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Extended Kappa Distribution in the Presence of Censored Data

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Abstract: In this study, we introduce a new extension of three-parameter Kappa distribution using the Topp-Leone family of distributions, called Topp-Leone Kappa distribution which offers distribution with more flexible and applicable study of the highly-skewed data. Some statistical properties of the proposed distribution are studied. Based on censoring schemes, the maximum likelihood estimation is considered for the new model parameters. A simulation study is carried out to evaluate the performance of the maximum likelihood estimates based on their biases and mean square errors. The new model is fitted to a real data set and it is shown that the distribution is better fits than some other competitive distributions.

Keywords: Kappa distribution, Topp-Leone family, Maximum likelihood estimates, Type I censored sample, Type II censored sample.

1 Introduction

The kappa (K) distribution is a family of positively skewed distributions which is an important probability distribution which can be used in modelling data with extreme values. It provides a good fit to several extreme events like heavy rains is necessary in the planning of water relevant setups, cultivation of crops in agriculture, floods basic control systems, climatic conditions and overseeing environmental changes. It is suitable model to explain rainfall data and weather modifications which received attention from the hydrologist as mentioned by [1-2]. Let X is a three-parameter kappa (K3) random variable with shape parameters and scale parameter, its probability density function (pdf) is given by:

$$g(x)=\frac{\alpha\theta}{\beta}(x/\beta)^{\theta-1}\left[\alpha+(x/\beta)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \quad , x>0. \tag{1}$$

The corresponding cumulative distribution function (cdf) is

$$G(x)=\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}} \quad . \tag{2}$$

The unknown parameters of the (K3) distribution estimated by [3] using maximum likelihood estimation, moment estimation, and L-moment estimation and use the Monte Carlo simulation for performance evaluation of these estimators. The unknown parameters of the (K3) distribution estimated by [4] under Type II censored samples.

The kappa distribution families have closed algebraic expressions that can easily be analyzed than some common traditional distributions. In the literature, some extensions of the kappa distribution are available such as Exponentiated generalized kappa, Kumaraswamy generalized kappa, and McDonald generalized kappa distribution see [5], [6] studied the Marshall-Olkin kappa distribution, while [7] offered the Kumaraswamy generalized kappa and [8] introduced the odd

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kappa-G family.

The paper is organized as follows. In section 2, we introduced a new generalization of Kappa distribution, namely, the Topp-Leone kappa (TLK3) distribution. In Section 3, useful expansions of (TLK3) pdf and cdf are obtained. Some statistical properties are derived in Section 4. Based on two censored samples, the population parameters are estimated in Section 5. Simulation study is carried out to illustrate theoretical results In Section 6. In Section 7, real data set is used to illustrate the usefulness of the (TLK3) distribution followed by conclusion.

2 The New Distribution

The new family of distributions called Topp Leone-G (TL-G) family of distributions suggested by [9], the corresponding pdf and cdf with $b>0$ shape parameter are given by, respectively

$$f_{TL-G}(x)=2b\,g(x)\,\overline{G}(x)\left[1-\left(\overline{G}(x)\right)^2\right]^{b-1}\quad ,\ x\in R. \tag{3}$$

and

$$F_{TL-G}(x)=\left[1-\left(\overline{G}(x)\right)^2\right]^{b}\quad , \tag{4}$$

where $\overline{G}(x)=1-G(x)$. Inserting (2) in (4), the cdf of the (TLK3) distribution is

$$F_{TL-K3}(x)=\left[1-\left(1-\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right)^2\right]^{b},\alpha,\theta,\beta,b>0,\ x>0. \tag{5}$$

The corresponding pdf is obtained by differentiating (5) and is given as

$$f_{TL-K3}(x)=2b\frac{\alpha\theta}{\beta}(x/\beta)^{\theta-1}\left[\alpha+(x/\beta)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)}\left\{1-\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right\}$$

$$\times\left[1-\left(1-\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right)^2\right]^{b-1},\ \alpha,\theta,\beta,b>0,\ x>0. \tag{6}$$

The (TLK3) distribution has the following distributions as special sub-models: For $\theta=1$, the (TLK3) distribution reduces to the Topp Leone-two parameter Kappa (TLK2) distribution with parameters α, β, and b. The (TLK3) distribution reduces to the Topp Leone-one parameter Kappa (TLK1) distribution with parameters α and b, for $\theta=\beta=1$.

The survival (S) and hazard (h) functions corresponding to (6) are given by:

$$S_{TL-K3}(x)=1-\left[1-\left(1-\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right)^2\right]^{b},$$

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and

$$h_{TL-K3}(x)=2b\frac{\alpha\theta}{\beta}(x/\beta)^{\theta-1}\left[\alpha+(x/\beta)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)}\left\{1-\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right\}$$

$$\times\left[1-\left(1-\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right)^{2}\right]^{b-1}\left[1-\left[1-\left(1-\left(\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right)^{\frac{1}{\alpha}}\right)^{2}\right]^{b}\right]^{-1}.$$

Figures 1and 2 illustrates the plots of the pdf and hazard function of (TLK3) distribution for various parameter values.

