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Fractional version of Hermite-Hadamard-Mercer inequalities for convex stochastic processes via ψ_κ-Riemann-Liouville fractional integrals and its applications

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Abstract: In the present paper, authors derive some new Hermite-Hadamard-Mercer type inequalities for convex stochastic processes using ψ_κ-Riemann-Liouville fractional integrals. Furthermore, to civilized the paper we prove different lemmas to present unique refinements of Hermite-Hadamard-Mercer type inequalities. Also, we discuss some special cases of our proven results. These new inequalities yield several generalizations of previously known results. Finally, we develop some applications to special means.

Keywords: convex stochastic processes, Jensen inequality, Jensen-Mercer inequality, ψ_κ-Riemann-Liouville fractional integrals, Hermite-Hadamard inequality, improved power-mean inequality, Hölder inequality

1 Introduction and preliminaries

Despite the fact that fractional calculus has the same historic antique origins of the classical calculus, it has become of extreme interest in the past few years for the researchers, in several and numerous science areas. In recent years, the theory and applications of fractional derivatives and integrals [1] have been widely formulated by different pure and applied mathematicians. Among a lot of determinations, we can assert that the researcher's community noticed that fractional differential equations and fractional integral results yield a natural framework for the description and the research of real phenomena, for example, those that exist in ecology, biology, and neuroscience [2, 3, 4].

The Hermite-Hadamard inequality is the fundamental result for convex functions along with a natural geometrical interpretation and has several applications. Different mathematicians have been concerned about their efforts to extend, generalize and refine it for numerous classes of functions such as convex mappings. In literature, C. Hermite and J. Hadamard discovered these inequalities for convex functions [5, 6, 7]. These inequalities state that: Let $\xi : I \to \mathbb{R}$ be a convex function in I and $x_1, y_1 \in I$ for $x_1 < y_1$, then

$$\xi\left(\frac{x_1+y_1}{2}\right) \leq \frac{1}{y_1-x_1}\int_{x_1}^{y_1}\xi(q)dq \leq \frac{\xi(x_1)+\xi(y_1)}{2}.$$

For more recent developments of Hermite-Hadamard inequality, one can consult [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

The stochastic process may be defined in a general sense and it has different applications such as mathematics, engineering, physics, and economics, therefore K. Nikodem introduced the idea of convex stochastic processes and also described their properties in 1980 [19, 20]. Also, A. Skorownski presented some more results using convex stochastic processes which generalize to some known results about classical convex mappings in [21, 22]. Later, D. Kotrys developed Hermite-Hadamard inequality using convex stochastic processes [23, 24, 25]. The well-known Hermite-Hadamard inequality for convex stochastic processes is as follows: Let $\xi : I \to \mathbb{R}$ be a convex

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stochastic process in the interval I and $x_1, y_1 \in I$ with $x_1 < y_1$, then holds almost everywhere

$$\xi\left(\frac{x_1+y_1}{2},\cdot\right) \leq \frac{1}{y_1-x_1}\int_{x_1}^{y_1}\xi(q,\cdot)dq \leq \frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2}.$$

For more on these inequalities, we refer [26, 27, 28, 29].

Definition 1.[30] *A stochastic process is a family of random variables $\xi(x_1)$ parametrized by $x_1 \in I$, with $I \subset \mathbb{R}$. When $I = \{1,2,...\}$, then $\xi(x_1)$ is known as a stochastic process in discrete time (i.e. a sequence of random variables). When $I \in \mathbb{R}$ $(I = [0,\infty))$, then $\xi(x_1)$ is a stochastic process in continuous time.*

For every $w \in \Omega$ the function

$$I \ni x_1 \longmapsto \xi(x_1,w)$$

is termed as a path or sample path of $\xi(x_1)$.

Definition 2.[30] *A family $\mathbb{F}_{x_1}$ of σ-fields on Ω parametrized by $x_1 \in I$, where $I \subset \mathbb{R}$, is called a filtration if*

$$F_{y_1} \subset \mathbb{F}_{x_1} \subset F$$

for any $y_1, x_1 \in I$ such that $y_1 \leq x_1$.

Definition 3.[30] *A sochastic process $\xi(x_1)$ parametrized by $x_1 \in T$ is termed as a martingale (supermartingale, submartingale) with respect to a filtration $\mathbb{F}_{x_1}$ if*

1) $\xi(x_1)$ is integrable for each $x_1 \in I$;

2) $\xi(x_1)$ is F_l-measurable for each $x_1 \in I$;

3) $\xi(y_1) = E(\xi(x_1)|F_s)$ (respectively, $\leq$ or $\geq$) for every $y_1, x_1 \in I$ such that $y_1 \leq x_1$.

Definition 4.[31] *Let (Ω, A, P) be an arbitrary probability space and $I \subset \mathbb{R}$. A stochastic process $\xi : I \times \Omega \to \mathbb{R}$ is known as*

(1) *Stochastically continuous in I, if $\forall\, x_\circ \in I$*

$$P - \lim_{l \to x_\circ} \xi(x_1,\cdot) = \xi(x_\circ,\cdot),$$

where $P - \lim$ shows the limit in probability.
(2) *Mean-square continuous in I, if $\forall\, x_\circ \in I$*

$$\lim_{l \to x_\circ} \mathbb{E}(\xi(x_1,\cdot) - \xi(x_\circ,\cdot))^2 = 0,$$

where $\mathbb{E}(\xi(x_1,\cdot))$ shows the expectation value of the random variable $\xi(x_1,\cdot)$.
(3) *Increasing (decreasing) if $\forall\, x_1, y_1 \in I$ with $x_1 < y_1$*

$$\xi(x_1,\cdot) \leq \xi(y_1,\cdot), \qquad \xi(x_1,\cdot) \geq \xi(y_1,\cdot).$$

(4) *Monotonic if it is increasing or decreasing.*
(5) *Mean square differentiable at a point $x_1 \in I$, If there exist a random variable $\xi'(x_1,\cdot) : I \times \Omega \to \mathbb{R}$ such that*

$$\xi'(x_1,\cdot) = P - \lim_{x_1 \to x_\circ} \frac{\xi(x_1,\cdot) - X(x_\circ,\cdot)}{x_1 - x_\circ}.$$

A stochastic process $\xi : I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of interval I.

Definition 5.[31] *Assume that (Ω, A, P) be a probability space and $I \subset \mathbb{R}$ with $E(X_1(\rho)^2) < \infty$ $\forall$ $\rho \in I$. If $[x_1, y_1] \subset I, x_1 = \rho_0 < \rho_1 < \rho_2 < ... < \rho_n = y_1$ be a partition of $[x_1, y_1]$ and $\Theta \in [\rho_{\kappa-1}, \rho_\kappa]$ for $\kappa = 1,2,...,n$. A random variable $Z_1 : \Omega \to \mathbb{R}$ is termed as mean-square integral of the process $X_1(\rho,\cdot)$ on $[x_1, y_1]$ if*

$$\lim_{n\to\infty} E\left[\sum_{\kappa=1}^{\infty} X_1(\Theta_\kappa,\cdot)(\rho_\kappa,\rho_{\kappa-1}) - Z(\cdot)\right]^2 = 0,$$

then

$$\int_{x_1}^{y_1} X_1(\rho,\cdot)d\rho = Z_1(\cdot) \ \ (a.e.).$$

Also, mean square integral operator is increasing,

$$\int_{x_1}^{y_1} X_1(\rho,\cdot)d\rho \leq \int_{x_1}^{y_1} Y_1(\rho,\cdot) \ \ (a.e.),$$

where $X(\rho,\cdot) \leq Y(\rho,\cdot)$ in $[x_1, y_1]$.

For more on stochastic processes, one can consult (see [32, 33, 34, 35, 36]).
First, we give the definition of convex stochastic processes as follows:

Definition 6.*A function $\xi : I \to \mathbb{R}$ is said to be a convex stochastic process if*

$$\xi(p\phi_1 + (1-\varsigma)\phi_2,\cdot) \leq p\xi(\phi_1,\cdot) + (1-\varsigma)\xi(\phi_2,\cdot) \ (a.e.),$$

holds for all $\phi_1, \phi_2 \in I$ and $p \in [0,1]$.

Numerous researchers have worked on huge applications of different inequalities. Jensen inequality and Hermite-Hadamard inequality are highly notable problems in the literature. Jensen inequality is one of the famous and essential inequalities in the mathematical study. Jensen inequality and Jensen inequality of Mercer type developed in the year 2003 and 2006 [37, 38]. In 2009, Mercer's results are generalized by M. Niezgoda to higher dimensions [39]. Further, several authors have discussed Jensen inequality, Jensen-Mercer operator inequalities, and reverse Jensen-Mercer operator type inequalities using super-quadratic functions derive in [40, 41, 42]. H. R. Moradi and S. Furuichi derive different improvements for Jensen-Mercer type inequalities in 2019,[43]. In 1980, K. Nikodem developed a Jensen inequality for convex stochastic processes. Now, we establish a Jensen-Mercer inequality with the help of Jensen inequality for convex stochastic processes. Next, we derive Hermite-Hadamard-Mercer inequalities by using Jensen-Mercer inequality in the setting of convex stochastic processes.

Here, we give definitions of fractional calculus theory.

Definition 7.[44] *Let* $(\phi_1,\phi_2)(-\infty \leq \phi_1 < \phi_2 \leq \infty)$ *and* $\alpha > 0$. *Also, consider* ψ *be an increasing positive monotone function on* $[\phi_1,\phi_2]$, *having a continuous derivative* ψ' *on* (ϕ_1,ϕ_2). *Then left and right sided* ψ*-Riemann-Liouville fractional integrals of a function* ξ *with respect to another function* ψ *on* $[\phi_1,\phi_2]$ *are defined by*

$$\left(I^{\alpha:\psi}_{\phi_1^+}\right)\xi(x_1) = \frac{1}{\Gamma(\alpha)}\int_{\phi_1}^{x_1}\psi'(p)\left(\psi(x_1)-\psi(p)\right)^{\alpha-1}\xi(p)d\varsigma,\quad \phi_1 < x_1,$$

and

$$\left(I^{\alpha:\psi}_{\phi_2^-}\right)\xi(x_1) = \frac{1}{\Gamma(\alpha)}\int_{x_1}^{\phi_2}\psi'(p)\left(\psi(p)-\psi(x_1)\right)^{\alpha-1}\xi(p)d\varsigma,\quad x_1 < \phi_2,$$

respectively.

