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The Lifetime Performance Index for Stacy Distribution Under Progressive First-Failure Type II Right Censoring Scheme Applied to Medical and Engineering Data

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Abstract: In this study, he statistical inference of the lifetime performance index for the Stacy distribution using a first-failure progressive right type II censored sample is achieved. Two real-life medical and engineering applications, as well as a simulated one, are developed to illustrate the applicability of the suggested technique. The findings demonstrated the capability of the presented inference technique and its usefulness in making appropriate conclusions in many fields.

Keywords: lifetime Performance, Censoring, First-Failure Progressive, Stacy Distribution.

1 Introduction

The main goal of studying the processing capacity via PCIs indices is to compute the prospective capabilities and performance of a process. Dimensions of the product quality have different descriptions and are evaluated via several issues for example; performance, reliability, and conformance to the standards. Process capability indices (PCIs) have been used to determine product quality. There are three sorts of PCIs indices: the first measures the target-the-better type of quality feature, the second the larger-the-better type, and the third the smaller-the-better type see [1], [2].

Because of time limits or other material resources, concerns about funding, or data collecting challenges, the tester in lifetime testing may not be able to monitor the lifetimes of all the objects or products on tests. As a result, censored samples may be encountered in actuality. When just a lower lifetime limit is identified, right censoring occurs. In other ways, whether certain units are still operational at the time of termination. One type of correct censoring is "Type II censoring," which happens when the test is discontinued after a certain number of units fail. The progressive Type-II censoring system consists of the following steps: first, monitoring n items until the mth failure happens, after which the test is completed. Second, once ith item fails$(i = 1,2,\dots,m-1)$, the surviving items "r_i" are excluded from the test. Finally, when the mth item fails all remaining units $r_m = n - m - \sum_{i=1}^{m-1} r_i$ are eliminated. In [3], the first-failure scheme is given as follows: $m \times n$ items are separated into m equal groups, and the test is run by checking each group sequentially and ending when the first failure in each grou progressive first-failure p is detected. To generalize all the schemes above, the first-failure progressive type II censoring is used , see [1], [4]. There are "N" items, which are separated into "n" distinct groups, each of which has "k" items $(N = n \times k)$ placed on a test. At the m^{th} failure occurs the life test is completed. Choose the group which contains the i^{th} item and select randomly "R_i" groups and remove them from the test when the i^{th} item fails $\{i = 1,2,\dots m-1\}$, when the mth failure occurs, delete all remaining groups from the test.

There are several papers in the literature on various censoring schemes for examining performance indices, for example; under progressive first-failure censoring, statistical inference of the Lindley distribution's lifetime performance index [5]. The performance index of the Burr XII distribution is implemented in [6] and [7] under progressive censoring. Life performance index estimation using the Weibull distribution with progressive first-failure censoring in [8]. The power Lomax distribution's lifetime performance index is based on a progressive first-failure censoring scheme [9]. Additionally, many papers have been published to investigate progressive censoring and evaluate the performance index under exponential distribution for various censoring schemes, for example; [10], [11], [12]. Moreover, infer the lifetime performance index with a power Rayleigh distribution [13]. Assessing the lifetime performance index for Kumaraswamy distribution under the first-failure progressive censoring scheme for ball bearing revolutions [14]. Furthermore,[15] proposed lifetime performance index inferences for the Lomax distribution in the case of progressively type-II censored data

Amoroso distribution was introduced in 1925 and was originally applied to model lifetimes [3]. It happened as the Weibullization of the standard Gamma distribution [16] and, with integer α , in extreme value statistics [17]. Stacy distribution was firstly studied in [18] which is a special case of the Amoroso distribution in which the location parameter α is set to zero.

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An important and useful feature of the Amoroso distribution [19] is that many interesting and common probability distributions arise as special cases or limits such that Chi-square, Gamma, Erlang, Inverse chi, and Maxwell distribution.

The purpose of this study is to calculate the lifetime performance index and evaluate its statistical inference on a first-failure gradually type II right censoring sample using the Stacy distribution.

This paper is organized as follows: Section 2 is concerned with the lifetime performance index for Stacy distribution. The conforming rate of Stacy distribution versus the performance indexes is defined in section 3. In Section 4, the maximum likelihood estimation of the performance index is studied. The testing procedure owed to the lifetime performance index is completed in Section 5. Finally, Section 6 is devoted to demonstrating the potentiality of the suggested approach under Stacy distribution, medical and engineering real-life applications, and a simulated example is being applied.

2 The Lifetime Performance Index of Stacy Distribution

A process capabilities index C_L is established by [2] to evaluate the larger-the better quality property. A longer lifetime has a higher quality. In general, the lifetime must exceed L unit times where L is the *known lower specification limit*. The item's performance is then evaluated using the lifetime performance index C_L, as shown below.

$$C_L = \frac{\mu - L}{\sigma} \tag{1}$$

where μ, σ represent the process mean and process standard deviation respectively.