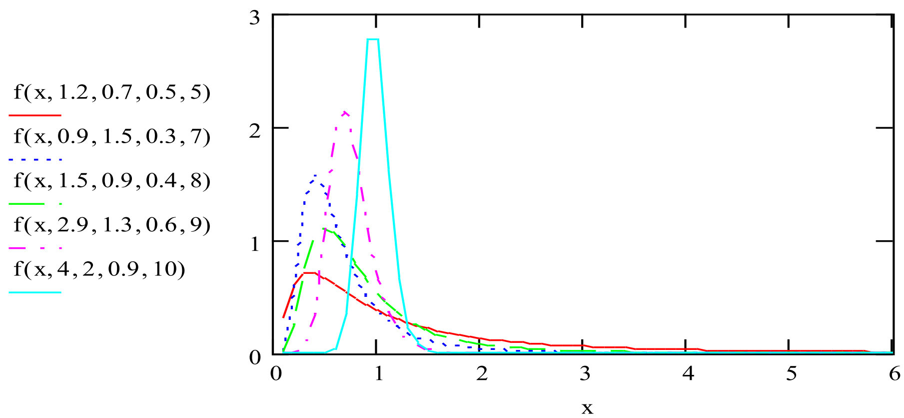

Fig 1. Pdf plot of (TLK3) distribution for various values of parameters

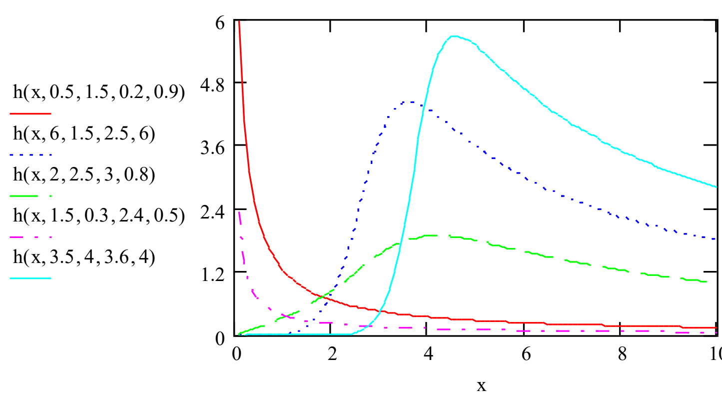

Fig 2. hf plot of (TLK3) distribution for various values of parameters

3 Useful Expansions

Here two important expansions are deduced for (TLK3) pdf and cdf distribution using infinite mixture forms of (K3) distribution, which are more efficient than computing those directly by numerical integration of its density function.

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Since, if $w \succ 0$ is real non-integer and $|z|<1$, the generalized binomial theorem is written as follows:

$$(1-z)^{w-1}=\sum_{i=0}^{\infty}(-1)^i\binom{w-1}{i}z^i. \qquad (7)$$

Then, by applying the binomial theorem (7) two times in (6), the (TLK3) pdf distribution becomes

$$f_{TL-K3}(x)=2b\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{2i+1}{j}\left(\frac{\alpha\,\theta}{\beta}\right)\left(\frac{x}{\beta}\right)^{\theta(j+1)-1}\left[\alpha+\left(\frac{x}{\beta}\right)^{\alpha\,\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1}.$$

Hence, the (TLK3) pdf can be expressed as infinite mixture of (K3) pdf as follows:

$$f_{TL-K3}(x)=\sum_{i,j=0}^{\infty}\psi_{i,j}\,\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta(j+1)-1}\left[\alpha+\left(\frac{x}{\beta}\right)^{\alpha\,\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1}, \qquad (8)$$

where

$$\psi_{i,j}=2b(-1)^{i+j}\binom{b-1}{i}\binom{2i+1}{j}.$$

Additionally, using the exponential expansion for $\left[F_{TL-K3}(x)\right]^s$, where s is positive integer, becomes

$$\left[F_{TL-K3}(x)\right]^s=\sum_{l,k=0}^{\infty}\omega_{l,k}\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{k\alpha^{-1}}, \qquad (9)$$

where $\omega_{l,k}=(-1)^{l+k}\binom{b\,s}{l}\binom{2l}{k}$.

4 Mathematical Properties

In this section, we obtained some mathematical properties of the (TLK3) distribution.

4.1 Quantile function

The quantile $Q(q)=F^{-1}(q)$ of the (TLK3) distribution, if q has a Uniform distribution $U(0,1)$, is given as

$$Q(q)=\beta\,\alpha^{\frac{1}{\alpha\,\theta}}\left\{1-\sqrt{1-q^{\frac{1}{b}}}\right\}^{\frac{1}{\theta}}\left[1-\left\{1-\sqrt{1-q^{\frac{1}{b}}}\right\}^{\alpha}\right]^{\frac{-1}{\alpha\,\theta}}, \quad 0<q<1 \qquad (10)$$

Put $q=0.5,$ in (10) one gets the median of the (TL-K3) distribution.

4.2 Moments and moment generating function

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The r[th] moment of the (TLk3) distribution is given by using the following relation:

$$\mu_r^{'} = E\left(X^r\right) = \int_0^\infty x^r\, f_{TL-K3}(x)\,dx.$$

Using (8), we have

$$\mu_r^{'} = \sum_{i,j=0}^{\infty} \int_0^\infty x^r\, \psi_{i,j}\, \frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta(j+1)-1}\left[\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1} dx\ .$$

Let $z = \left(\frac{x}{\beta}\right)^{\alpha\theta}$, after simplification, we get

$$\mu_r^{'} = \sum_{i,j=0}^{\infty} \psi_{i,j}\, \frac{\beta^r}{\alpha^{\frac{1}{\alpha}(j+1)+1}} \int_0^\infty z^{\frac{r}{\alpha\theta}+\frac{1}{\alpha}(j+1)-1}\left[1-\left(1-\left(1+\frac{z}{\alpha}\right)^{-1}\right)\right]^{\frac{1}{\alpha}(j+1)+1} dz\ .$$