Definition 8.[45] *Diaz et al. defined the* κ*-gamma* $\Gamma_\kappa(\cdot)$ *a generalization of classical gamma function, which is expressed as*

$$\Gamma_\kappa(q) = \lim_{n\to+\infty}\frac{n!\kappa^n(n\kappa)^{\frac{q}{\kappa}-1}}{(q)_{n,\kappa}},\quad \kappa > 0.$$

It is note that, Mellin transform of the exponential function $e^{-\frac{p^\kappa}{\kappa}}$ *is the* κ*-gamma presented by*

$$\Gamma_\kappa(\alpha) = \int_0^\infty e^{-\frac{p^\kappa}{\kappa}}p^{\alpha-1}d\varsigma.$$

Clearly, $\Gamma_\kappa(q+\kappa) = q\Gamma_\kappa(q)$, $\Gamma(q) = \lim_{\kappa\to 1}\Gamma_\kappa(q)$ *and* $\Gamma_\kappa(q) = \kappa^{\frac{q}{\kappa}-1}\Gamma\left(\frac{q}{\kappa}\right)$.

Definition 9.[46] *Let* $(\phi_1,\phi_2)(-\infty \leq \phi_1 < \phi_2 \leq \infty)$ *and* $\alpha,\kappa > 0$. *Also, consider* ψ *be an increasing positive monotone function on* $[\phi_1,\phi_2]$, *having a continuous derivative* ψ' *on* (ϕ_1,ϕ_2). *Then left and right sided* ψ_κ*-Riemann-Liouville fractional integrals of a function* ξ *with respect to another function* ψ *on* $[\phi_1,\phi_2]$ *are defined by*

$$\left({}_\kappa I^{\alpha:\psi}_{\phi_1^+}\right)\xi(x_1) = \frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_{\phi_1}^{x_1}\psi'(p)\left(\psi(x_1)-\psi(p)\right)^{\frac{\alpha}{\kappa}-1}\xi(p)d\varsigma,\quad \phi_1 < x_1$$

and

$$\left({}_\kappa I^{\alpha:\psi}_{\phi_2^-}\right)\xi(x_1) = \frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_{x_1}^{\phi_2}\psi'(p)\left(\psi(p)-\psi(x_1)\right)^{\frac{\alpha}{\kappa}-1}\xi(p)d\varsigma,\quad x_1 < \phi_2,$$

respectively.

The present article is organized as follows: In section 2, we established Jensen-Mercer inequality via convex stochastic processes. In section 3, we present the Hermite-Hadamard-Mercer type inequalities using of Jensen-Mercer inequality with the help of convex stochastic processes. In section 4, we derive some new inequalities via improved power-mean and Hölder İşcan inequality and also obtain different inequalities for a differentiable function whose first derivative in absolute value are convex stochastic processes. In section 5, we discussed some applications to special means and at last, we write concluding remarks related to our present paper.

2 Jensen-Mercer inequality for convex stochastic process

In this section, we will obtain Jensen-Mercer inequality by using two technical Lemmas for convex stochastic processes.

Lemma 1.[20] *Suppose that* $\xi : I\times\Omega \to R$ *be a convex stochastic process, then for all* $x_1,x_2,\ldots,x_n \in I$ *and* $\forall$ $\varsigma_1,\varsigma_2,\ldots,\varsigma_n \in Q\cap[0,1]$ *such that* $\varsigma_1+\varsigma_2+\ldots+\varsigma_n = 1$, *then*

$$\xi\left(\sum_{i=1}^n \varsigma_i x_i,\cdot\right) \leq \sum_{i=1}^n \varsigma_i\xi(x_i,\cdot)$$

holds almost everywhere.

To prove Jensen-Mercer inequality for convex stochastic process first we prove lemma 2.
Let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$ and take $\varsigma_i (1\leq i\leq n)$ be positive weights associated with these ς_i and $\sum_{i=1}^n x_i = 1$.

Lemma 2.*Suppose that* $\xi : I\times\Omega \to R$ *be a convex stochastic process, then the following inequality holds almost everywhere*

$$\xi(x_1+x_n-x_i,\cdot) \leq \xi(x_1,\cdot)+\xi(x_n,\cdot)-\xi(x_i,\cdot), \tag{1}$$

$\forall\, x_i \in I$, $1\leq i\leq n$ *and* $\forall\,\varsigma\in(0,1)$.

*Proof.*Note that $y_i = x_1+x_n-x_i$. Then $x_1+x_n = x_i+y_i$ so that the pairs x_1,x_n and x_i,y_i posses the same mid point. Since that is the case there exist ς such that

$$x_i = \varsigma x_1+(1-\varsigma)x_n,$$
$$y_i = (1-\varsigma)x_1+\varsigma x_n,$$

where $0\leq\varsigma\leq 1$ and $1\leq i\leq n$.
Thus, from the definition of convex stochastic process yields

$$\begin{aligned}\xi(y_i,\cdot) &= \xi((1-\varsigma)x_1+\varsigma x_n,\cdot)\\ &\leq (1-\varsigma)\xi(x_1,\cdot)+\varsigma\xi(x_n,\cdot)\\ &= \xi(x_1,\cdot)+\xi(x_n,\cdot)-[\varsigma\xi(x_1,\cdot)+(1-\varsigma)\xi(x_n,\cdot)]\\ &\leq \xi(x_1,\cdot)+\xi(x_n,\cdot)-\xi(\varsigma x_1+(1-\varsigma)x_n,\cdot)\\ &= \xi(x_1,\cdot)+\xi(x_n,\cdot)-\xi(x_i,\cdot)\ \ (a.e.).\end{aligned}$$

Take $y_i = x_1+x_n-x_i$, then

$$\xi(x_1+x_n-x_i,\cdot) \leq \xi(x_1,\cdot)+\xi(x_n,\cdot)-\xi(x_i,\cdot)\ \ (a.e.).$$

This completes the proof.

Theorem 1.*Suppose that $\xi : I \times \Omega \to R$ be a convex stochastic process and take $0 < x_1 \le s_2 \le \ldots \le x_n$ be real numbers in I. If $\varsigma_i (1 \le i \le n)$ be positive weights associated with these ς_i and $\sum_{i=1}^{n} x_i = 1$. Then*

$$\xi\left(x_1 + x_n - \sum_{i=1}^{n} \varsigma_i x_i, \cdot\right) \le \xi(x_1,\cdot) + \xi(x_n,\cdot) - \sum_{i=1}^{n} \varsigma_i \xi(x_i,\cdot) \quad (a.e.). \tag{2}$$

Proof.

$$\xi\left(x_1 + x_n - \sum_{i=1}^{n} \varsigma_i x_i, \cdot\right) = \xi\left(\sum_{i=1}^{n} \varsigma_i (x_1 + x_n - x_i, \cdot)\right)$$
$$\le \sum_{i=1}^{n} \varsigma_i \xi(x_1 + x_n - x_i, \cdot) \quad \text{from Lemma 1}$$
$$\le \sum_{i=1}^{n} \varsigma_i [\xi(x_1,\cdot) + \xi(x_n,\cdot) - \xi(x_i,\cdot)] \quad \text{from Lemma 2}$$
$$= \xi(x_1,\cdot) + \xi(x_n,\cdot) - \sum_{i=1}^{n} \varsigma_i \xi(x_i,\cdot).$$

This completes the proof.

3 Hermite-Hadamard-Mercer type fractional integral inequalities

In this section, we derive some inequalities of Hermite-Hadamard type. Also develop some new Lemmas using ψ_κ-Riemann-Liouville fractional integrals and obtain related fractional integral inequalities. Throughout the paper, we use the following assumption:
M_1 : Let $\xi : [\phi_1, \phi_2] \times \Omega \to \mathbb{R}$ be a convex stochastic processes on $[\phi_1, \phi_2]$ with $0 \le \phi_1 < \phi_2$ and $\xi' \in L_1[\phi_1, \phi_2]$. Also, suppose that $\psi(\cdot)$ is an increasing and positive monotone on $[\phi_1, \phi_2]$ and ψ' is a mean square differentiable (continuous) on (ϕ_1, ϕ_2) and $\alpha, \kappa > 0$.

Theorem 2.*Suppose M_1 holds, then*

$$\xi\left(\phi_1 + \phi_2 - \frac{x_1 + y_1}{2}, \cdot\right) \le [\xi(\phi_1,\cdot) + \xi(\phi_2,\cdot)] - \frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1 - x_1)^{\frac{\alpha}{\kappa}}} \times \left\{ \left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(x_1)^+}\right) \left((\xi \circ \psi)\left(\psi^{-1}(y_1,\cdot)\right)\right) + \left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(y_1)^-}\right) \left((\xi \circ \psi)\left(\psi^{-1}(x_1,\cdot)\right)\right) \right\}$$
$$\le [\xi(\phi_1,\cdot) + \xi(\phi_2,\cdot)] - \xi\left(\frac{x_1 + y_1}{2}, \cdot\right) \quad (a.e.), \tag{3}$$

$\forall\ x_1, y_1 \in [\phi_1, \phi_2]$ with $x_1 < y_1$, and $\Gamma_\kappa(\cdot)$ is κ-gamma function.