Assume that an item's lifetime X has a Stacy distribution with parameters α, θ *and* β with the probability density function and the cumulative distribution function defined respectively as;

$$f(x) = \frac{1}{\Gamma[\alpha]}\left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{(\alpha\beta-1)} e^{-\left(\frac{x}{\theta}\right)^{\beta}}, x > 0, \alpha, \theta, \beta > 0 \tag{2}$$

$$F(x) = \frac{\Gamma[\alpha] - \Gamma[\alpha, \left(\frac{x}{\theta}\right)^{\beta}]}{\Gamma[\alpha]} \qquad , x > 0, \alpha, \theta, \beta > 0 \tag{3}$$

with mean μ and standard deviation σ as indicated here

$$\mu = \frac{\theta\Gamma[\alpha + \frac{1}{\beta}]}{\Gamma[\alpha]} \quad , \ \alpha, \theta, \beta > 0 \tag{4}$$

$$\sigma = \frac{\theta^2\left(-\Gamma\left[\alpha + \frac{1}{\beta}\right]^2 + \Gamma[\alpha]\Gamma\left[\alpha + \frac{2}{\beta}\right]\right)}{\Gamma[\alpha]^2} \ , \ \alpha, \theta, \beta > 0 \tag{5}$$

Then the lifetime performance index C_L is obtained as

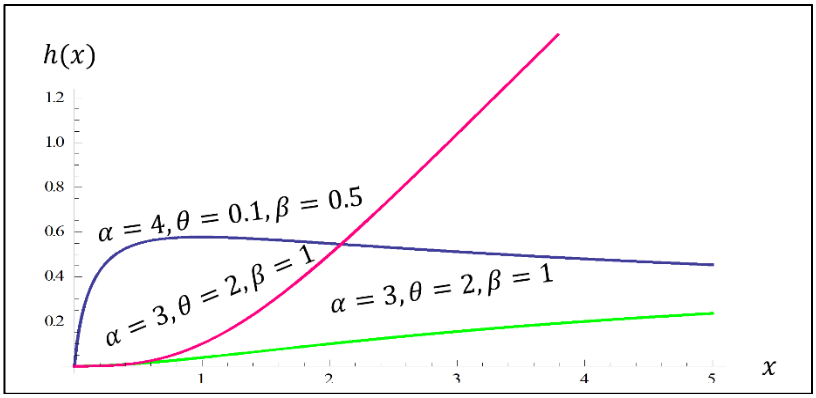
$$C_L = \frac{-L + \frac{\theta\Gamma\left[\alpha + \frac{1}{\beta}\right]}{\Gamma[\alpha]}}{\sqrt{\frac{\theta^2(-\Gamma[\alpha + \frac{1}{\beta}]^2 + \Gamma[\alpha]\Gamma[\alpha + \frac{2}{\beta}])}{\Gamma[\alpha]^2}}} \tag{6}$$

where $$-\infty < C_L < \frac{\theta\Gamma\left[\alpha + \frac{1}{\beta}\right]}{\Gamma[\alpha]\sqrt{\frac{\theta^2\left(-\Gamma\left[\alpha + \frac{1}{\beta}\right]^2 + \Gamma[\alpha]\Gamma\left[\alpha + \frac{2}{\beta}\right]\right)}{\Gamma[\alpha]^2}}}, \alpha, \theta, \beta > 0$$

The hazard rate $h(x)$ of Stacy distribution is

$$h(x) = \frac{e^{-\left(\frac{x}{\theta}\right)^{\beta}}\beta\left(\frac{x}{\theta}\right)^{\alpha\beta}}{x\Gamma\left[\alpha, \left(\frac{x}{\theta}\right)^{\beta}\right]}, x > 0, \alpha, \theta, \beta > 0 \tag{7}$$

When the process mean $\frac{\theta\Gamma[\alpha + \frac{1}{\beta}]}{\Gamma[\alpha]} > L$, following that the performance index $C_L > 0$. According to Equation (7), we note that the hazard function is an increasing function in x see (plot 2.1). From Equations (4), (5), and (6), we can see that the mean $\frac{\theta\Gamma[\alpha + \frac{1}{\beta}]}{\Gamma[\alpha]}$ is inversely related to failure rate and directly proportional to the greater lifetime performance index $C_L > 0$. Therefore, C_L provides a reliable estimation of new product lifetime performance.



Plot 2.1. Hazard function of Stacy distribution

3 The Conforming Rate of Stacy Distribution

In this section, we obtain the reliability function, hazard rate and Reverse hazard rate functions of the Proposed weighted power Shanker distribution.

3.1 Reliability Function

If the product's lifetime exceeds the lower specification limit L, it is considered a conforming product. The conforming product ratio, often known as the conforming rate, can be calculated for $X \sim Stacy(\alpha, \theta, \beta)$as follows,

$$P_r = P(X \geq L)$$

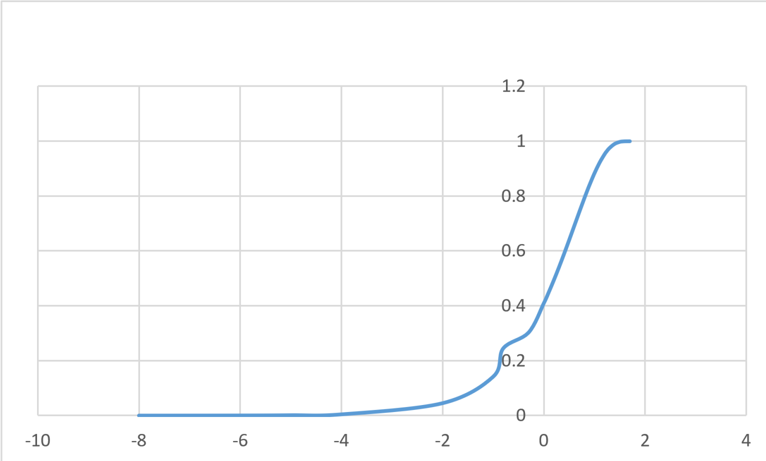
$$= \frac{\Gamma\left[\alpha, \left(\frac{\frac{\Gamma\left[\alpha+\frac{1}{\beta}\right]}{\Gamma[\alpha]} - \sqrt{\frac{\theta^2\left(-\Gamma\left[\alpha+\frac{1}{\beta}\right]^2 + \Gamma[\alpha]\Gamma\left[\alpha+\frac{2}{\beta}\right]\right)}{\Gamma[\alpha]^2}}C_L}{\theta}\right)^\beta\right]}{\Gamma[\alpha]} \qquad (8)$$