Let $y = \left(1-\left(1+\frac{z}{\alpha}\right)^{-1}\right)$, after simplification, we get

$$\mu_r^{'} = \sum_{i,j=0}^{\infty} \psi_{i,j}\, \beta^r \alpha^{\frac{r}{\alpha\theta}-1} \int_0^1 y^{\frac{r}{\alpha\theta}+\frac{1}{\alpha}(j+1)-1}\left[1-y\right]^{1-\frac{r}{\alpha}-1}\ dy\ .$$

Then

$$\mu_r^{'} = E(X^r) = \sum_{i,j=0}^{\infty} \psi_{i,j}\, \beta^r \alpha^{\frac{r}{\alpha\theta}-1} B\left(\frac{r}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{r}{\alpha\theta}\right),\ r = 1,2,3,4. \qquad (11)$$

Where $B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}\,dw$, is the beta function. From (11), the mean and variance of the (TLk3) distribution can be obtained, respectively as follows:

$$E(X) = \sum_{i,j=0}^{\infty} \psi_{i,j}\, \beta\alpha^{\frac{1}{\alpha\theta}-1} B\left(\frac{1}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{1}{\alpha\theta}\right),$$

and

$$Var(X) = \sum_{i,j=0}^{\infty} \psi_{i,j}\, \beta^2 \alpha^{\frac{2}{\alpha\theta}-1} B\left(\frac{2}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{2}{\alpha\theta}\right) - \left[\sum_{i,j=0}^{\infty} \psi_{i,j}\, \beta\, \alpha^{\frac{1}{\alpha\theta}-1} B\left(\frac{1}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{1}{\alpha\theta}\right)\right]^2,$$

Where $\psi_{i,j} = 2b(-1)^{i+j}\binom{b-1}{i}\binom{2i+1}{j}$. Based on the first four moments of the(TLk3) distribution, the measures of skewness and kurtosis of the (TLk3) distribution can be obtained as:

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$$S=\frac{\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3}{\left(\mu_2'-\mu_1'^2\right)^{3/2}},\text{and}\quad K=\frac{\mu_4'-4\mu_3'\mu_1'+6\mu_2'\mu_1'^2-3\mu_1'^4}{\left(\mu_2'-\mu_1'^2\right)^2}.$$

The moment generating function and characteristic function of the (TLk3) distribution can be obtained, respectively as follows:

$$M_X(t)=E\left(e^{tx}\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'=\sum_{r,i,j=0}^{\infty}\frac{t^r}{r!}\psi_{i,j}\,\beta^r\alpha^{\frac{r}{\alpha\theta}-1}B\left(\frac{r}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{r}{\alpha\theta}\right),$$

and

$$\varphi_X(t)=E\left(e^{itx}\right)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\mu_r'=\sum_{r,i,j=0}^{\infty}\frac{(it)^r}{r!}\psi_{i,j}\,\beta^r\alpha^{\frac{r}{\alpha\theta}-1}B\left(\frac{r}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{r}{\alpha\theta}\right).$$

4.3 Mean deviation

The amount of scattering in a population can be measured by the totality of deviations from the mean and the median. Let μ, M are the mean and median respectively of the (TLk3) distribution. The mean deviation about the mean and the mean deviation about the median are, respectively, defined by

$$\delta_1(\mu)=2[\mu F(\mu)-J(\mu)] \text{ and } \delta_2(M)=\mu-2J(M), \qquad (12)$$

Where $J(t)=\int_0^t x\, f_{TL-K3}(x)\,dx$ is the first incomplete moment of the(TL-k3) distribution. From (8) we have

$$J(t)=\sum_{i,j=0}^{\infty}\psi_{i,j}\frac{\alpha\theta}{\beta}\int_0^t x\left(\frac{x}{\beta}\right)^{\theta(j+1)-1}\left[\alpha+\left(\frac{x}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1}dx\,.$$

Let $z=\left(\frac{x}{\beta}\right)^{\alpha\theta}$, after simplification, we get

$$J(t)=\sum_{i,j=0}^{\infty}\psi_{i,j}\,\beta\alpha^{\frac{1}{\alpha\theta}-1}B_y\left(\frac{1}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{1}{\alpha\theta}\right),\quad y=\left(1-\left(1+\frac{z}{\alpha}\right)^{-1}\right),$$

Where $B_y(a,b)=\int_0^y w^{a-1}(1-w)^{b-1}dw$, is the incomplete beta function. Using (9) with $s=1$, one can easily find $\delta_1(\mu)$ and $\delta_2(M)$ from (12).

Another application of the first incomplete moment, it can be used to determine some inequality measures, namely Lorenz and Bonferroni curves. These curves are very useful in a wide variety of fields such as economics, reliability, demography, and medicine.

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The Lorenz $L(p)$ and Bonferroni $B(p)$ curves of the (TLk3) distribution are, respectively, given by using the following relation:

$$L(p)=J(p)/\mu=\frac{1}{\mu}\int_0^p x\, f_{TL-K3}(x)\,dx=\frac{\sum_{i,j=0}^{\infty}\psi_{i,j}\,B_y\left(\frac{1}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{1}{\alpha\theta}\right)}{\sum_{i,j=0}^{\infty}\psi_{i,j}\,B\left(\frac{1}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{1}{\alpha\theta}\right)},$$

And

$$B(P)=\frac{L(P)}{F_{TL-K3}(x)}$$

$$=\left[\sum_{l,k=0}^{\infty}\omega_{l,k}\left[\frac{\left(\frac{x}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{x}{\beta}\right)^{\alpha\theta}}\right]^{k\alpha^{-1}}\right]^{-1}\left[\frac{\sum_{i,j=0}^{\infty}\psi_{i,j}\,B_y\left(\frac{1}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{1}{\alpha\theta}\right)}{\sum_{i,j=0}^{\infty}\psi_{i,j}\,B\left(\frac{1}{\alpha\theta}+\frac{1}{\alpha}(j+1),1-\frac{1}{\alpha\theta}\right)}\right],$$

where $\omega_{l,k}=(-1)^{l+k}\binom{b}{l}\binom{2l}{k}$.