*Proof.*From Jensen-Mercer inequality (2), we have

$$\xi\left(\phi_1 + \phi_2 - \frac{v_1 + v_2}{2}, \cdot\right) \le \xi(\phi_1,\cdot) + \xi(\phi_2,\cdot) - \frac{\xi(v_1,\cdot) + \xi(v_2,\cdot)}{2},$$

$\forall\ v_1, v_2 \in [\phi_1, \phi_2]$.
By substituting $v_1 = \varsigma x_1 + (1-\varsigma) y_1$ and $v_2 = (1-\varsigma) x_1 + \varsigma y_1$, for all $x_1, y_1 \in [\phi_1, \phi_2]$ and $\varsigma \in [0,1]$, we get

$$\xi\left(\phi_1 + \phi_2 - \frac{x_1 + y_1}{2}, \cdot\right) \le \xi(\phi_1,\cdot) + \xi(\phi_2,\cdot) - \frac{\xi(\varsigma x_1 + (1-\varsigma) y_1, \cdot) + \xi((1-\varsigma) x_1 + \varsigma y_1, \cdot)}{2}. \tag{4}$$

Multiplying both sides of (4) by $\varsigma^{\frac{\alpha}{\kappa}-1}$ and then integrate with respect to ς over $[0,1]$ yields that

$$\frac{\kappa}{\alpha} \xi\left(\phi_1 + \phi_2 - \frac{x_1 + y_1}{2}, \cdot\right) \le \frac{\kappa}{\alpha} [\xi(\phi_1,\cdot) + \xi(\phi_2,\cdot)] - \frac{1}{2} \left\{ \int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1} \Big(\xi(\varsigma x_1 + (1-\varsigma) y_1, \cdot) + \xi((1-\varsigma) x_1 + \varsigma y_1, \cdot) \Big) d\varsigma \right\},$$

where

$$\frac{\alpha}{2\kappa} \left\{ \int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1} \Big(\xi(\varsigma x_1 + (1-\varsigma) y_1, \cdot) + \xi((1-\varsigma) x_1 + \varsigma y_1, \cdot) \Big) d\varsigma \right\}$$
$$= \frac{\alpha}{2\kappa} \int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1} \xi(\varsigma x_1 + (1-\varsigma) y_1, \cdot) d\varsigma + \frac{\alpha}{2\kappa} \int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1} \xi((1-\varsigma) x_1 + \varsigma y_1, \cdot) d\varsigma$$
$$= \frac{\alpha}{2\kappa} \int_{\psi^{-1}(x_1)}^{\psi^{-1}(y_1)} \left(\frac{y_1 - \psi(\Upsilon)}{y_1 - x_1}\right)^{\frac{\alpha}{\kappa}-1} \xi(\psi(\Upsilon), \cdot) \frac{\psi'(\Upsilon)}{y_1 - x_1} d\Upsilon + \frac{\alpha}{2\kappa} \int_{\psi^{-1}(x_1)}^{\psi^{-1}(y_1)} \left(\frac{\psi(\Upsilon) - x_1}{y_1 - x_1}\right)^{\frac{\alpha}{\kappa}-1} \xi(\psi(\Upsilon), \cdot) \frac{\psi'(\Upsilon)}{y_1 - x_1} d\Upsilon.$$

Thus,

$$\frac{\alpha}{2\kappa} \left\{ \int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1} \Big(\xi(\varsigma x_1 + (1-\varsigma) y_1, \cdot) + \xi((1-\varsigma) x_1 + \varsigma y_1, \cdot) \Big) d\varsigma \right\} = \frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1 - x_1)^{\frac{\alpha}{\kappa}}} \times \left\{ \left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(x_1)^+}\right) \Big(\xi(y_1,\cdot)\Big) + \left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(y_1)^-}\right) \Big(\xi(x_1,\cdot)\Big) \right\}.$$

By rearranging first inequality of (3) is proved.
Now, we prove the second part of (3). Since ξ is convex stochastic process, then for $\varsigma \in [0,1]$, we get

$$\xi\left(\frac{x_1 + y_1}{2}, \cdot\right) = \xi\left(\frac{\varsigma x_1 + (1-\varsigma) y_1 + (1-\varsigma) x_1 + p y_1}{2}, \cdot\right) \le \frac{\xi(\varsigma x_1 + (1-\varsigma) y_1, \cdot) + \xi((1-\varsigma) x_1 + p y_1, \cdot)}{2}. \tag{5}$$

Both sides of inequality (5) is multiplying by $\varsigma^{\frac{\alpha}{\kappa}-1}$ and then integrate with respect to ς over $[0,1]$, and let $\psi(\Upsilon) =$

$\varsigma x_1+(1-\varsigma)y_1$, and $\psi(\alpha)=(1-\varsigma)x_1+py_1$ yields that

$$\frac{\kappa}{\alpha}\xi\left(\frac{x_1+y_1}{2},\cdot\right)\le\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}\left(\xi\left(\varsigma x_1+(1-\varsigma)y_1,\cdot\right)+\xi\left((1-\varsigma)x_1+py_1,\cdot\right)\right)d\varsigma.$$

Implies that

$$\xi\left(\frac{x_1+y_1}{2},\cdot\right)=\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\left\{\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(x_1)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(y_1,\cdot)\right)\right)+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(y_1)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(x_1,\cdot)\right)\right)\right\}.$$

Both sides of above inequality multiplying by (-1) and then add $\xi(\phi_1)+\xi(\phi_2)$, we get

$$\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\xi\left(\frac{x_1+y_1}{2},\cdot\right)\ge\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\left\{\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(x_1)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(y_1,\cdot)\right)\right)+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(y_1)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(x_1,\cdot)\right)\right)\right\}.$$

Thus, we get second part of (3).

Corollary 1. *By the assumption of Theorem 2, substituting $\psi(\Upsilon)=\Upsilon$, we get*

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\le[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)]-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\left\{{}_\kappa I^{\alpha}_{(x_1)^+}\xi(y_1,\cdot)+{}_\kappa I^{\alpha}_{(y_1)^-}\xi(x_1,\cdot)\right\}\le[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)]-\xi\left(\frac{x_1+y_1}{2},\cdot\right).$$

Theorem 3. *Suppose M_1 holds, then*

$$\begin{aligned}\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)&\le\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\times\Big[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\\&\quad+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\Big]\\&\le\frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2}\\&\le[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)]-\frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2},\end{aligned}\tag{6}$$

for all $x_1,y_1\in[\phi_1,\phi_2]$ with $x_1<y_1$ and $\Gamma_\kappa(\cdot)$ is κ-gamma function.

Proof. Since ξ is a convex stochastic processes yields that

$$\xi\left(\phi_1+\phi_2-\frac{v_1+v_2}{2},\cdot\right)=\xi\left(\frac{\phi_1+\phi_2-v_1+\phi_1+\phi_2-v_2}{2},\cdot\right)\le\frac{\xi(\phi_1+\phi_2-v_1,\cdot)+\xi(\phi_1+\phi_2-v_2,\cdot)}{2},$$

for all $v_1,v_2\in[\phi_1,\phi_2]$.
By putting $v_1=\varsigma(\phi_1+\phi_2-x_1)+(1-\varsigma)(\phi_1+\phi_2-y_1)$ and $v_2=(1-\varsigma)(\phi_1+\phi_2-x_1)+\varsigma(\phi_1+\phi_2-y_1)$, for all $x_1,y_1\in[\phi_1,\phi_2]$ and $\varsigma\in[0,1]$, we have

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\le\frac{1}{2}\Big[\xi\left(p(\phi_1+\phi_2-x_1)+(1-\varsigma)(\phi_1+\phi_2-y_1),\cdot\right)+\xi\left((1-\varsigma)(\phi_1+\phi_2-x_1)+\varsigma(\phi_1+\phi_2-y_1),\cdot\right)\Big].\tag{7}$$

Multiplying both sides of inequality (7) by $\varsigma^{\frac{\alpha}{\kappa}-1}$ and then integrate with respect to ς over $[0,1]$, and $\psi(\Upsilon)=\varsigma(\phi_1+\phi_2-x_1)+(1-\varsigma)(\phi_1+\phi_2-y_1)$, and $\psi(\alpha)=(1-\varsigma)(\phi_1+\phi_2-x_1)+\varsigma(\phi_1+\phi_2-y_1)$, we can write

$$\begin{aligned}\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)&\le\frac{\alpha}{2\kappa}\Big[\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}\times\xi\left(p(\phi_1+\phi_2-x_1)+(1-\varsigma)(\phi_1+\phi_2-y_1),\cdot\right)d\varsigma\\&\quad+\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}\xi\left((1-\varsigma)(\phi_1+\phi_2-x_1)+\varsigma(\phi_1+\phi_2-y_1),\cdot\right)d\varsigma\Big]\\&=\frac{\alpha}{2\kappa}\times\int_{\psi^{-1}(\phi_1+\phi_2-x_1)}^{\psi^{-1}(\phi_1+\phi_2-y_1)}\left(\frac{(\phi_1+\phi_2-y_1)-\psi(\Upsilon)}{y_1-x_1}\right)^{\frac{\alpha}{\kappa}-1}\xi(\psi(\Upsilon),\cdot)\times\frac{\psi'(\Upsilon)}{y_1-x_1}d\Upsilon\\&\quad+\frac{\alpha}{2\kappa}\int_{\psi^{-1}(\phi_1+\phi_2-x_1)}^{\psi^{-1}(\phi_1+\phi_2-y_1)}\left(\frac{\psi(\Upsilon)-(\phi_1+\phi_2-x_1)}{y_1-x_1}\right)^{\frac{\alpha}{\kappa}-1}\times\xi(\psi(\Upsilon),\cdot)\frac{\psi'(\Upsilon)}{y_1-x_1}d\Upsilon,\end{aligned}$$

implies that

$$\begin{aligned}\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)&\le\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\times\Big[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\\&\quad+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\Big].\end{aligned}$$

Thus, we proved the first inequality of (6).
Now, we prove second inequality of (6). Since ξ is a convex stochastic processes, then for $\varsigma\in[0,1]$ yields that

$$\begin{aligned}&\xi\left(\varsigma(\phi_1+\phi_2-x_1)+(1-\varsigma)(\phi_1+\phi_2-y_1),\cdot\right)\le p\xi(\phi_1+\phi_2-x_1,\cdot)+(1-\varsigma)\xi(\phi_1+\phi_2-y_1,\cdot),\\&\xi\left((1-\varsigma)(\phi_1+\phi_2-x_1)+\varsigma(\phi_1+\phi_2-y_1),\cdot\right)\le(1-\varsigma)\xi(\phi_1+\phi_2-x_1,\cdot)+p\xi(\phi_1+\phi_2-y_1,\cdot).\end{aligned}$$

Adding above inequalities and employing Jensen-Mercer inequality (2), we have

$$\begin{aligned}&\xi\left(\varsigma(\phi_1+\phi_2-x_1)+(1-\varsigma)(\phi_1+\phi_2-y_1),\cdot\right)+\xi\left((1-\varsigma)(\phi_1+\phi_2-x_1)+\varsigma(\phi_1+\phi_2-y_1),\cdot\right)\\&\le\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)\\&\le2[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)]-[\xi(x_1,\cdot)+\xi(y_1,\cdot)].\end{aligned}\tag{8}$$

Both sides of (8) multiplying by $\varsigma^{\frac{\alpha}{\kappa}-1}$ and then integrate with respect to ς over $[0,1]$ yields the second and third inequalities of (6).