where
$$-\infty < C_L < \frac{\theta\Gamma\left[\alpha+\frac{1}{\beta}\right]}{\Gamma[\alpha]\sqrt{\frac{\theta^2\left(-\Gamma\left[\alpha+\frac{1}{\beta}\right]^2 + \Gamma[\alpha]\Gamma\left[\alpha+\frac{2}{\beta}\right]\right)}{\Gamma[\alpha]^2}}}, \alpha, \theta, \beta > 0,$$

Table 1: The lifetime performance index C_L vs the conforming rate P_r for Stacy distribution with $(\hat{\alpha}, \hat{\theta}, \hat{\beta}) = (8.90077, 7.21435, 0.57933)$

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.0000	-0.3	0.3029	0.7	0.7416	1.25	0.9652
$\mathbf{-8}$	0.0001	0	0.4095	0.8	0.7917	1.3	0.9757
$\mathbf{-6}$	0.0005	0.1	0.4507	0.82	0.8014	1.4	0.9905
$\mathbf{-5}$	0.0014	0.2	0.4945	0.85	0.8159	1.5	0.9977
$\mathbf{-4}$	0.0045	0.3	0.5409	0.9	0.8393	1.55	0.9992
$\mathbf{-2}$	0.0452	0.4	0.5893	1	0.8830	1.6	0.9998
$\mathbf{-1}$	0.1419	0.5	0.6395	1.1	0.9213	1.67	1.0000
$\mathbf{-0.5}$	0.2454	0.6	0.6905	1.2	0.9525	1.7	1

Note that: $C_L \to\approx 1.7058 \Rightarrow P_r \to 1.0$.

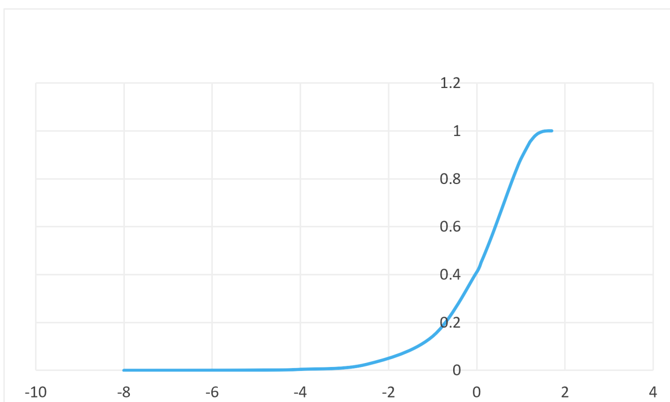


Plot 3.1. The lifetime performance index C_L vs the conforming rate P_r for Stacy distribution

Table 2: The lifetime performance index C_L vs the conforming rate P_r for Stacy distribution with $(\hat{\alpha}, \hat{\theta}, \hat{\beta}) = (8.34687, 9.0908, 0.5979)$.

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.0000	0.2	0.4949	0.85	0.8151	1.5	0.9975
−8	0.0000	0.3	0.5412	0.9	0.8385	1.6	0.9998
−5	0.0014	0.4	0.5895	1	0.8821	1.65	1.0000
−4	0.0044	0.6	0.6903	1.2	0.9517	1.7	1
−2.5	0.0252	0.7	0.7412	1.22	0.9570		
−1	0.1422	0.8	0.7910	1.3	0.9750		
0	0.4101	0.82	0.8008	1.35	0.9836		
0.1	0.4512	0.84	0.8104	1.4	0.9901		

Note that: $C_L \rightarrow\approx 1.706 \Rightarrow P_r \rightarrow 1.0$



Plot 3.2. The lifetime performance index C_L vs the conforming rate P_r for Stacy distribution

4 Maximum Likelihood Estimator of the Stacy Distribution's Lifetime Performance Index

Let $X_{1:m:n:k}, X_{2:m:n:k}, \ldots\ldots, X_{m:m:n:k}$ be the progressive first-failure type II right censored sample from a continuous population with $p.d.f$ and $c.d.f$ $\boldsymbol{f_X(.;\ \Theta)}$ **and** $\boldsymbol{F_X(.;\ \Theta)}$ **respectively, where** $\boldsymbol{\Theta}$ **is a vector of parameters.** Following [4], the associated likelihood function of the observed data $X = (x_{1:m:n:k}, x_{2:m:n:k}, \ldots\ldots, x_{m:m:n:k})$ is given by

$$L(\boldsymbol{\Theta}, X) = Ck^m \prod_{i=1}^{m} f_X(x_{i:m:n:k};\ \boldsymbol{\Theta})(1 - F_X(x_{i:m:n:k};\ \boldsymbol{\Theta}))^{k(R_i+1)-1} \tag{9}$$

where $0 < x_{1:m:n:k} < x_{2:m:n:k} < \cdots \ldots < x_{m:m:n:k} < \infty$

and $C = n(n - R_1 - 1)(n - R_2 - 1) \ldots\ldots\ldots (n - \sum_{i=1}^{m-1} R_i - m + 1)$.

