $\mathscr{G}(f) = \{\mathscr{J}(0),\ \mathscr{J}(1),\ \mathscr{J}(2),\ \mathscr{J}(3),\ \mathscr{J}(4),\ \mathscr{J}(5)\}$. Furthermore,

$$E(\mathscr{J}(0)) = \{23, 02, 25, 42, 43\},$$
$$E(\mathscr{J}(1)) = \{34, 13, 30, 53, 54\},$$
$$E(\mathscr{J}(2)) = \{45, 24, 41, 04, 05\},$$
$$E(\mathscr{J}(3)) = \{50, 35, 52, 15, 10\},$$
$$E(\mathscr{J}(4)) = \{01, 40, 03, 20, 21\},$$
$$E(\mathscr{J}(5)) = \{12, 51, 14, 31, 32\}.$$

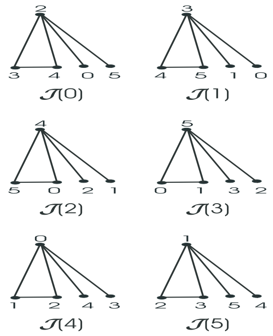

Fig. 6: $C_3 \cup^v K_{1,2}$-design for $Cay(\mathbb{Z}_6,\ A = \mathbb{Z}_6 \backslash \{0\})$.

Consequently, The incidence matrix $\mathscr{I}$ of $GD(\mathbb{Z}_n, C_3 \cup^v K_{1,n-4},\ A)$ is a 6×15 binary matrix.

$$\begin{matrix} \mathscr{J}(0) \rightarrow \\ \mathscr{J}(1) \rightarrow \\ \mathscr{J}(2) \rightarrow \\ \mathscr{J}(3) \rightarrow \\ \mathscr{J}(4) \rightarrow \\ \mathscr{J}(5) \rightarrow \end{matrix} \underbrace{\begin{pmatrix} 0&1&0&0&0&0&0&0&0&1&1&1&1&0&0 \\ 0&0&1&0&0&0&1&0&0&0&0&0&1&1&1 \\ 0&0&0&1&1&0&0&1&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0&1&0&0&1&0&1&0 \\ 1&1&1&1&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&1&1&1&0&0&0&0&0 \end{pmatrix}}_{\text{The incidence matrix I for the graph design in Example 5}}.$$

Every column from $\mathscr{I}$ corresponds to a unique edge from the set $E(Cay(\mathbb{Z}_n,\ A = \mathbb{Z}_n \backslash \{0\}))$ and every row refers to a unique block from the collection $\mathscr{G}(f)$. Observing the incidence matrix $\mathscr{I}$. We can see that the properties of a $GD(\mathbb{Z}_n, G, A)$ of $Cay(\mathbb{Z}_n,\ A)$ are valid on $\mathscr{I}$. Every edge from $Cay(\mathbb{Z}_n,\ A)$ occurs in exactly two blocks. Looking at $\mathscr{I}$ each column has exactly two positions of ones. From the design, any two blocks intersect in exactly one edge, that is any two rows from $\mathscr{I}$ have exactly one common position of ones. Figure 6, shows the graph design $(\mathbb{Z}_6, C_3 \cup^v K_{1,n-4},\ A)$ of $Cay(\mathbb{Z}_6,\ A = \mathbb{Z}_6 \backslash \{0\})$.

The code

$$\mathscr{R} = \left\{ \begin{matrix} 010000000111100, 001000100000111, \\ 000110010010001, 100010001001010 \\ 111101000000000, 000001111100000 \end{matrix} \right\}$$

Thus $\mathscr{R}$ has 6 codewords each of length 15 and the weight (the number of 1' bits in the codeword) of each codeword is 5. Besides that each codeword in $\mathscr{R}$ can be represented as $C_3 \cup^v K_{1,2}$. The minimum distance $d(\mathscr{R}) = 2(5-1) = 8$. Hence $\mathscr{R}$ can detect up to 7 errors and can correct up to 4 errors.