Corollary 2. *Substituting $\psi(\Upsilon)=\Upsilon$, we get*

$$\begin{aligned}
&\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\\
&\le \frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\left[\left({}_{\kappa}I^{\alpha}_{(\phi_1+\phi_2-x_1)^+}\right)\xi\left(\phi_1+\phi_2-y_1,\cdot\right)\right.\\
&\quad\left.+\left({}_{\kappa}I^{\alpha}_{(\phi_1+\phi_2-y_1)^-}\right)\xi\left(\phi_1+\phi_2-x_1,\cdot\right)\right]\\
&\le \frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2}\\
&\le [\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)]-\frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2}.
\end{aligned}$$

Remark. 1. For $\alpha=\kappa=1$, $\psi(\Upsilon)=\Upsilon$, $x_1=\phi_1$ and $y_1=\phi_2$ in Theorem 3 yields the Hermite-Hadamard inequality for convex stochastic processes [23].

Theorem 4. *Let M_1 holds, then*

$$\begin{aligned}
&\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\\
&\le \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}\\
&\times\left[\left({}_{\kappa}I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\right.\\
&\left.+\left({}_{\kappa}I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\right]\\
&\le \xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left(\frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2}\right),
\end{aligned}$$

for all $x_1,y_1\in[\phi_1,\phi_2]$ with $x_1<y_1$ and $\Gamma_\kappa(\cdot)$ is κ-gamma function.

Proof. We prove first inequality of (9), since ξ is convex stochastic process yields that

$$\xi\left(\phi_1+\phi_2-\frac{v_1+v_2}{2},\cdot\right)\le\frac{\xi(\phi_1+\phi_2-v_1,\cdot)+\xi(\phi_1+\phi_2-v_2,\cdot)}{2},$$

for all $v_1,v_2\in[\phi_1,\phi_2]$.
By setting $v_1=\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1$ and $v_2=\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1$, $\varsigma\in[0,1]$, we get

$$\begin{aligned}
&\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\\
&\le\frac{1}{2}\xi\left(\phi_1+\phi_2-\left(\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1\right),\cdot\right)\\
&+\frac{1}{2}\xi\left(\phi_1+\phi_2-\left(\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1\right),\cdot\right).
\end{aligned}\tag{9}$$

Multiplying both sides of (9) by $\varsigma^{\frac{\alpha}{\kappa}-1}$ and then integrate with respect to ς over $[0,1]$, and set $\psi(\Upsilon)=\left(\phi_1+\phi_2-\left(\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1\right)\right)$, and $\psi(\alpha)=\left(\phi_1+\phi_2-\left(\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1\right)\right)$ yields that

$$\begin{aligned}
&\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}d\varsigma\\
&\le\frac{1}{2}\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}\times\left[\xi\left(\phi_1+\phi_2-\left(\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1\right),\cdot\right)\right.\\
&\left.+\xi\left(\phi_1+\phi_2-\left(\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1\right),\cdot\right)\right]d\varsigma.
\end{aligned}$$

Implies that

$$\begin{aligned}
&\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\\
&\le\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}\\
&\times\left[\left({}_{\kappa}I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\right.\\
&\cdot\left.+\left({}_{\kappa}I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\right].
\end{aligned}$$

Thus, the first part of inequality (9) proved.
Now, we prove second part of (9). Since ξ is convex stochastic process and applying Jensen-Mercer inequality (2), then for $\varsigma\in[0,1]$, we can write

$$\begin{aligned}
&\xi\left(\phi_1+\phi_2-\left(\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1\right),\cdot\right)\\
&\le\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left[\frac{\varsigma}{2}\xi(x_1,\cdot)+\frac{2-\varsigma}{2}\xi(y_1,\cdot)\right],
\end{aligned}\tag{10}$$

and

$$\begin{aligned}
&\xi\left(\phi_1+\phi_2-\left(\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1\right),\cdot\right)\\
&\le\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left[\frac{2-\varsigma}{2}\xi(x_1,\cdot)+\frac{\varsigma}{2}\xi(y_1,\cdot)\right].
\end{aligned}\tag{11}$$

Adding (10) and (11) yields that

$$\begin{aligned}
&\xi\left(\phi_1+\phi_2-\left(\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1\right),\cdot\right)\\
&+\xi\left(\phi_1+\phi_2-\left(\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1\right),\cdot\right)\\
&\le 2[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)]-[\xi(x_1,\cdot)+\xi(y_1,\cdot)].
\end{aligned}$$

Multiplying both sides of (12) by $\varsigma^{\frac{\alpha}{\kappa}-1}$ and then integrate with respect to ς over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}\left[\xi\left(\phi_1+\phi_2-\left(\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1\right),\cdot\right)\right.\\
&\left.+\xi\left(\phi_1+\phi_2-\left(\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1\right),\cdot\right)\right]d\varsigma\\
&\le\{2[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)]-[\xi(x_1,\cdot)+\xi(y_1,\cdot)]\}\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}d\varsigma.
\end{aligned}$$

Thus, we can write

$$\frac{2^{\frac{\alpha}{\kappa}}\kappa\Gamma_\kappa(\alpha)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}$$
$$\times\Bigg[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)$$
$$+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\Bigg]$$
$$\leq\{2[\xi(\phi_1)+\xi(\phi_2)]-[\xi(x_1)+\xi(y_1)]\}\frac{\kappa}{\alpha}.$$

Multiplying by $\frac{\alpha}{2\kappa}$ to both sides of above inequality yields that

$$\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}$$
$$\times\Bigg[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)$$
$$+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\Bigg]$$
$$\leq\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left(\frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2}\right).$$

This completes the proof.

Corollary 3. *By setting $\psi(\Upsilon)=\Upsilon$, we get*

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)$$
$$\leq\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}\Bigg[\left({}_\kappa I^{\alpha}_{(\phi_1+\phi_2-\frac{x_1+y_1}{2})^+}\right)\xi(\phi_1+\phi_2-x_1,\cdot)$$
$$+\left({}_\kappa I^{\alpha}_{(\phi_1+\phi_2-\frac{x_1+y_1}{2})^-}\right)\xi(\phi_1+\phi_2-y_1,\cdot)\Bigg]$$
$$\leq\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left(\frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2}\right).$$

Theorem 5. *Let M_1 satisfied, then following inequality will be of the form:*

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)$$
$$\leq\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}\Bigg[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)$$
$$\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)$$
$$+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)$$
$$\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)\Bigg]$$
$$\leq\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left(\frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2}\right),\tag{12}$$

for all $x_1,y_1\in[\phi_1,\phi_2]$ with $x_1<y_1$ and $\Gamma_\kappa(\cdot)$ is κ-gamma function.

Proof. To prove first part of (12), take the definition of ξ convex stochastic process, we have

$$\xi\left(\phi_1+\phi_2-\frac{v_1+v_2}{2},\cdot\right)$$
$$\leq\frac{\xi(\phi_1+\phi_2-v_1,\cdot)+\xi(\phi_1+\phi_2-v_2,\cdot)}{2},$$

for all $v_1,v_2\in[\phi_1,\phi_2]$.
By setting $v_1=\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1$ and $v_2=\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1$, $\varsigma\in[0,1]$ yields that

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)$$
$$\leq\frac{1}{2}\xi\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1\right),\cdot\right)$$
$$+\frac{1}{2}\xi\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1\right),\cdot\right).\tag{13}$$

Multiplying both sides of (13) by $\varsigma^{\frac{\alpha}{\kappa}-1}$ and then integrate with respect to ς over $[0,1]$, and let $\psi(\Upsilon)=\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1\right)\right)$, and $\psi(\alpha)=\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1\right)\right)$ yields that

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}d\varsigma$$
$$\leq\frac{1}{2}\int_0^1\varsigma^{\frac{\alpha}{\kappa}-1}\Bigg[\xi\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1\right),\cdot\right)$$
$$+\xi\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1\right),\cdot\right)\Bigg]d\varsigma.$$

Implies that

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\leq\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}$$
$$\times\Bigg[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)$$
$$+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-\frac{x_1+y_1}{2})^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\Bigg].$$

The first part of (9) is proved.
Now, we prove second inequality of (9). Tke the definition of ξ convex stochastic process and Jensen-Mercer inequality (2), then for $\varsigma\in[0,1]$ yields that

$$\xi\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1\right),\cdot\right)$$
$$\leq\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left[\frac{1+\varsigma}{2}\xi(x_1,\cdot)+\frac{1-\varsigma}{2}\xi(y_1,\cdot)\right],\tag{14}$$

and

$$\xi\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1\right),\cdot\right)$$
$$\leq\xi(\phi_1)+\xi(\phi_2)-\left[\frac{1-\varsigma}{2}\xi(x_1)+\frac{1+\varsigma}{2}\xi(y_1)\right].\tag{15}$$