Consider that the progressive first-failure type II right censoring sample from a life test of n products whose lifetimes follow $Stacy(\alpha,\theta,\beta)$ distribution. From (2) and (3), the likelihood function is as follows

$$L(\alpha,\theta,\beta;X) = C\,k^m \prod_{i=1}^{m} \frac{1}{\Gamma[\alpha]}\left(\frac{\beta}{\theta}\right)\left(\frac{x_{i:m:n:k}}{\theta}\right)^{(\alpha\beta-1)} e^{-\left(\frac{x_{i:m:n:k}}{\theta}\right)^{\beta}}\left[\frac{\Gamma[\alpha]-\Gamma[\alpha,\left(\frac{x_{i:m:n:k}}{\theta}\right)^{\beta}]}{\Gamma[\alpha]}\right]^{k(R_i+1)-1} \tag{10}$$

The natural Logarithm of $L(\alpha,\theta,\beta;X)$ is obtained as

$$ln\big(L(\alpha,\theta,\beta;X)\big) = \text{Log}[c] + m\text{Log}[k] - m\text{Log}[\Gamma[\alpha]] + m\text{Log}[\beta] - m\text{Log}[\theta] + \sum_{i=1}^{m}(\alpha\beta-1)\text{Log}[\frac{x_{i:m:n:k}}{\theta}] - \sum_{i=1}^{m}(\frac{x_{i:m:n:k}}{\theta})^{\beta} + \sum_{i=1}^{m}(k(R_i+1)-1)\text{Log}\left[\frac{\Gamma\left[\alpha,\left(\frac{x_{i:m:n:k}}{\theta}\right)^{\beta}\right]}{\Gamma[\alpha]}\right] \tag{11}$$

The MLE $\widehat{\boldsymbol{\Theta}} = (\hat{\alpha},\hat{\theta},\hat{\beta})$ can be obtained by equating the first partial derivative of (11) concerning $\alpha,\theta,$ and β. Then the likelihood equations for the parameters α,θ and β is obtained as

$$\frac{\partial\, ln\big(L(\alpha,\theta,\beta;X)\big)}{\partial\alpha} = -m\text{Poly}\Gamma[0,\alpha] + \sum_{i=1}^{m}\beta\text{Log}[\frac{x_{i:m:n:k}}{\theta}] + \sum_{i=1}^{m}\left(L[\alpha](1/[\alpha]\,(\Gamma[\alpha,\left(\frac{x_{i:m:n:k}}{\theta}\right)^{\beta}]\text{Log}[\left(\frac{x_{i:m:n:k}}{\theta}\right)^{\beta}] + \text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,\alpha\},\{\}\},\left(\frac{x_{i:m:n:k}}{\theta}\right)^{\beta}]) - \frac{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]\text{Poly}\Gamma[0,\alpha]}{\Gamma\alpha]}\right)(-1+k(1+R_i))) \Big/ \Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}] \tag{12}$$

$$\frac{\partial\, ln\big(L(\alpha,\theta,\beta;X)\big)}{\partial\theta} = -\frac{m\alpha\beta}{\theta} - \sum_{i=1}^{m} - \frac{\beta x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}}{\theta^2} + \sum_{i=1}^{m}\frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\beta(-1+k(1+R_i))x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+\alpha}}{\theta^2\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} \tag{13}$$

$$\frac{\partial\, ln\big(L(\alpha,\theta,\beta;X)\big)}{\partial\beta} = \frac{m}{\beta} + \sum_{i=1}^{m}\alpha\text{Log}[\frac{x_{i:m:n:k}}{\theta}] - \sum_{i=1}^{m}\text{Log}[\frac{x_{i:m:n:k}}{\theta}](\frac{x_{i:m:n:k}}{\theta})^{\beta} + \sum_{i=1}^{m} - \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\text{Log}[\frac{x_{i:m:n:k}}{\theta}](-1+k(1+R_i))((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{\alpha}}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} \tag{14}$$

Hence;

The closed-form of the above equation is very hard to analytically solved, hence, these non-linear equations will be solved numerically.

Following [20], the invariance property of the MLE is satisfied, then the MLE of C_L has a form

$$\widehat{C_L} = \frac{\hat{\theta}\Gamma\left[\hat{\alpha}+\frac{1}{\hat{\beta}}\right]}{\Gamma[\hat{\alpha}]\sqrt{\frac{\hat{\theta}^2\left(-\Gamma\left[\hat{\alpha}+\frac{1}{\hat{\beta}}\right]^2+\Gamma[\hat{\alpha}]\Gamma\left[\hat{\alpha}+\frac{2}{\hat{\beta}}\right]\right)}{\Gamma[\hat{\alpha}]^2}}} \tag{15}$$

Following [4] and [21], the asymptotic normal distribution has been obtained for the MLEs in Appendix A.