4.4 Probability weighted moments

The probability weighted moments (PWMs) can be obtained using the following relation:

$$\tau_{r,s}=E\left[X^r F(x)^s\right]=\int_{-\infty}^{\infty} x^r F(x)^s f(x)\,dx \quad . \tag{13}$$

The PWMs of the (TLk3) distribution is obtained by substituting (8) and (9) into (13), as follows:

$$\tau_{r,s}=\int_0^{\infty}\sum_{i,j,l,k=0}^{\infty}\psi_{i,j}\,\omega_{l,k}\,x^r\,\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta(j+k+1)-1}\left[\alpha+\left(\frac{x}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{j+k+1}{\alpha}\right)-1}dx,$$

$$=\sum_{i,j,l,k=0}^{\infty}\psi_{i,j}\,\omega_{l,k}\,\beta^r\alpha^{\frac{r}{\alpha\theta}-1}B\left(\frac{r}{\alpha\theta}+\frac{1}{\alpha}(j+k+1),1-\frac{r}{\alpha\theta}\right),$$

Where

$$\psi_{i,j}=2b(-1)^{i+j}\binom{b-1}{i}\binom{2i+1}{j},\ \omega_{l,k}=(-1)^{l+k}\binom{bs}{l}\binom{2l}{k}.$$

4.5 Order Statistics

Let $X_{(1)},X_{(2)},...,X_{(n)}$ be the order statistics corresponding to a simple random sample of size n from the (TLk3) distribution. The pdf of $X_{(k)}$, $k=1,2,...,n$, the kth order statistics, is given by

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$$f_{X_{(k)}}(x)=\sum_{t=0}^{n-k}\frac{(-1)^t}{B(k,n-k+1)}\binom{n-k}{t}\left[F_{TL-K3}(x)\right]^{k+t-1} f_{TL-K3}(x). \tag{14}$$

Inserting pdf and cdf of (TLk3) distribution into (14) and simplifying we get:

$$f_{X_{(k)}}(x)=\frac{1}{B(k,n-k+1)}\sum_{t=0}^{n-k}\sum_{i,j=0}^{\infty}(-1)^t\binom{n-k}{t}\psi_{i,j}\frac{\alpha\theta}{\beta}(x/\beta)^{\theta(j+1)-1}\left[\alpha+(x/\beta)^{\alpha\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1}$$
$$\times\left[1-\left(1-\left[\frac{(x/\beta)^{\alpha\theta}}{\alpha+(x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right)^2\right]^{b+k+t-1}.$$

The pdf of the first order statistics can be obtained by using (k=1) and the pdf of the nth order statistics can be defined by replacing (k=n).

5 Parameter Estimation

In this section, the estimators of the (TLk3) distribution parameters using maximum likelihood (ML) method are obtained under two different types of censoring schemes.

5.1 ML Estimators under Type I Censoring

In a typical life test, n times whose lifetimes follow the (TLk3) distribution (6) with parameter vector $\Theta=\{b,\alpha,\beta,\theta\}$ are placed on test and each failure occurs is noted. At a predetermined fixed time T, the test is terminated. The data collected consists of items $x_{(1)}<x_{(2)}<...<x_{(r)}$plus (n-r) items survived beyond the time of termination, T. clearly the number of failures r is a random variable. [10] gave the log-likelihood function, under Type I censored, as follows:

$$\ln L_1(\Theta)=\ln\left(\frac{n!}{(n-r)!}\right)+r\ln\left(\frac{2b\alpha\theta}{\beta^\theta}\right)+(\theta-1)\sum_{i=1}^{r}\ln\left(x_{(i)}\right)-\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\ln\left[\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right]$$
$$+\sum_{i=1}^{r}\ln(1-Z_i)+(b-1)\sum_{i=1}^{r}\ln\left[1-(1-Z_i)^2\right]+(n-r)\ln\left[1-\left\{1-(1-W)^2\right\}^b\right],$$

Where

$$Z_i=\left[\frac{\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}}\right]^{\frac{1}{\alpha}},\quad W=\left[\frac{\left(\frac{T}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{T}{\beta}\right)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}.$$

The components of the score vector $U(\Theta)$ are given by:

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$$U(b)=\frac{r}{b}+\sum_{i=1}^{r}\ln\left[1-(1-Z_i)^2\right]-(n-r)\left(1-\left\{1-(1-W)^2\right\}^b\right)^{-1}\left[\left\{1-(1-W)^2\right\}^b\ln\left\{1-(1-W)^2\right\}\right], \tag{15}$$

$$U(\alpha)=\frac{r}{\alpha}+\frac{1}{\alpha^2}\sum_{i=1}^{r}\ln\left(\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)-\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\left(\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)^{-1}-\theta\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\left[Z_i^{\alpha}\ln\left(\frac{x_{(i)}}{\beta}\right)\right]$$
$$-\sum_{i=1}^{r}\left[Z'_{i\alpha}(1-Z_i)^{-1}\right]+2(b-1)\sum_{i=1}^{r}\left[Z'_{i\alpha}(1-Z_i)\left[1-(1-Z_i)^2\right]^{-1}\right]-2b(n-r)W'_{\alpha}(1-W)\times$$
$$\left\{1-(1-W)^2\right\}^{b-1}\left(1-\left\{1-(1-W)^2\right\}^b\right)^{-1}, \tag{16}$$