Adding (14) and (15) yields that

$$\xi\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1\right),\cdot\right)+\xi\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1\right),\cdot\right)\le 2\left[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)\right]-\left[\xi(x_1,\cdot)+\xi(y_1,\cdot)\right]. \quad (16)$$

Multiplying both sides of (16) by $\varsigma^{\frac{\alpha}{\kappa}-1}$ and then integrate with respect to ς over $[0,1]$, we have

$$\int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1}\left[\xi\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1\right),\cdot\right)+\xi\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1\right),\cdot\right)\right]d\varsigma \le \left\{2\left[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)\right]-\left[\xi(x_1,\cdot)+\xi(y_1,\cdot)\right]\right\}\int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1}d\varsigma.$$

Thus,

$$\frac{2^{\frac{\alpha}{\kappa}}\kappa\Gamma_\kappa(\alpha)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)\right]\le\left\{2\left[\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)\right]-\left[\xi(x_1,\cdot)+\xi(y_1,\cdot)\right]\right\}\frac{\kappa}{\alpha}.$$

Multiplying by $\frac{\alpha}{2\kappa}$ to both sides of above inequality yields that

$$\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)\right]\le\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left(\frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2}\right).$$

The second part of (9) is proved.

Corollary 4. *By setting $\psi(\Upsilon)=\Upsilon$, we obtain*

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\le\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-x_1)^{\frac{\alpha}{\kappa}}}\times\left[\left({}_\kappa I^{\alpha}_{(\phi_1+\phi_2-y_1)^+}\right)\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)+\left({}_\kappa I^{\alpha}_{(\phi_1+\phi_2-x_1)^-}\right)\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right]\le\xi(\phi_1,\cdot)+\xi(\phi_2,\cdot)-\left(\frac{\xi(x_1,\cdot)+\xi(y_1,\cdot)}{2}\right).$$

Lemma 3. *Consider $\xi: I\times\Omega\to\mathbb{R}$ be a mean square differentiable stochastic process on I° and ξ' is a mean square integrable on $[\phi_1,\phi_2]$, where $\phi_1,\phi_2\in I$ with $0\le\phi_1<\phi_2$. Also, suppose that $\psi(\cdot)$ is an increasing and positive monotone on $[\phi_1,\phi_2]$ and ψ' is a mean square continuous $(differentiable)$ on (ϕ_1,ϕ_2) and $\alpha,\kappa>0$, then*

$$\frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2}-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-y_1)^{\frac{\alpha}{\kappa}}}\times\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\right]=\frac{1}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\int_{\psi^{-1}(\phi_1+\phi_2-y_1)}^{\psi^{-1}(\phi_1+\phi_2-x_1)}\left((\psi(\Upsilon)-(\phi_1+\phi_2-y_1))^{\frac{\alpha}{\kappa}}-((\phi_1+\phi_2-x_1)-\psi(\Upsilon))^{\frac{\alpha}{\kappa}}\right)\left(\left(\xi'\circ\psi\right)(\Upsilon,\cdot)\right)\psi'(\Upsilon)d\Upsilon \quad (17)$$

for all $x_1,y_1\in[\phi_1,\phi_2]$ with $x_1<y_1$, and $\Gamma_\kappa(\cdot)$ is κ-gamma function.

Proof. Note that

$$I=\frac{\xi(\phi_1+\phi_2-x_1,\cdot)-\xi(\phi_1+\phi_2-y_1,\cdot)}{2}-[I_1+I_2], \quad (18)$$

where

$$I_1=\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\times\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\right]$$
$$=\frac{\alpha}{2\kappa(y_1-x_1)^{\frac{\alpha}{\kappa}}}\int_{\psi^{-1}(\phi_1+\phi_2-y_1)}^{\psi^{-1}(\phi_1+\phi_2-x_1)}\psi'(\Upsilon)\times((\phi_1+\phi_2-x_1)-\psi(\Upsilon))^{\frac{\alpha}{\kappa}-1}\left((\xi\circ\psi)(\Upsilon,\cdot)\right)d\Upsilon$$
$$=\frac{\alpha}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\left[\xi(\phi_1+\phi_2-y_1)(y_1-x_1)^{\frac{\alpha}{\kappa}}\right]+\int_{\psi^{-1}(\phi_1+\phi_2-y_1)}^{\psi^{-1}(\phi_1+\phi_2-x_1)}\psi'(\Upsilon)\times((\phi_1+\phi_2-x_1)-\psi(\Upsilon))^{\frac{\alpha}{\kappa}-1}\left(\left(\xi'\circ\psi\right)(\Upsilon,\cdot)\right)d\Upsilon, \quad (19)$$

and

$$I_1=\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\times\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\right]$$
$$=\frac{\alpha}{2\kappa(y_1-x_1)^{\frac{\alpha}{\kappa}}}\int_{\psi^{-1}(\phi_1+\phi_2-y_1)}^{\psi^{-1}(\phi_1+\phi_2-x_1)}\psi'(\Upsilon)\times(-(\phi_1+\phi_2-x_1)+\psi(\Upsilon))^{\frac{\alpha}{\kappa}-1}\left((\xi\circ\psi)(\Upsilon,\cdot)\right)d\Upsilon$$

$$= \frac{\alpha}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\left[\xi(\phi_1+\phi_2-x_1)(y_1-x_1)^{\frac{\alpha}{\kappa}}\right]$$
$$-\int_{\psi^{-1}(\phi_1+\phi_2-y_1)}^{\psi^{-1}(\phi_1+\phi_2-x_1)}\psi'(\Upsilon)$$
$$\times(-(\phi_1+\phi_2-y_1)+\psi(\Upsilon))^{\frac{\alpha}{\kappa}-1}\left(\left(\xi'\circ\psi\right)(\Upsilon,\cdot)\right)d\Upsilon. \quad (20)$$

By setting (19) and (20) in (18) yields (17).

Theorem 6. *Consider M_1 holds. Also, suppose that $|\xi'|$ is convex stochastic process on $[\phi_1,\phi_2]$, then*

$$\Bigg|\frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2}-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-y_1)^{\frac{\alpha}{\kappa}}}$$
$$\times\Bigg[\left({}_{\kappa}I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)$$
$$+\left({}_{\kappa}I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\Bigg]\Bigg|$$
$$\le\left(\frac{y_1-x_1}{\frac{\alpha}{\kappa}+1}\right)\left(1-\frac{1}{2^{\frac{\alpha}{\kappa}}}\right)$$
$$\times\left\{\left[\left|\xi'(\phi_1,\cdot)\right|+\xi'|(\phi_2,\cdot)|\right]-\left(\frac{\left|\xi'(x_1,\cdot)\right|+\left|\xi'(y_1,\cdot)\right|}{2}\right)\right\}, \quad (21)$$

for all $x_1,y_1\in[\phi_1,\phi_2]$ with $x_1<y_1$, and $\Gamma_\kappa(\cdot)$ is κ-gamma function.

Proof. By using Lemma 3, Jensen-Mercer inequality (2) and the fact that $\left|\xi'\right|$ is convex stochastic processes yields that

$$\Bigg|\frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2}-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-y_1)^{\frac{\alpha}{\kappa}}}$$
$$\times\Bigg[\left({}_{\kappa}I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)$$
$$+\left({}_{\kappa}I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\Bigg]\Bigg|$$
$$\le\int_{\psi^{-1}(\phi_1+\phi_2-y_1)}^{\psi^{-1}(\phi_1+\phi_2-x_1)}\times$$
$$\left|\left((\psi(\Upsilon)-(\phi_1+\phi_2-y_1))^{\frac{\alpha}{\kappa}}-\left((\phi_1+\phi_2-x_1)-\psi(\Upsilon)\right)^{\frac{\alpha}{\kappa}}\right)\right.$$
$$\times\left|\frac{1}{2(y_1-x_1)^{\frac{\alpha}{\kappa}}}\right|\left(\xi'\circ\psi\right)(\Upsilon,\cdot)\left|\psi'(\Upsilon)d\Upsilon\right.$$
$$=\frac{y_1-x_1}{2}\int_0^1\left|\varsigma^{\frac{\alpha}{\kappa}}-(1-\varsigma)^{\frac{\alpha}{\kappa}}\right|$$
$$\times\left|\xi'(\phi_1+\phi_2-(\varsigma x_1+(1-\varsigma)y_1,\cdot))\right|d\varsigma$$
$$=\frac{y_1-x_1}{2}\int_0^1\left|\varsigma^{\frac{\alpha}{\kappa}}-(1-\varsigma)^{\frac{\alpha}{\kappa}}\right|$$
$$\times\left\{\left|\xi'(\phi_1,\cdot)\right|+\left|\xi'(\phi_2,\cdot)\right|\right.$$
$$\left.-\left(\varsigma\left|\xi'(x_1,\cdot)\right|+(1-\varsigma)\left|\xi'(y_1,\cdot)\right|\right)\right\}$$
$$=\frac{y_1-x_1}{2}[I_1+I_2], \quad (22)$$