According to [21]under some regularity conditions, the asymptotic normality of MLE of γ is

$$\hat{\gamma}\sim N(\gamma, I(\gamma)^{-1}) \tag{16}$$

where $I(\gamma)$ is the Fisher information matrix. By considering the approximate information matrix $I_O(\hat{\gamma})$ which is defined by

$$I_O(\hat{\gamma}) = -\begin{bmatrix} \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\alpha^2} & \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\alpha\partial\theta} & \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\alpha\partial\beta} \\ \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\theta\partial\alpha} & \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\theta^2} & \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\theta\partial\beta} \\ \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\beta\partial\alpha} & \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\beta\partial\theta} & \frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\beta^2} \end{bmatrix}_{\hat{\gamma}}$$

$$= \begin{bmatrix} v_{\alpha\alpha} & v_{\alpha\theta} & v_{\alpha\beta} \\ v_{\theta\alpha} & v_{\theta\theta} & v_{\theta\beta} \\ v_{\beta\alpha} & v_{\beta\theta} & v_{\beta\beta} \end{bmatrix}_{\hat{\gamma}} \tag{17}$$

By using the variance-covariance matrix $I_O(\hat{\gamma})^{-1}$to estimate $I(\gamma)^{-1}$.

Let $C_L \equiv C(\gamma)$, and due to [22] the multivariate delta method stated that the asymptotic normal distribution of $C(\hat{\gamma})$ is

$$\widehat{C_L} \equiv C(\hat{\gamma}) \sim N(C_L, \Psi_\gamma) \tag{18}$$

The approximate asymptotic variance-covariance matrix $\Psi_{\hat{\gamma}}$ of $C(\gamma)$ to esti mate Ψ_γ and it is defined by

$$\Psi_{\hat{\gamma}} = \begin{pmatrix} \frac{\partial C(\gamma)}{\partial\alpha} & \frac{\partial C(\gamma)}{\partial\theta} & \frac{\partial C(\gamma)}{\partial\beta} \end{pmatrix} I_O(\gamma)^{-1} \begin{pmatrix} \frac{\partial C(\gamma)}{\partial\alpha} \\ \frac{\partial C(\gamma)}{\partial\theta} \\ \frac{\partial C(\gamma)}{\partial\beta} \end{pmatrix}_{\gamma=\hat{\gamma}} \tag{19}$$

5 Testing Technique for the Lifetime Performance Index

The statistical hypothesis testing approach is used to determine whether the lifetime performance index meets the required level. The hypothesis testing and confidence interval can be conducted by considering (18), and (19). Assume that the needed index value for lifetime performance is greater than c^*, where c^*indicate the target value. Hence, the null and the alternative hypothesis is executed as

$$H_o: \; C_L \leq c^*$$

against $\quad H_1: \; C_L > c^*$

For a given specified significance level α, the critical value c_0 is obtained as [7]

$$P\left(\frac{\widehat{C_L} - C_L}{\sqrt{\Sigma_{\hat{\gamma}}}} \leq \frac{C_O - c^*}{\sqrt{\Sigma_{\hat{\gamma}}}}\right) = 1 - \alpha$$

where $\frac{\widehat{C_L} - C_L}{\sqrt{\Sigma_{\hat{\gamma}}}} \sim N(0,1)$ and $\Sigma_{\hat{\gamma}}$ as described in (19).

Then, the critical value is obtained from

$$C_O = c^* + z_\alpha \sqrt{\Sigma_{\hat{\gamma}}} \tag{20}$$

Thus, the $100(1-\alpha)\%$ one-sided confidence interval of C_L is

$$C_L \geq \widehat{C_L} - z_\alpha \sqrt{\Sigma_{\hat{\gamma}}}$$

and the $100(1-\alpha)\%$ lower confidence bound for C_L is

$$\underline{LB} = \widehat{C_L} - z_\alpha \sqrt{\Sigma_{\hat{\gamma}}} \tag{21}$$

6 Real-life Data Application

This section described how to use the above technique in two separate applications, as well as another simulated example that used the Stacy distribution with first-failure progressive type-II censored data.

Example 6.1.

To illustrate the statistical inference for the lifetime performance index, consider a real data set from organ transplant recipients mentioned in [23] and [24]. The data was collected over $N = 56$ blood samples. Using a standard approved procedure to analyse an aliquot of each sample liquid chromatography with high performance (HPIC).

The data is fitted to the Stacy distribution and then compared to various well-known lifetime models using the goodness of fit criteria, thus according [23]

Suppose the first failure, after sorting the data into $n = 14$ groups with $k = 4$ items inside each group, progressive right censoring type II is applied to the data.

The first failure censored is as follows

{35, 71, 77, 87, 99, 109, 129, 148, 162, 185, 198, 203, 241, 275}

The total data set is fitted to the Stacy distribution using goodness of fit criteria $\mathrm{KS} = 0.094043$ and $\mathrm{P-value} = 0.70497$.

Using the next two progressive censoring schemes with $m = 10$ shown in Table (3) and (4) as follows:

Table 3: Censoring Scheme II: $\underline{R_2} = (4,0*9)$

x_i	35	109	129	148	162	185	198	203	241	275
R_i	4	0	0	0	0	0	0	0	0	0

Table 4: Censoring Scheme I: $\underline{R_1} = (2,2,0*8)$

x_i	35	87	129	148	162	185	198	203	241	275
R_i	2	2	0	0	0	0	0	0	0	0

Therefore, for the given schemes, the suggested testing technique for the lifetime performance index C_L is implemented as follows.