$$U(\beta)=\frac{-r\theta}{\beta}+\left(1+\frac{1}{\alpha}\right)\left(\frac{\alpha\theta}{\beta}\right)\sum_{i=1}^{r}\left(Z_i^{\alpha}\right)-\sum_{i=1}^{r}\left[Z'_{i\beta}(1-Z_i)^{-1}\right]+2(b-1)\sum_{i=1}^{r}\left[Z'_{i\beta}(1-Z_i)\left[1-(1-Z_i)^2\right]^{-1}\right]$$
$$-2b(n-r)W'_{\beta}(1-W)\left\{1-(1-W)^2\right\}^{b-1}\left(1-\left\{1-(1-W)^2\right\}^b\right)^{-1}, \tag{17}$$

and

$$U(\theta)=\frac{r}{\theta}(1-\theta\ln\beta)+\sum_{i=1}^{r}\ln\left(x_{(i)}\right)-(1+\alpha)\sum_{i=1}^{r}\left[Z_i^{\alpha}\ln\left(\frac{x_{(i)}}{\beta}\right)\right]-\sum_{i=1}^{r}\left[Z'_{i\theta}(1-Z_i)^{-1}\right]$$
$$+2(b-1)\sum_{i=1}^{r}\left[Z'_{i\theta}(1-Z_i)\left[1-(1-Z_i)^2\right]^{-1}\right]-2b(n-r)W'_{\theta}(1-W)\times$$
$$\left\{1-(1-W)^2\right\}^{b-1}\left(1-\left\{1-(1-W)^2\right\}^b\right)^{-1}. \tag{18}$$

This system of non-linear equations (15-18) cannot be solved analytically, therefore we apply statistical software to solve them numerically to obtain the ML estimates (MLEs) of the four unknown parameters b, α, β, and θ of the (TLk3) distribution under type I censored, called $\left(\hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\theta}\right)$.

5.2 ML Estimators under Type II Censoring

Suppose n times whose lifetimes follow the (TLk3) distribution (6) are placed on test. The life test is terminated after predetermined number of failures r has occurred. The log-likelihood function, under Type II censored, is given by [10] as follows:

$$\ln L_2(\Theta)=\ln\left(\frac{n!}{(n-r)!}\right)+r\ln\left(\frac{2b\alpha\theta}{\beta^{\theta}}\right)+(\theta-1)\sum_{i=1}^{r}\ln\left(x_{(i)}\right)-\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\ln\left[\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right]$$
$$+\sum_{i=1}^{r}\ln(1-Z_i)+(b-1)\sum_{i=1}^{r}\ln\left[1-(1-Z_i)^2\right]+(n-r)\ln\left[1-\left\{1-(1-Z_r)^2\right\}^b\right].$$

The components of the score vector $U^*(\Theta)$ are given by:

$$U^*(b)=\frac{r}{b}+\sum_{i=1}^{r}\ln\left[1-(1-Z_i)^2\right]-(n-r)\left(1-\left\{1-(1-Z_r)^2\right\}^b\right)^{-1}\left[\left\{1-(1-Z_r)^2\right\}^b\ln\left\{1-(1-Z_r)^2\right\}\right], \tag{19}$$

$$U^*(\alpha)=\frac{r}{\alpha}+\frac{1}{\alpha^2}\sum_{i=1}^{r}\ln\left(\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)-\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\left(\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)^{-1}-\theta\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\left[Z_i^{\alpha}\ln\left(\frac{x_{(i)}}{\beta}\right)\right]$$
$$-\sum_{i=1}^{r}\left[Z'_{i\alpha}(1-Z_i)^{-1}\right]+2(b-1)\sum_{i=1}^{r}\left[Z'_{i\alpha}(1-Z_i)\left[1-(1-Z_i)^2\right]^{-1}\right]-2b(n-r)Z'_{r\alpha}(1-Z_r)\times$$
$$\left\{1-(1-Z_r)^2\right\}^{b-1}\left(1-\left\{1-(1-Z_r)^2\right\}^{b}\right)^{-1}, \tag{20}$$

$$U^*(\beta)=\frac{-r\theta}{\beta}+\left(1+\frac{1}{\alpha}\right)\left(\frac{\alpha\theta}{\beta}\right)\sum_{i=1}^{r}\left(Z_i^{\alpha}\right)-\sum_{i=1}^{r}\left[Z'_{i\beta}(1-Z_i)^{-1}\right]+2(b-1)\sum_{i=1}^{r}\left[Z'_{i\beta}(1-Z_i)\left[1-(1-Z_i)^2\right]^{-1}\right]$$
$$-2b(n-r)Z'_{r\beta}(1-Z_r)\left\{1-(1-Z_r)^2\right\}^{b-1}\left(1-\left\{1-(1-Z_r)^2\right\}^{b}\right)^{-1}, \tag{21}$$

And

$$U^*(\theta)=\frac{r}{\theta}(1-\theta\ln\beta)+\sum_{i=1}^{r}\ln\left(x_{(i)}\right)-(1+\alpha)\sum_{i=1}^{r}\left[Z_i^{\alpha}\ln\left(\frac{x_{(i)}}{\beta}\right)\right]-\sum_{i=1}^{r}\left[Z'_{i\theta}(1-Z_i)^{-1}\right]$$
$$+2(b-1)\sum_{i=1}^{r}\left[Z'_{i\theta}(1-Z_i)\left[1-(1-Z_i)^2\right]^{-1}\right]-2b(n-r)Z'_{r\theta}(1-Z_r)\times$$
$$\left\{1-(1-Z_r)^2\right\}^{b-1}\left(1-\left\{1-(1-Z_r)^2\right\}^{b}\right)^{-1}. \tag{22}$$