where

$$I_1=\int_0^{\frac{1}{2}}\left((1-\varsigma)^{\frac{\alpha}{\kappa}}-\varsigma^{\frac{\alpha}{\kappa}}\right)\left\{\left|\xi'(\phi_1,\cdot)\right|+\left|\xi'(\phi_2,\cdot)\right|\right.$$
$$\left.-\left(\varsigma\left|\xi'(x_1,\cdot)\right|+(1-\varsigma)\left|\xi'(y_1,\cdot)\right|\right)\right\}$$
$$=\left(\left|\xi'(\phi_1,\cdot)\right|+\left|\xi'(\phi_2,\cdot)\right|\right)\left(\frac{1}{\left(\frac{\alpha}{\kappa}+1\right)}-\frac{2^{-\frac{\alpha}{\kappa}}}{\left(\frac{\alpha}{\kappa}+1\right)}\right)$$
$$-\left\{\left|\xi'(x_1,\cdot)\right|\left(\frac{1}{\left(\frac{\alpha}{\kappa}+1\right)(\alpha+2)}-\frac{2^{-\frac{\alpha}{\kappa}-1}}{\left(\frac{\alpha}{\kappa}+1\right)}\right)\right.$$
$$\left.+\left|\xi'(y_1,\cdot)\right|\left(\frac{1}{\left(\frac{\alpha}{\kappa}+2\right)}-\frac{2^{-\frac{\alpha}{\kappa}-1}}{\left(\frac{\alpha}{\kappa}+1\right)}\right)\right\}, \quad (23)$$

and

$$I_2=\int_{\frac{1}{2}}^1\left(\varsigma^{\frac{\alpha}{\kappa}}-(1-\varsigma)^{\frac{\alpha}{\kappa}}\right)$$
$$\times\left\{\left|\xi'(\phi_1,\cdot)\right|+\left|\xi'(\phi_2,\cdot)\right|\right.$$
$$\left.-\left(\varsigma\left|\xi'(x_1,\cdot)\right|+(1-\varsigma)\left|\xi'(y_1,\cdot)\right|\right)\right\}$$
$$=\left(\left|\xi'(\phi_1,\cdot)\right|+\left|\xi'(\phi_2,\cdot)\right|\right)\left(\frac{1}{\left(\frac{\alpha}{\kappa}+1\right)}-\frac{2^{-\frac{\alpha}{\kappa}}}{\left(\frac{\alpha}{\kappa}+1\right)}\right)$$
$$-\left\{\left|\xi'(x_1,\cdot)\right|\left(\frac{1}{\left(\frac{\alpha}{\kappa}+2\right)}-\frac{2^{-\frac{\alpha}{\kappa}-1}}{\left(\frac{\alpha}{\kappa}+1\right)}\right)\right.$$
$$\left.+\left|\xi'(y_1,\cdot)\right|\left(\frac{1}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}-\frac{2^{-\frac{\alpha}{\kappa}-1}}{\left(\frac{\alpha}{\kappa}+1\right)}\right)\right\}. \quad (24)$$

Substituting (23) and (24) in (22) yields (21).

Corollary 5. *By putting $\psi(\Upsilon)=\Upsilon$, we get the following inequality*

$$\Bigg|\frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2}-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(y_1-y_1)^{\frac{\alpha}{\kappa}}}$$
$$\times\Bigg[\left({}_{\kappa}I^{\alpha}_{(\phi_1+\phi_2-y_1)^+}\right)\xi(\phi_1+\phi_2-x_1,\cdot)$$
$$+\left({}_{\kappa}I^{\alpha}_{(\phi_1+\phi_2-x_1)^-}\right)\xi(\phi_1+\phi_2-y_1,\cdot)\Bigg]\Bigg|$$
$$\le\left(\frac{y_1-x_1}{\frac{\alpha}{\kappa}+1}\right)\left(1-\frac{1}{2^{\frac{\alpha}{\kappa}}}\right)$$
$$\times\left\{\left[\left|\xi'(\phi_1,\cdot)\right|+\xi'|(\phi_2,\cdot)|\right]-\left(\frac{\left|\xi'(x_1,\cdot)\right|+\left|\xi'(y_1,\cdot)\right|}{2}\right)\right\}.$$

Lemma 4. *Consider $\xi: I\times\Omega\to\mathbb{R}$ be a mean square differentiable stochastic process on I° and ξ' is a mean square* *integrable on* $[\phi_1,\phi_2]$, *where* $\phi_1,\phi_2 \in I$ *with* $0 \le \phi_1 < \phi_2$. *Also, suppose that* $\psi(\cdot)$ *is an increasing and positive monotone on* $[\phi_1,\phi_2]$ *and* ψ' *is a mean square continuous* $(differentiable)$ *on* (ϕ_1,ϕ_2) *and* $\alpha,\kappa > 0$, *then*

$$\frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2} - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\Bigg[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right) + \left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)\Bigg]$$
$$= \frac{(y_1-x_1)}{4}\left[\int_0^1 \varsigma^{\frac{\alpha}{\kappa}}\xi'\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1,\cdot\right)\right)d\varsigma - \int_0^1 \varsigma^{\frac{\alpha}{\kappa}}\xi'\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1,\cdot\right)\right)d\varsigma\right], \quad (25)$$

for all $x_1,y_1 \in [\phi_1,\phi_2]$ *with* $x_1 < y_1$, *and* $\Gamma_\kappa(\cdot)$ *is* κ*-gamma function.*

*Proof.*Note that

$$I = \frac{y_1-x_1}{4}\{I_1-I_2\}, \quad (26)$$

where

$$I_1 = \int_0^1 \varsigma^{\frac{\alpha}{\kappa}}\xi'\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1,\cdot\right)\right)d\varsigma$$
$$= \frac{2}{y_1-x_1}\xi(\phi_1+\phi_2-x_1,\cdot) - \frac{2\frac{\alpha}{\kappa}}{y_1-x_1}\times\int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1}\xi\left(\phi_1+\phi_2-\left(\frac{1+\varsigma}{2}x_1+\frac{1-\varsigma}{2}y_1,\cdot\right)\right)d\varsigma$$
$$= \frac{2}{y_1-x_1}\xi(\phi_1+\phi_2-x_1,\cdot) - \frac{2^{\frac{\alpha}{\kappa}+1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}+1}}\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)\right], \quad (27)$$

and

$$I_1 = \int_0^1 \varsigma^{\frac{\alpha}{\kappa}}\xi'\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1,\cdot\right)\right)d\varsigma$$
$$= -\frac{2}{y_1-x_1}\xi(\phi_1+\phi_2-y_1,\cdot) - \frac{2\frac{\alpha}{\kappa}}{y_1-x_1}\times\int_0^1 \varsigma^{\frac{\alpha}{\kappa}-1}\xi\left(\phi_1+\phi_2-\left(\frac{1-\varsigma}{2}x_1+\frac{1+\varsigma}{2}y_1,\cdot\right)\right)d\varsigma$$
$$= -\frac{2}{y_1-x_1}\xi(\phi_1+\phi_2-y_1,\cdot) - \frac{2^{\frac{\alpha}{\kappa}+1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}+1}}\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)\right]. \quad (28)$$

Setting (27) and (28) in (26) yields (25).

Theorem 7.*Let* M_1 *hold. Also, suppose that* $|\xi'|$ *is a convex stochastic process on* $[\phi_1,\phi_2]$, *then*

$$\Bigg|\frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2} - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\Bigg[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-y_1)^+}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right) + \left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}(\phi_1+\phi_2-x_1)^-}\right)\times\left((\xi\circ\psi)\left(\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right)\right)\Bigg]\Bigg|$$
$$\le \frac{(y_1-x_1)}{2\left(\frac{\alpha}{\kappa}+1\right)}\times\left\{\left[\left|\xi'(\phi_1,\cdot)\right|+\xi'|(\phi_2,\cdot)|\right]-\left(\frac{\left|\xi'(x_1,\cdot)\right|+\left|\xi'(y_1,\cdot)\right|}{2}\right)\right\}, \quad (29)$$

for all $x_1,y_1 \in [\phi_1,\phi_2]$ *with* $x_1 < y_1$, *and* $\Gamma_\kappa(\cdot)$ *is* κ*-gamma function.*

*Proof.*By using Lemma 4, Jensen-Mercer inequality (2), properties of absolute value and the fact that $\left|\xi'\right|$ is a convex stochastic process yields (29).

Corollary 6.*By putting* $\psi(\Upsilon) = \Upsilon$ *yields the following inequality*

$$\Bigg|\frac{\xi(\phi_1+\phi_2-x_1,\cdot)+\xi(\phi_1+\phi_2-y_1,\cdot)}{2} - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\times\left[\left({}_\kappa I^{\alpha}_{(\phi_1+\phi_2-y_1)^+}\right)\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right) + \left({}_\kappa I^{\alpha}_{(\phi_1+\phi_2-x_1)^-}\right)\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\right]\Bigg|$$
$$\le \frac{(y_1-x_1)}{2\left(\frac{\alpha}{\kappa}+1\right)}\times\left\{\left[\left|\xi'(\phi_1,\cdot)\right|+\xi'|(\phi_2,\cdot)|\right]-\left(\frac{\left|\xi'(x_1,\cdot)\right|+\left|\xi'(y_1,\cdot)\right|}{2}\right)\right\}.$$

Lemma 5.*Consider* $\xi : I \times \Omega \to \mathbb{R}$ *be a mean square differentiable stochastic process on* I° *and* ξ' *is a mean square integrable on* $[\phi_1,\phi_2]$, *where* $\phi_1,\phi_2 \in I$ *with* $0 \le \phi_1 < \phi_2$. *Also, suppose that* $\psi(\cdot)$ *is an increasing and positive monotone on* $[\phi_1,\phi_2]$ *and* ψ' *is a mean square continuous* $(differentiable)$ *on* (ϕ_1,ϕ_2) *and* $\alpha,\kappa > 0$, *then*

$$\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right) - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\times\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1)\right)\right) + \left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1)\right)\right)\right]$$
$$= \frac{(y_1-x_1)}{4}\left[\int_0^1 \varsigma^{\frac{\alpha}{\kappa}}\xi'\left(\phi_1+\phi_2-\left(\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1\right)\right)d\varsigma - \int_0^1 \varsigma^{\frac{\alpha}{\kappa}}\xi'\left(\phi_1+\phi_2-\left(\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1\right)\right)d\varsigma\right], \quad (30)$$

for all $x_1, y_1 \in [\phi_1, \phi_2]$ *with* $x_1 < y_1$, *and* $\Gamma_\kappa(\cdot)$ *is* κ*-gamma function.*

Proof. The proof of this Lemma is similar to the proof of Lemma 4.