Find the solution of Equations (12,13,14) to get the MLE estimates of α, θ *and* β. The obtained MLE estimates of the Stacy parameters α, θ *and* β are shown in Table (5).

Assume that the lower lifetime L is 35. To solve specific concerns about operational performance, the conforming rate P_r must be greater than 80%. Tables (1) and (2), show that the targeted value is $c^* = 0.82$ for all censoring schemes. Then, the required test is

$$H_o:\ C_L \le 0.82$$

$$\text{against} \qquad H_1:\ C_L > 0.82 \tag{22}$$

Specifying the significance level $\alpha = 0.05$.

Through Equations (18, 19, 21), lifetime performance estimate $\widehat{C_L}$, the asymptotic variance $\Sigma_{\hat{\gamma}}$, the 95% lower confidence interval bound for C_L and the 95% one-sided confidence interval for C_L is $[\underline{LB}, \infty)$ are established in Table (5).

The observed results show that the performance index value $c^* = 0.82 \notin [\underline{LB}, \infty)$, indicating that H_o is rejected. Consequently, there is evidence to indicate that the lifetime performance index of HQIC achieves the needed threshold.

Moreover, by using Equation (18, 20), $\widehat{C_L} > C_0 = c^* + z_\alpha\sqrt{\Sigma_{\hat{\gamma}}}$ as shown in Table (5), H_o is rejected so the same conclusion is achieved.

Table 5: The Real Data Application's Results

Scheme	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\widehat{C_L}$	$\Sigma_{\hat{\gamma}}$	$\underline{LB}$	C_0
I	8.9008	7.2144	0.5793	1.5280	0.1280	0.8267	1.5213
II	8.3469	9.0909	0.5979	1.5297	0.1201	0.8504	1.4993

Example 6.2.

A real-life data set from Nicholas and Padgett [25]. Data on the tensile strength of 100 carbon fiber observations. The first failure censored data is:

0.39, 1.08, 1.22, 1.47, 1.59, 1.69, 1.84, 1.92, 2.05 2.17, 2.43, 2.53, 2.59, 2.74, 2.81, 2.85, 2.95, 3.09, 3.15 , 3.22, 3.31, 3.51, 3.68, 4.2, 4.9

The entire data set is fitted to the Stacy distribution using goodness-of-fit criteria $\mathrm{KS} = 0.06423$ and $\mathrm{P-value} = 0.80374$.

Then, the progressive censoring scheme with $m = 21$ is as follows:

Table 6: Censoring Scheme: $\underline{R} = (4,0*20)$

x_i	0.39	1.69	1.84	1.92	2.05	2.17	2.43	2.53	2.59	2.74	
R_i	4	0	0	0	0	0	0	0	0	0	
x_i	2.81	2.85	2.95	3.09	3.15	3.22	3.31	3.51	3.68	4.2	4.9
R_i	0	0	0	0	0	0	0	0	0	0	0

For the above scheme, the statistical inference procedure for the lifetime performance index C_L is as follows. **Firstly, by solving the MLE** equations of the Stacy distribution parameters (12), (13), and (14) numerically with the following results $\hat{\alpha} = 0.7451$, $\hat{\theta} = 5.14435$, $\hat{\beta} = 4.02793$. Consequently, suppose the lifetime limit is 0.39, if the lifetime exceeds 0.39 then the product is considered a conforming product. To satisfy product purchasers' concerns regarding lifetime performance, the conforming rate P_r of items must be more than 80%. Looking at Table (7), the performance index value is set at $c^* = 0.88$ and the testing of hypothesis: H_o: $C_L \leq 0.88$ against H_1: $C_L > 0.88$.

For the significance level $\alpha^* = 0.05$, using Equations (16), (24), and (26), the lower confidence interval bound is calculated as $\underline{LB} = \widehat{C_L} - z_{\alpha^*}\sqrt{\Psi_{\hat{\gamma}}}$ = 2.7316 − 1.96($\sqrt{0.2883}$) = 1.679. Therefore, the 95% one-sided confidence interval for C_L is $[\underline{LB}, \infty) = [1.679, \infty)$. Accordingly, the performance index $c^* = 0.88 \notin [\underline{LB}, \infty) = [1.679, \infty)$ then, H_o: $C_L \leq 0.88$ is rejected. Thus, the lifetime performance index of the product meets the required level. Also, from (16) and (26) $\widehat{C_L} = 1.81139 > C_0 = c^* + z_{\alpha^*}\sqrt{\Psi_{\hat{\gamma}}} = 0.88 + 1.96(\sqrt{0.2883}) =\approx 1.932394$. Then, the decision is to reject H_o: $C_L \leq 0.88$ and the lifetime performance index of the product meets the required level.