Hereafter, we show an application of a graph design $GD(\mathbb{Z}_n, G, A)$ of $Cay(\mathbb{Z}_n,\ A)$ in experiments design.

*Example 6.*A psychiatric clinic has 91 patients were diagnosed with depression. At some stage of treatment, the patients should partioned into a set of groups therapy. Such partions have to satisfy the below constraints:

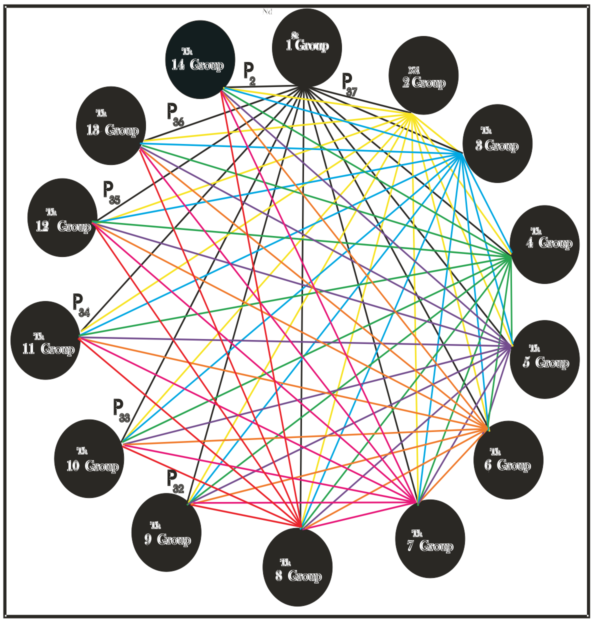

Fig. 7: $Cay(\mathbb{Z}_{14},\ A = \mathbb{Z}_{14}\backslash\{0\})$

1. Each group consistis of 13 patients.
2. For medical reasons, each patient has to join two different groups with totally different people in each group.
3. For more interaction among patients of the same group, the pineapple shape of the group is preferred.

By the help of cayley graph and its graph designs we show how to solve this problem. Firstly we consider cayley graph to represent the groups and the relations among them. The vertices of cayley graph would refer to the groups and the edges represent the relations between groups. Hence, $Cay(\mathbb{Z}_{14},\ A = 13)$ is the graph representation for this problem and such representation is illustrated in Figure 7. Now we claim to build a graph design of the obtained cayley graph. Such design produces 14 blocks (groups) each group has $|A| = 13$ patients (edges) . To have a guarantee that the structure of each group is a pineapple, we emphasize on building a graph design $GD\left(\mathbb{Z}_{14}, C_3 \cup^2 K_{1,10}, A\}\right)$ of $Cay(\mathbb{Z}_{14},\ A = \mathbb{Z}_{14}\backslash\{0\})$. using the starter map f defined as

$$f(a) = \begin{cases} 0 & \text{if} \quad a = 2, \\ 4 & \text{if} \quad a \in \{n-2, n-1\}, \\ 2 & \quad otherwise. \end{cases}$$

The whole design of the required groups (14 groups therapy) and the members (Patients such that each patient has a unique number from the set $\{1,2,\cdots,91\}$) of each group is presented in Table 1.

Table 1: The whole design of groups required in Example 6

1^{st} group	2, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37
2^{nd} group	3, 15, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47
4^{th} group	4,16, 27, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
5^{th} group	5,17, 28, 38, 56, 57, 58, 59, 60, 6, 62, 63, 64
6^{th} group	6, 18, 29, 39, 48, 64, 65, 66, 67, 68, 69, 70, 71
7^{th} group	7, 19, 30, 40, 49, 57, 71, 72, 73, 74, 75, 76, 77
8^{th} group	8, 20, 31, 41, 50, 58, 68, 77, 78, 79, 80, 81, 82
9^{th} group	9, 21, 32, 42, 51, 59, 66, 72, 82, 83, 84, 85, 86
10^{th} group	10, 22, 33, 43, 52, 60, 67, 73, 78, 86, 87, 88, 89
11^{th} group	11, 23, 34, 44, 53, 61, 68, 74, 79, 83, 89, 90, 91
12^{th} group	12, 13, 24, 35, 45, 54, 62, 69, 75, 80, 84, 87, 91
13^{th} group	1,13, 24, 36, 46, 55, 63, 70, 76, 81, 85, 88, 90
14^{th} group	1,2,3,4,5,6,7,8,9,10,11,12,14