Theorem 8. *Let* M_1 *hold. Also, consider that* $|\xi'|$ *is convex stochastic process on* $[\phi_1, \phi_2]$, *then*

$$\begin{aligned}&\left|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\right.\\&\times\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\right.\\&+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\\&\left.\left.\times\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\right]\right|\\&\le\frac{(y_1-x_1)}{2\left(\frac{\alpha}{\kappa}+1\right)}\\&\times\left\{\left[\left|\xi'(\phi_1,\cdot)\right|+\xi'|(\phi_2,\cdot)|\right]-\left(\frac{\left|\xi'(x_1,\cdot)\right|+\left|\xi'(y_1,\cdot)\right|}{2}\right)\right\},\end{aligned}\tag{31}$$

for all $x_1, y_1 \in [\phi_1, \phi_2]$ *with* $x_1 < y_1$, *and* $\Gamma_\kappa(\cdot)$ *is* κ*-gamma function.*

Proof. By Lemma 5, Jensen-Mercer inequality (2), properties of absolute value and the fact that $\left|\xi'\right|$ is convex stochastic process yields that

$$\begin{aligned}&\left|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\right.\\&\times\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\right.\\&\left.\left.+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\right]\right|\\&\le\frac{(y_1-x_1)}{4}\left[\int_0^1\varsigma^{\frac{\alpha}{\kappa}}\left|\xi'\left(\phi_1+\phi_2-\left(\frac{\varsigma}{2}x_1+\frac{2-\varsigma}{2}y_1,\cdot\right)\right)\right|d\varsigma\right.\\&\left.+\int_0^1\varsigma^{\frac{\alpha}{\kappa}}\left|\xi'\left(\phi_1+\phi_2-\left(\frac{2-\varsigma}{2}x_1+\frac{\varsigma}{2}y_1,\cdot\right)\right)\right|d\varsigma\right]\\&\le\frac{(y_1-x_1)}{4}\left[\int_0^1\varsigma^{\frac{\alpha}{\kappa}}\left\{\left|\xi'(\phi_1,\cdot)\right|+\xi'|(\phi_2,\cdot)|\right.\right.\\&\left.-\left(\frac{\varsigma}{2}\xi'|(x_1,\cdot)|+\frac{2-\varsigma}{2}\xi'|(y_1,\cdot)|\right)\right.\\&+\int_0^1\varsigma^{\frac{\alpha}{\kappa}}\left\{\left|\xi'(\phi_1,\cdot)\right|+\xi'|(\phi_2,\cdot)|\right.\\&\left.\left.-\left(\frac{2-\varsigma}{2}\xi'|(x_1,\cdot)|+\frac{\varsigma}{2}\xi'|(y_1,\cdot)|\right)\right\}\right]d\varsigma.\end{aligned}\tag{32}$$

After simplification, we obtain (31).

Corollary 7. *By choosing* $\psi(\Upsilon) = \Upsilon$ *yields the following inequality*

$$\begin{aligned}&\left|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\right.\\&\times\left[\left({}_\kappa I^{\alpha}_{\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\xi(\phi_1+\phi_2-x_1,\cdot)\right.\\&\left.\left.+\left({}_\kappa I^{\alpha}_{\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\xi(\phi_1+\phi_2-y_1,\cdot)\right]\right|\\&\le\frac{(y_1-x_1)}{2\left(\frac{\alpha}{\kappa}+1\right)}\\&\times\left\{\left[\left|\xi'(\phi_1,\cdot)\right|+\xi'|(\phi_2,\cdot)|\right]-\left(\frac{\left|\xi'(x_1,\cdot)\right|+\left|\xi'(y_1,\cdot)\right|}{2}\right)\right.\end{aligned}$$

Theorem 9. *Suppose* M_1 *hold. If* $\left|\xi'\right|^q$ *is a convex stochastic process on* $[\phi_1, \phi_2]$, *for* $q \ge 1$ *with* $\frac{1}{\mu}+\frac{1}{q}=1$, *then*

$$\begin{aligned}&\left|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\right.\\&\times\left[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\right.\\&\left.\left.+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\right]\right|\\&\le\frac{(y_1-x_1)}{4}\left(\frac{\kappa}{\mu\alpha+\kappa}\right)^{\frac{1}{\mu}}\times\left[\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right.\right.\\&\left.-\left(\frac{1}{4}\left|\xi'(x_1,\cdot)\right|^q+\frac{3}{4}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}}\\&\left.+\left(\left|\xi'(\phi_1)\right|^q+\left|\xi'(\phi_2)\right|^q-\left(\frac{3}{4}\left|\xi'(x_1)\right|^q+\frac{1}{4}\left|\xi'(y_1)\right|^q\right)\right)^{\frac{1}{q}}\right],\end{aligned}\tag{33}$$

for all $x_1, y_1 \in [\phi_1, \phi_2]$ *with* $x_1 < y_1$, *and* $\Gamma_\kappa(\cdot)$ *is* κ*-gamma function.*

Proof. Applying Lemma 5, Hölder integral inequality, Jensen-Mercer inequality (2), properties of absolute value and the fact that $\left|\xi'\right|^q$ is convex stochastic process yields the desired inequality (33).

Corollary 8. *Substituting* $\psi(\Upsilon) = \Upsilon$ *yields the following inequality*

$$\begin{aligned}&\left|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\right.\\&\times\left[\left({}_\kappa I^{\alpha}_{\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\xi(\phi_1+\phi_2-x_1)\right.\\&\left.\left.+\left({}_\kappa I^{\alpha}_{\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\xi(\phi_1+\phi_2-y_1)\right]\right|\\&\le\frac{(y_1-x_1)}{4}\left(\frac{\kappa}{\mu\alpha+\kappa}\right)^{\frac{1}{\mu}}\left[\left(\left|\xi'(\phi_1)\right|^q+\left|\xi'(\phi_2)\right|^q\right.\right.\\&\left.-\left(\frac{1}{4}\left|\xi'(x_1)\right|^q+\frac{3}{4}\left|\xi'(y_1)\right|^q\right)\right)^{\frac{1}{q}}+\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right.\\&\left.\left.-\left(\frac{3}{4}\left|\xi'(x_1,\cdot)\right|^q+\frac{1}{4}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}}\right].\end{aligned}$$

4 New results related to improved Hölder inequalities

Theorem 10. *Suppose M_1 hold. If $\left|\xi'\right|^q$ is a convex stochastic process on $[\phi_1,\phi_2]$, for $q\geq 1$, then*

$$\begin{aligned}
&\Bigg|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\\
&-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\Bigg[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\\
&\times\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\\
&+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\\
&\times\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\Bigg]\Bigg|\\
&\leq\frac{y_1-x_1}{4}\Bigg[\Bigg\{\left(\frac{1}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}-\left(\frac{\left|\xi'(x_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}\right.\right.\\
&\left.\left.+\frac{\left|\xi'(y_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}\right)\right)^{\frac{1}{q}}+\left(\frac{1}{\left(\frac{\alpha}{\kappa}+2\right)}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}\right.\\
&\left.-\left(\frac{\left(\frac{\alpha}{\kappa}+5\right)\left|\xi'(x_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}+\frac{\left|\xi'(y_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}\right)\right)^{\frac{1}{q}}\Bigg\}\\
&+\Bigg\{\left(\frac{1}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}\right.\\
&\left.-\left(\frac{\left(\frac{\alpha}{\kappa}+5\right)\left|\xi'(x_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}+\frac{\left|\xi'(y_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}\right)\right)\\
&+\left(\frac{1}{\left(\frac{\alpha}{\kappa}+2\right)}\right)^{1-\frac{1}{q}}\left(\frac{\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)}{\left(\frac{\alpha}{\kappa}+2\right)}\right.\\
&\left.-\left(\frac{\left(\frac{\alpha}{\kappa}+4\right)\left|\xi'(x_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}+\frac{\left|\xi'(y_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+3\right)}\right)\right)^{\frac{1}{q}}\Bigg\},
\end{aligned}\tag{34}$$

for all $x_1,y_1\in[\phi_1,\phi_2]$ with $x_1<y_1$, and $\Gamma_\kappa(\cdot)$ is κ-gamma function.

Proof. It follows from Lemma 5, improved power-mean integral inequality, properties of absolute value, Jensen-Mercer inequality (2) and the fact that $\left|\xi'\right|^q$ is convex stochastic process yields the required result.