Table 7: The lifetime performance index C_L versus the conforming rate P_r for Stacy distribution with $(\hat{\alpha}, \hat{\theta}, \hat{\beta}) = (0.7451, 5.14435, 4.02793)$

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.0000	0.3	0.6182	0.88	0.8005	1.55	0.9334
−2	0.0185	0.4	0.6533	0.9	0.8057	1.6	0.9399
−1	0.1652	0.5	0.6871	1	0.8308	1.67	0.9483
−0.5	0.3207	0.6	0.7194	1.1	0.8538	1.7	0.9517
−0.3	0.3936	0.7	0.7500	1.2	0.8748	2.2	0.9885
0	0.5072	0.8	0.7788	1.3	0.8939	2.5	0.9971
0.1	0.5449	0.82	0.7844	1.4	0.9111	3	1
0.2	0.5820	0.85	0.7925	1.5	0.9264		

Example 6.3: Simulated data set

Table (7) shows that the censoring scheme for progressive first-failure type II censored data with $n = 40, m = 10, k = 4$ was generated from Stacy distribution.

Table 8: Simulated data of Stacy distribution

27.2416	**27.3196**	**18.5581**	**31.6103**	**27.5968**	**11.565**	**43.1924**	**27.6558**	**56.2843**	**24.8551**
25.389	41.5905	26.4119	27.7718	32.6278	29.2143	39.3499	35.8745	32.8341	51.3031
12.463	33.6912	31.1299	**19.0596**	**19.0155**	58.093	**29.8197**	28.1314	42.8176	**17.7475**
37.3228	34.2583	45.3663	24.0338	29.2738	25.0105	39.8337	**21.1778**	**18.5072**	28.2563

and the first failure censored data shown as

{11.565, 12.463, 17.7475, 18.5072, 18.5581, 19.0155, 19.0596, 21.1778, 27.3196, 29.8197}

Table 9: The simulated progressive first failure censored data scheme

i	1	2	3	4	5	6
$X_{i:m:n:k}$	11.565028	19.0155	19.0596	21.1778	27.3196	29.8197
R_i	4	0	0	0	0	0

Then, the testing technique of C_L based on a confidence interval is defined as follows:
According to the censoring data above in Table (9), the MLE of the Stacy distribution parameters are obtained from equations (13), (14), and (15) and the results are: $\hat{\alpha} = 6.7241, \hat{\theta} = 6.45993, \hat{\beta} = 0.873268$. Let the lifetime limit is 11.565 and if the lifetime is more than 11.565 then the product is then classified as a conforming product. The conforming rate P_r of products must be greater than 80%. Referring to Table (10). The target value of C_L must be greater than 0.9. Thus, $c^* = 0.9$ and the testing of hypothesis: H_o: $C_L \leq 0.9$ versus H_1: $C_L > 0.9$. The significance level is $\alpha^* =$

0.05. And from Equations (19), (20), and (21), The bound of the lower confidence interval is calculated. from $\underline{LB} = \widehat{C_L} - z_{\alpha^*}\sqrt{\Psi_{\hat{\gamma}}} = 1.685430479 - 1.96(\sqrt{0.120839}) = 1.004097$. Hence, the 95% one-sided confidence interval for C_L is $[\underline{LB}, \infty) = [1.004097, \infty)$. **Moreover,** the performance index $c^* = 0.9 \notin [\underline{LB}, \infty) = [1.004097, \infty)$ then, H_o: $C_L \leq 0.9$ is rejected. Thus, the product's lifetime performance index reaches the required level. Moreover, $\widehat{C_L} = 1.81139 > C_0 = c^* + z_{\alpha^*}\sqrt{\Psi_{\hat{\gamma}}} = 0.9 + 1.96\left(\sqrt{0.120839}\right) = 1.581333$. Then, the decision is to reject H_o: $C_L \leq 0.9$ and the product's lifetime performance index is at the appropriate level.

Table 10: The lifetime performance index C_L against the conforming rate P_r for Stacy's distribution with $\left(\hat{\alpha}, \hat{\theta}, \hat{\beta}\right) = (6.7241, 6.45993, 0.87326)$.

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.0000	0.3	0.5781	0.9	0.8142	1.5	0.9600
-5	0.0001	0.4	0.6198	1	0.8468	1.8	0.9888
-4	0.0011	0.6	0.7019	1.2	0.9025	2	0.9965
-2.5	0.0167	0.7	0.7413	1.22	0.9073	2.1	0.9983
-1	0.1554	0.8	0.7789	1.3	0.9252	2.2	0.9993
0	0.4555	0.8	0.7861	1.3	0.9293	2.4	0.9999
0.1	0.4955	0.8	0.7933	1.4	0.9443	2.6	1
0.2	0.5365	0.9	0.8003	1.5	0.9526	2.7	1

7 Conclusions

In lifetime testing studies, the experimenter may not be able to detect the lives of all things under test, therefore a process capability analysis is always required to quantify the performance and prospective capacities of a process. As a result, this study aims to evaluate the lifetime performance index C_L of items under first-failure progressive right censoring data using the Stacy distribution. The maximum likelihood of C_L is performed and the testing of the hypothesis procedure is completed. The hypothesis testing process not only evaluates lifetime performance but also serves as an experimenter's supplier selection criteria. The proposed technique is clarified using an actual data application, and the results show that the goal of assessing the lifetime performance index has been achieved.