Setting equations (19-22) to zero and solving the resulting system of non-linear equations simultaneously to obtain the MLs of the unknown parameters of the (TLk3) distribution under type II censored, called $\left(\tilde{b},\tilde{\alpha},\tilde{\beta},\tilde{\theta}\right)$. For interval estimation of the unknown parameters, we require the two observed 4×4 information matrix under type I censored or type II censored are given in Appendix. The $100(1-\gamma)\%$ confidence intervals for the parameters $b,\alpha,\beta,$and θ of the (TLk3) distribution of two types of censored are respectively given by:

$$\hat{b}\pm z_{\frac{\gamma}{2}}\sqrt{\mathrm{var}(\hat{b})},\ \hat{\alpha}\pm z_{\frac{\gamma}{2}}\sqrt{\mathrm{var}(\hat{\alpha})},\ \hat{\beta}\pm z_{\frac{\gamma}{2}}\sqrt{\mathrm{var}(\hat{\beta})},\ \hat{\theta}\pm z_{\frac{\gamma}{2}}\sqrt{\mathrm{var}(\hat{\theta})}$$, and

$$\tilde{b}\pm z_{\frac{\gamma}{2}}\sqrt{\mathrm{var}(\tilde{b})},\ \tilde{\alpha}\pm z_{\frac{\gamma}{2}}\sqrt{\mathrm{var}(\tilde{\alpha})},\ \tilde{\beta}\pm z_{\frac{\gamma}{2}}\sqrt{\mathrm{var}(\tilde{\beta})},\ \tilde{\theta}\pm z_{\frac{\gamma}{2}}\sqrt{\mathrm{var}(\tilde{\theta})}$$,

where $Z_{\gamma/2}$ is the upper $(\gamma/2)^{th}$ percentile of the standard normal distribution and the $Var(.)'s$ denote the diagonal elements of $I(\hat{\Theta})^{-1}$ and $I(\tilde{\Theta})^{-1}$ corresponding to parameters (b,α,β,θ).

6 Simulation Study

In this section, a simulation study is performed to evaluate the performance of the MLE for the (TLk3) distribution parameters based on two types of censoring schemes (Type I and Type II). The simulation study is designed as follows:

Step 1: Generate 1000 random samples of size 80, 100, 120 and 150 from the (TLk3) distribution under Type I and Type II censored samples.

Step 2: Two set of parameters values are selected as: Case I $(b=0.5, \alpha=0.5, \beta=0.5, \quad \theta=0.5)$ and Case II $(b=0.5, \alpha=0.5, \beta=1.5, \theta=0.5)$.

Step 3: In Type I censoring, two terminated times are selected as T=5 and T=10. Also, for Type II censoring, the number of failure items; r, is selected based on two levels of censoring as 70% and 90%.

Step 4: The MLEs of the four unknown parameters are obtained by solving the non-linear equations (15-18) based on Type I censoring. Also, the non-linear equations (19-22) are solved, to obtain the MLEs of the unknown parameters under Type II censoring.

Step 5: Compute the biases, mean square errors (MSEs), and confidence interval length (CL) of MLEs of the unknown parameters.

The numerical results for two types of censoring schemes are summarized in Tables 1 and 2. For different choices of sample sizes, the estimates are work well and MSE and bias decreases as the sample size increases. Generally the results corresponding to type II censoring scheme are better than those corresponding to type I censoring scheme in the sense of MSE, Bias, and CL.

7 Application to Real Data

In this section, one data set will be analyzed to show the applicability of the (TLk3) distribution. The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, analyzed by [11]. The data are:

In this section, one data set will be analyzed to show the applicability of the (TLk3) distribution. The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, analyzed by [11]. The data are:

0.1	0.33	0.44	0.56	0.59	0.72	0.74	0.77	0.92	0.93	0.96	1
1	1.02	1.05	1.07	0.07	0.08	1.08	1.08	1.09	1.12	1.13	1.15
1.16	1.2	1.21	1.22	1.22	1.24	1.3	1.34	1.36	1.39	1.44	1.46
1.53	1.59	1.6	1.63	1.63	1.68	1.71	1.72	1.76	1.83	1.95	1.96
1.97	2.02	2.13	2.15	2.16	2.22	2.3	2.31	2.4	2.45	2.51	2.53
2.54	2.54	2.78	2.93	3.27	3.42	3.47	3.61	4.02	4.32	4.58	5.55

Here, these data will be fitted to three distributions namely: Kumaraswamy generalized Kappa (KGK), three parameter Kappa (K3), and Two parameter Kappa (K2), The MLEs of the unknown parameters will be calculated for each competitive distribution. To compare (TLK3) distribution with these distributions that represent this data the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Kolmogorov-Smirnov (KS) distances between the fitted distribution and the empirical distribution function with corresponding p-value are calculated as shown in table (3).