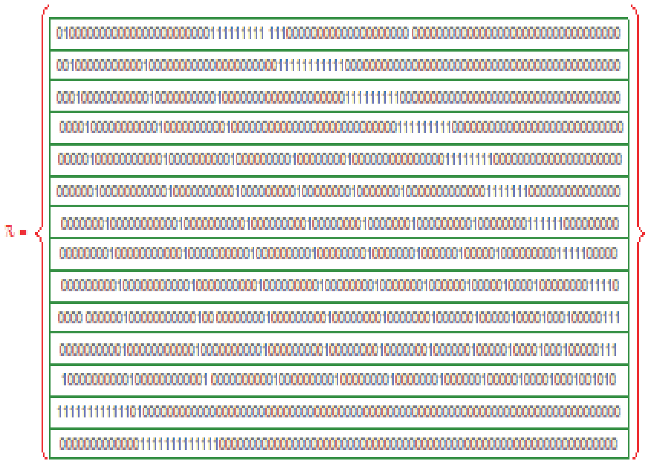

Fig. 8: Code $\mathscr{R}$ induced from the graph design of example 6.

This design can be converted into a code $\mathscr{R}$, see Figure 8. Thus, the code $\mathscr{R}$ helps in preserving privacy of data while sending and receiving among different departments of the clinic and intelligible only to those concerned with these patients.

5 Conclusion

The paper investigates the the graph design (H, G, l). We proposed an approach that can generate an (H, G, l) graph design for some graphs G and cayley graph H. Using the presented approach, we constructed graph designs ($G-$design) of cayley graphs where G is isomorphic to $C_l \cup^{\nu} K_{1,m}$. Such designs are summarized in Table 2. In addition, we studied the application of $G-$designs in constructing binary codes with weights divisible by the cardinality of the edges set of G. We showed that the induced binary codes from designs have efficient properties in detecting and correcting errors that may occur during data transmission.

Table 2: Summary of the introduced graph designs

$H = Cay(\mathbb{Z}_n, A)$	GD $(\mathbb{Z}_n, G, A)$
$A = \mathbb{Z}_n \backslash \{0\}$	$(\mathbb{Z}_n, C_3 \cup^2 K_{1,n-4}, A)$
$A = \mathbb{Z}_n \backslash \{0, 3, n-3\}$	$(\mathbb{Z}_n, C_4 \cup^2 K_{1,n-7}, A)$
$A = \mathbb{Z}_n \backslash \{0, 4, n-4\}$	$(\mathbb{Z}_n, C_5 \cup^0 K_{1,n-8}, A)$
$A = \mathbb{Z}_n \backslash \{0\}$	$(\mathbb{Z}_n, C_6 \cup^3 K_{1,n-7}, A)$
$A = \mathbb{Z}_{2m+1} \backslash \{0, 4, 6, 2m-5, 2m-3\})$	$(\mathbb{Z}_n, C_7 \cup^0 K_{1,2m-11}, A)$
$A = \mathbb{Z}_n \backslash \{0, 2, 4, n-2, n-4\})$	$(\mathbb{Z}_n, C_8 \cup^6 K_{1,n-13}, A)$
$A = \mathbb{Z}_{2m+1} \backslash \{0, 4, 8, 2m-7, 2m-3\})$	$(\mathbb{Z}_n, C_9 \cup^0 K_{1,2m-13}, A)$

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