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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Appendix

The asymptotic normal distribution for the MLEs

$$\frac{\partial^2 \, ln\left(L(\alpha,\beta,\lambda;X)\right)}{\partial \alpha^2} = -m\text{Poly}\Gamma[1,\alpha] + \sum_{i=1}^{m}\Bigg(\frac{\text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2} - \frac{k\text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2} - \frac{2\text{MeijerG}[\{\{\},\{1,1,1\}\},\{\{0,0,0,\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} + \frac{2k\text{MeijerG}[\{\{\},\{1,1,1\}\},\{\{0,0,0,\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} - \frac{k\text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2 R_i}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2} + \frac{2k\text{MeijerG}[\{\{\},\{1,1,1\}\},\{\{0,0,0,\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]R_i}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} + \text{Poly}\Gamma[1,\alpha](1-k-kR_i)\Bigg),$$

$$\frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\alpha\partial\theta} = -\frac{m\beta}{\theta} + \sum_{i=1}^{m} \Bigg(\frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\beta\,\mathrm{MeijerG}[\{\{\},\{1+\alpha,1+\alpha\}\},\{\{\alpha,\alpha,2\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]}{\theta\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2} - \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}k\beta\,\mathrm{MeijerG}[\{\{\},\{1+\alpha,1+\alpha\}\},\{\{\alpha,\alpha,2\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]}{\theta\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2} - \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}k\beta\,\mathrm{MeijerG}[\{\{\},\{1+\alpha,1+\alpha\}\},\{\{\alpha,\alpha,2\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]R_i}{\theta\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2}\Bigg)$$

$$\frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\alpha\partial\beta} = \sum_{i=1}^{m}\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}] + \sum_{i=1}^{m}\Bigg(-\frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}]\mathrm{MeijerG}[\{\{\},\{1+\alpha,1+\alpha\}\},\{\{\alpha,\alpha,2\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2} + \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}k\,\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}]\mathrm{MeijerG}[\{\{\},\{1+\alpha,1+\alpha\}\},\{\{\alpha,\alpha,2\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2} + \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}k\,\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}]\mathrm{MeijerG}[\{\{\},\{1+\alpha,1+\alpha\}\},\{\{\alpha,\alpha,2\alpha\},\{\}\},(\frac{x_{i:m:n:k}}{\theta})^{\beta}]R_i}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2}\Bigg),$$

$$\frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\theta^2} = \frac{m\alpha\beta}{\theta^2} - \sum_{i=1}^{m}\Bigg(\frac{(-1+\beta)\beta x_{i:m:n:k}{}^2(\frac{x_{i:m:n:k}}{\theta})^{-2+\beta}}{\theta^4} + \frac{2\beta x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}}{\theta^3}\Bigg) + \sum_{i=1}^{m}\Bigg(-\frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}(-1+\alpha)\beta^2(-1+k(1+R_i))x_{i:m:n:k}{}^2(\frac{x_{i:m:n:k}}{\theta})^{-2+2\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-2+\alpha}}{\theta^4\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} - \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}(-1+\beta)\beta(-1+k(1+R_i))x_{i:m:n:k}{}^2(\frac{x_{i:m:n:k}}{\theta})^{-2+\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+\alpha}}{\theta^4\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} - \frac{2e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\beta(-1+k(1+R_i))x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+\alpha}}{\theta^3\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} + \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\beta^2(-1+k(1+R_i))x_{i:m:n:k}{}^2(\frac{x_{i:m:n:k}}{\theta})^{-2+2\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+\alpha}}{\theta^4\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} - \frac{e^{-2(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\beta^2(-1+k(1+R_i))x_{i:m:n:k}{}^2(\frac{x_{i:m:n:k}}{\theta})^{-2+2\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-2+2\alpha}}{\theta^4\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2}\Bigg)$$

$$\frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\theta\partial\beta} = -\frac{m\alpha}{\theta} - \sum_{i=1}^{m}\Bigg(-\frac{x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}}{\theta^2} - \frac{\beta\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}]x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}}{\theta^2}\Bigg) + \sum_{i=1}^{m}\Bigg(\frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}(-1+k(1+R_i))x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+\alpha}}{\theta^2\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} + \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\beta\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}](-1+k(1+R_i))x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+\alpha}}{\theta^2\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} + \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}(-1+\alpha)\beta\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}](-1+k(1+R_i))x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+\alpha}}{\theta^2\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} - \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\beta\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}](-1+k(1+R_i))x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+2\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+\alpha}}{\theta^2\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} + \frac{e^{-2(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\beta\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}](-1+k(1+R_i))x_{i:m:n:k}(\frac{x_{i:m:n:k}}{\theta})^{-1+\beta}((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{-1+2\alpha}}{\theta^2\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2}\Bigg),$$

$$\frac{\partial^2 \ln(L(\alpha,\theta,\beta;X))}{\partial\beta^2} = -\frac{m}{\beta^2} - \sum_{i=1}^{m}\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}]^2(\frac{x_i}{\theta})^{\beta} + \sum_{i=1}^{m}\Bigg(-\frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\alpha\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}]^2(-1+k(1+R_i))((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{\alpha}}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]} - \frac{e^{-2(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}]^2(-1+k(1+R_i))((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{2\alpha}}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]^2} + \frac{e^{-(\frac{x_{i:m:n:k}}{\theta})^{\beta}}\mathrm{Log}[\frac{x_{i:m:n:k}}{\theta}]^2(-1+k(1+R_i))((\frac{x_{i:m:n:k}}{\theta})^{\beta})^{1+\alpha}}{\Gamma[\alpha,(\frac{x_{i:m:n:k}}{\theta})^{\beta}]}\Bigg)$$