Corollary 9. *Substituting $\psi(\Upsilon)=\Upsilon$ in Theorem 10, we get*

$$\begin{aligned}
&\Bigg|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\\
&-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\Bigg[\left({}_\kappa I^{\alpha}_{\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\xi(\phi_1+\phi_2-x_1,\cdot)\\
&+\left({}_\kappa I^{\alpha}_{\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\xi(\phi_1+\phi_2-y_1,\cdot)\Bigg]\Bigg|\\
&\leq\frac{y_1-x_1}{4}\Bigg[\Bigg\{\left(\frac{1}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}-\left(\frac{\left|\xi'(x_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}\right.\right.\\
&\left.\left.+\frac{\left|\xi'(y_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}\right)\right)^{\frac{1}{q}}+\left(\frac{1}{\left(\frac{\alpha}{\kappa}+2\right)}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}\right.\\
&\left.-\left(\frac{\left(\frac{\alpha}{\kappa}+5\right)\left|\xi'(x_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}+\frac{\left|\xi'(y_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}\right)\right)^{\frac{1}{q}}\Bigg\}\\
&+\Bigg\{\left(\frac{1}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)}{\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)}\right.\\
&\left.-\left(\frac{\left(\frac{\alpha}{\kappa}+5\right)\left|\xi'(x_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+1\right)\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}+\frac{\left|\xi'(y_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}\right)\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{\left(\frac{\alpha}{\kappa}+2\right)}\right)^{1-\frac{1}{q}}\left(\frac{\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)}{\left(\frac{\alpha}{\kappa}+2\right)}\right.\\
&\left.-\left(\frac{\left(\frac{\alpha}{\kappa}+4\right)\left|\xi'(x_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+2\right)\left(\frac{\alpha}{\kappa}+3\right)}+\frac{\left|\xi'(y_1,\cdot)\right|^q}{2\left(\frac{\alpha}{\kappa}+3\right)}\right)\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Theorem 11. *Let M_1 hold. If $\left|\xi'\right|^q$ is a convex stochastic process on $[\phi_1,\phi_2]$, for $q\geq 1$ with $\frac{1}{\mu}+\frac{1}{q}=1$, then*

$$\begin{aligned}
&\Bigg|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)\\
&-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\Bigg[\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)\\
&\times\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-x_1,\cdot)\right)\right)\\
&+\left({}_\kappa I^{\alpha:\psi}_{\psi^{-1}\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\left((\xi\circ\psi)\left(\psi^{-1}(\phi_1+\phi_2-y_1,\cdot)\right)\right)\Bigg]\Bigg|
\end{aligned}$$

$$\leq \frac{(y_1-x_1)}{4}\Bigg[\Bigg\{\left(\frac{1}{\left(\frac{\alpha\mu}{\kappa}+1\right)\left(\frac{\alpha\mu}{\kappa}+2\right)}\right)^{\frac{1}{\mu}}$$
$$\times\left(\frac{1}{2}\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)\right.$$
$$\left.-\left(\frac{1}{12}\left|\xi'(x_1,\cdot)\right|^q+\frac{5}{12}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\left(\frac{\alpha\mu}{\kappa}+2\right)}\right)^{\frac{1}{\mu}}\left(\frac{1}{2}\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)\right.$$
$$\left.-\left(\frac{1}{6}\left|\xi'(x_1,\cdot)\right|^q+\frac{1}{3}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}}\Bigg\}$$
$$+\Bigg\{\left(\frac{1}{\left(\frac{\alpha\mu}{\kappa}+1\right)\left(\frac{\alpha\mu}{\kappa}+2\right)}\right)^{\frac{1}{\mu}}$$
$$\times\left(\frac{1}{2}\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)\right.$$
$$\left.-\left(\frac{5}{12}\left|\xi'(x_1,\cdot)\right|^q+\frac{1}{12}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\left(\frac{\alpha\mu}{\kappa}+2\right)}\right)^{\frac{1}{\mu}}\left(\frac{1}{2}\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)\right.$$
$$\left.-\left(\frac{1}{3}\left|\xi'(x_1,\cdot)\right|^q+\frac{1}{6}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}}\Bigg\}\Bigg],$$

for all $x_1,y_1\in[\phi_1,\phi_2]$ with $x_1<y_1$, and $\Gamma_\kappa(\cdot)$ is κ-gamma function.

Proof. By using Lemma 5, from Hölder-İşcan inequality, properties of absolute value, Jensen-Mercer inequality (2) and the fact that $\left|\xi'\right|^q$ is a convex stochastic process simultaneously yields the desired result.

Corollary 10. *Substituting $\psi(\Upsilon)=\Upsilon$ in Theorem 11, we get*

$$\Bigg|\xi\left(\phi_1+\phi_2-\frac{x_1+y_1}{2},\cdot\right)$$
$$-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(y_1-y_1)^{\frac{\alpha}{\kappa}}}\Bigg[\left({}_\kappa I^\alpha_{\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^+}\right)$$
$$\times\xi(\phi_1+\phi_2-x_1,\cdot)+\left({}_\kappa I^\alpha_{\left(\phi_1+\phi_2-\frac{x_1+y_1}{2}\right)^-}\right)\xi(\phi_1+\phi_2-y_1,\cdot)\Bigg]\Bigg|$$
$$\leq \frac{(y_1-x_1)}{4}\Bigg[\Bigg\{\left(\frac{1}{\left(\frac{\alpha\mu}{\kappa}+1\right)\left(\frac{\alpha\mu}{\kappa}+2\right)}\right)^{\frac{1}{\mu}}$$
$$\times\left(\frac{1}{2}\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)\right.$$
$$\left.-\left(\frac{1}{12}\left|\xi'(x_1,\cdot)\right|^q+\frac{5}{12}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}} \tag{35}$$
$$+\left(\frac{1}{\left(\frac{\alpha\mu}{\kappa}+2\right)}\right)^{\frac{1}{\mu}}\left(\frac{1}{2}\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)\right.$$
$$\left.-\left(\frac{1}{6}\left|\xi'(x_1,\cdot)\right|^q+\frac{1}{3}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}}\Bigg\}$$
$$+\Bigg\{\left(\frac{1}{\left(\frac{\alpha\mu}{\kappa}+1\right)\left(\frac{\alpha\mu}{\kappa}+2\right)}\right)^{\frac{1}{\mu}}\left(\frac{1}{2}\left(\left|\xi'(\phi_1,\cdot)\right|^q+\left|\xi'(\phi_2,\cdot)\right|^q\right)\right.$$
$$\left.-\left(\frac{5}{12}\left|\xi'(x_1,\cdot)\right|^q+\frac{1}{12}\left|\xi'(y_1,\cdot)\right|^q\right)\right)^{\frac{1}{q}}+\left(\frac{1}{\left(\frac{\alpha\mu}{\kappa}+2\right)}\right)^{\frac{1}{\mu}}$$
$$\times\left(\frac{1}{2}\left(\left|\xi'(\phi_1)\right|^q+\left|\xi'(\phi_2)\right|^q\right)\right.$$
$$\left.-\left(\frac{1}{3}\left|\xi'(x_1)\right|^q+\frac{1}{6}\left|\xi'(y_1)\right|^q\right)\right)^{\frac{1}{q}}\Bigg\}\Bigg].$$

5 Applications

Consider the following special means for arbitrary $l_1,l_2\in\mathbb{R}$, $l_1\neq l_2$:

Arithmetic mean

$$A(l_1,l_2)=\frac{l_1+l_2}{2},\quad l_1,l_2\in\mathbb{R}.$$

Harmonic mean

$$H(l_1,l_2)=\frac{2}{\frac{1}{l_1}+\frac{1}{l_2}},\quad l_1,l_2\in\mathbb{R}\setminus\{0\}.$$

Logarithmic mean

$$L(l_1,l_2)=\frac{l_2-l_1}{ln|l_2|-ln|l_1|}\quad |l_1|\neq|l_2|, l_1l_2\neq 0.$$

r-logarithmic mean

$$L_r(l_1,l_2)=\left[\frac{l_2^{r+1}-l_1^{r+1}}{(r+1)(l_2-l_1)}\right]^{\frac{1}{r}},$$
$$r\in Z\setminus\{-1,0\},\ l_1,l_2\in\mathbb{R},\ l_1\neq l_2.$$

Now, we give some applications to special means:

Proposition 1. *Let $\phi_1,\phi_2,x_1,y_1\in\mathbb{R}^+$, $\phi_1<\phi_2$, $0\notin[\phi_1,\phi_2]$, $c>0$, and $n\in\mathbb{Z}$, $|n|\geq 2$.*

$$\Bigg|A\left((2A(\phi_1,\phi_2)-x_1)^n,(2A(\phi_1,\phi_2)-y_1)^n\right)$$
$$-\left(L_n^n\left(2A(\phi_1,\phi_2)-y_1,2A(\phi_1,\phi_2)-x_1\right)\right)^n\Bigg|$$
$$\leq\left(\frac{y_1-x_1}{4}\right)$$
$$\times\left[n(\phi_1)^{n-1}+n(\phi_2)^{n-1}-\left(\frac{n(x_1)^{n-1}+n(y_1)^{n-1}}{2}\right)\right].$$

Proof. By substituting $\xi(x,\cdot)=x^n$, $\psi(x)=x$, $\alpha=\kappa=1$ in Theorem 6, we get the desired result.

Proposition 2.*Let $\phi_1, \phi_2, x_1, y_1 \in \mathbb{R}^+$, $\phi_1 < \phi_2$, $0 \notin [\phi_1, \phi_2]$, $c > 0$, and $n \in \mathbb{Z}$, $|n| \geq 2$.*

$$\left| H^{-1}\left((2A(\phi_1,\phi_2)-x_1)^n, (2A(\phi_1,\phi_2)-y_1)^n \right) - \left(L^{-1}\left(2A(\phi_1,\phi_2)-y_1, 2A(\phi_1,\phi_2)-x_1 \right) \right)^n \right| \leq \left(\frac{y_1-x_1}{4} \right) \left[\frac{1}{(\phi_1)^2} + \frac{1}{(\phi_2)^2} - \left(\frac{\frac{1}{(x_1)^2} + \frac{1}{(y_1)^2}}{2} \right) \right].$$

*Proof.*By choosing $\xi(x,\cdot) = \frac{1}{x}$, $\psi(x) = x$, $\alpha = \kappa = 1$ in Theorem 6 yields the desired result.

6 Conclusion

In the present note, we develop inequalities of the Hermite-Hadamard-Mercer type by using the Jensen-Mercer inequality for convex stochastic processes in the setting of ψ_κ-Riemann-Liouville fractional integrals. We also derive different inequalities of the Hermite-Hadamard-Mercer type with the help of Hölder integral and power-mean integral inequality. In literature, some known results become the special cases of these inequalities as mentioned in remarks. Some applications to special means are also developed. All inequalities and related results presented here are unique, fascinating, and important in the field of integral inequalities. We hope that our new ideas and techniques utilized in this paper may inspire several interested authors to explore new results. It is an interesting and new result that the upcoming researchers can offer the same inequalities for different types of convex stochastic processes in their future research.

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Conflict of Interest The authors declare that they have no conflict of interest.

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