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Table (1): Biases, MSEs and confidence interval length of TLK3 distribution parameters based on Type I censored data

n	**CaseI**	T	**Bias**	**MSE**	**CL**	**CaseII**	**T**	**Bias**	**MSE**	**CL**
80	b=0.5	5	-0.032	0.017	0.034	b=0.5	5	-0.029	0.065	0.031
		10	-0.047	0.038	0.026		10	0.049	0.035	0.238
	α=0.5	5	0.082	0.008	0.131	α=0.5	5	-0.026	0.076	0.138
		10	0.07	0.006	0.126		10	1.075	0.231	0.033
	β=0.5	5	-0.016	0.021	0.082	β=1.5	5	-0.123	0.096	0.034
		10	0.028	0.006	0.041		10	0.042	0.029	0.042
	θ=0.5	5	0.548	0.061	0.102	θ=0.5	5	0.126	0.085	0.107
		10	0.004	0.012	0.083		10	0.004	0.012	0.101
100	b=0.5	5	-0.076	0.009	0.024	b=0.5	5	0.068	0.052	0.027
		10	-0.146	0.028	0.019		10	0.042	0.025	0.132
	α=0.5	5	0.063	0.005	0.112	α=0.5	5	0.035	0.031	0.122
		10	0.053	0.005	0.106		10	0.842	0.027	0.023
	β=0.5	5	-0.042	0.011	0.052	β=1.5	5	0.024	0.042	0.032
		10	-0.014	0.012	0.031		10	0.027	0.016	0.032
	θ=0.5	5	-0.014	0.036	0.051	θ=0.5	5	0.019	0.026	0.089
		10	-0.015	0.009	0.031		10	0.013	0.009	0.073
120	b=0.5	5	-0.082	0.007	0.021	b=0.5	5	-0.124	0.044	0.021
		10	-0.162	0.026	0.016		10	0.007	0.019	0.061
	α=0.5	5	0.055	0.004	0.128	α=0.5	5	-0.013	0.023	0.089
		10	0.022	0.004	0.072		10	-0.047	0.018	0.019
	β=0.5	5	-0.061	0.009	0.041	β=1.5	5	-0.185	0.022	0.021
		10	-0.032	0.008	0.031		10	0.005	0.009	0.023
	θ=0.5	5	-0.024	0.036	0.032	θ=0.5	5	-0.162	0.016	0.087
		10	-0.027	0.008	0.023		10	-0.039	0.006	0.025
150	b=0.5	5	-0.119	0.006	0.071	b=0.5	5	-0.082	0.005	0.018
		10	-0.194	0.002	0.014		10	-0.015	0.009	0.029
	α=0.5	5	0.038	0.002	0.122	α=0.5	5	-0.123	0.001	0.076
		10	0.002	0.006	0.031		10	-0.057	0.003	0.018
	β=0.5	5	-0.109	0.007	0.015	β=1.5	5	-0.053	0.011	0.012
		10	-0.079	0.005	0.015		10	0.004	0.008	0.019
	θ=0.5	5	-0.142	0.023	0.026	θ=0.5	5	-0.183	0.006	0.041
		10	-0.097	0.001	0.013		10	-0.093	0.001	0.014

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Table (2): Biases, MSEs and confidence interval length of TLK3 distribution parameters based on Type II censored data

n	CaseI	x_r	Bias	MSE	CL	CaseII	x_r	Bias	MSE	CL
80	b=0.5	60	-0.048	0.014	0.016	b=0.5	60	-0.051	0.032	0.081
		80	-0.047	0.038	0.015		80	-0.161	0.003	0.019
	α=0.5	60	0.063	0.004	0.122	α=0.5	60	-0.138	0.045	0.012
		80	0.061	0.004	0.112		80	-0.227	0.026	0.003
	β=0.5	60	-0.217	0.014	0.011	β=1.5	60	-0.175	0.052	0.022
		80	0.028	0.012	0.016		80	-0.204	0.019	0.021
	θ=0.5	60	-0.153	0.025	0.012	θ=0.5	60	-0.188	0.034	0.014
		80	0.004	0.006	0.013		80	-0.209	0.009	0.012
100	b=0.5	60	-0.093	0.008	0.014	b=0.5	60	-0.081	0.015	0.017
		80	-0.161	0.021	0.014		80	0.031	0.002	0.013
	α=0.5	60	0.047	0.004	0.102	α=0.5	60	-0.081	0.017	0.009
		80	0.023	0.004	0.091		80	0.003	0.023	0.002
	β=0.5	60	-0.267	0.008	0.003	β=1.5	60	-0.149	0.034	0.016
		80	-0.087	0.007	0.011		80	0.015	0.003	0.012
	θ=0.5	60	-0.189	0.008	0.005	θ=0.5	60	-0.035	0.026	0.011
		80	-0.075	0.006	0.003		80	-0.016	0.006	0.005
120	b=0.5	60	-0.096	0.004	0.012	b=0.5	60	-0.292	0.007	0.041
		80	-0.168	0.021	0.013		80	-0.108	0.006	0.011
	α=0.5	60	0.046	0.002	0.109	α=0.5	60	-0.399	0.008	0.007
		80	0.021	0.003	0.009		80	-0.165	0.017	0.002
	β=0.5	60	-0.276	0.006	0.007	β=1.5	60	-0.677	0.012	0.013
		80	-0.109	0.007	0.009		80	-0.126	0.001	0.005
	θ=0.5	60	-0.189	0.006	0.006	θ=0.5	60	-1.241	0.008	0.005
		80	-0.091	0.003	0.005		80	-0.566	0.003	0.004
150	b=0.5	60	-0.132	0.002	0.021	b=0.5	60	-0.172	0.001	0.007
		80	-0.194	0.002	0.012		80	-0.046	0.004	0.006
	α=0.5	60	0.035	0.002	0.091	α=0.5	60	-0.274	0.001	0.003
		80	0.001	0.003	0.002		80	-0.134	0.002	0.001
	β=0.5	60	-0.376	0.003	0.006	β=1.5	60	-0.227	0.006	0.011
		80	-0.388	0.003	0.006		80	0.003	0.002	0.007
	θ=0.5	60	-0.247	0.002	0.003	θ=0.5	60	-0.255	0.002	0.005
		80	-0.263	0.001	0.002		80	-0.139	0.0002	0.004

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The (TLK3) distribution provides a better fit than the other tested distributions, because it has the smallest value among AIC, BIC, CAIC and KS and largest p-value.

Table 3: The MLEs and the values of AIC, BIC, CAIC and KS statistics.

Model	Parameters					AIC	BIC	CAIC	KS	p-value
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$					
TLK3	-	9.864	0.043	0.288	18.811	334.32	343.427	334.917	0.014	1
KGK	1.707	0.982	3.312	1.108	4.004	607.426	618.809	608.335	0.108	0.37
K3	-	-	0.168	2.834	3.917	353.048	359.878	353.401	0.097	0.507
K2	-	-	3.371	15.97	-	466.732	471.285	466.906	0.128	0.269

8 Conclusions

In this paper, we introduce a new four- parameter distribution called the Topp-Leone kappa distribution. Some mathematical properties of the proposed distribution such as quantile function, moments, mean deviation, and order statistics are derived. The maximum likelihood estimation is employed based on two censoring schemes to estimate the model parameters. Finally, a simulation study and real data set are considered.

Conflicts of Interest Statement: The authors declare no conflict of interest.

Appendix

The elements of the 4×4 observed information matrix under type I or type II censored for the (TLK3) distribution with parameters (b,α,β,θ) are given by:

$$U(bb)=\frac{-r}{b^2}-(n-r)\left\{1-(1-A)^2\right\}^b\left[\ln\left\{1-(1-A)^2\right\}\right]^2\left(1-\left\{1-(1-A)^2\right\}^b\right)^{-1}$$
$$+\left[1-\left\{1-(1-A)^2\right\}^b\right]^{-2}\left[\left\{1-(1-A)^2\right\}^b\ln\left\{1-(1-A)^2\right\}\right]^2,$$

$$U(b\alpha)=2\sum_{i=1}^{r}\left[\frac{Z'_{i\alpha}\left(1-Z_i\right)}{\left\{1-\left(1-Z_i\right)^2\right\}}\right]-2(n-r)(1-A)\,A'_\alpha\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^b\right)^{-1}$$
$$\times\left[1+b\ln\left\{1-(1-A)^2\right\}+b\left[1-\left\{1-(1-A)^2\right\}^b\right]^{-1}\left[\left\{1-(1-A)^2\right\}^b\ln\left\{1-(1-A)^2\right\}\right]\right],$$

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$$U(b\beta)=2\sum_{i=1}^{r}\left[\frac{Z'_{i\beta}\left(1-Z_i\right)}{\left\{1-\left(1-Z_i\right)^2\right\}}\right]-2(n-r)(1-A)\,A'_\beta\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^b\right)^{-1}$$
$$\times\left[1+b\ln\left\{1-(1-A)^2\right\}+b\left[1-\left\{1-(1-A)^2\right\}^b\right]^{-1}\left[\left\{1-(1-A)^2\right\}^b\ln\left\{1-(1-A)^2\right\}\right]\right],$$

$$U(b\theta)=2\sum_{i=1}^{r}\left[\frac{Z'_{i\theta}\left(1-Z_i\right)}{\left\{1-\left(1-Z_i\right)^2\right\}}\right]-2(n-r)(1-A)\,A'_\theta\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^b\right)^{-1}$$
$$\times\left[1+b\ln\left\{1-(1-A)^2\right\}+b\left[1-\left\{1-(1-A)^2\right\}^b\right]^{-1}\left[\left\{1-(1-A)^2\right\}^b\ln\left\{1-(1-A)^2\right\}\right]\right],$$

$$U(\alpha\alpha)=\frac{-r}{\alpha^2}-\frac{2}{\alpha^3}\sum_{i=1}^{r}\ln\left[\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right]+\frac{2}{\alpha^2}\sum_{i=1}^{r}\left[\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right]^{-1}+\frac{2\theta}{\alpha^2}\sum_{i=1}^{r}\left[Z_i^{\alpha}\ln\left(\frac{x_{(i)}}{\beta}\right)\right]$$
$$+\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\left\{\left[\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right]^{-2}+\theta Z_i^{2\alpha}\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}\ln\left(\frac{x_{(i)}}{\beta}\right)\right\}-\sum_{i=1}^{r}\left[\frac{Z''_{i\alpha}\left(1-Z_i\right)+\left(Z'_{i\alpha}\right)^2}{\left(1-Z_i\right)^2}\right]$$
$$-\theta\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\ln\left(\frac{x_{(i)}}{\beta}\right)\left[Z_i^{\alpha}\ln Z_i+\alpha Z_i^{\alpha-1}Z'_{i\alpha}\right]+2(b-1)\sum_{i=1}^{r}\left[\frac{Z''_{i\alpha}\left(1-Z_i\right)-\left(Z'_{i\alpha}\right)^2}{\left[1-\left(1-Z_i\right)^2\right]}\right]$$
$$-4(b-1)\sum_{i=1}^{r}\left(\frac{Z'_{i\alpha}\left(1-Z_i\right)}{\left[1-\left(1-Z_i\right)^2\right]}\right)^2-2b(n-r)\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^b\right)^{-1}\times$$
$$\left[A'^{\,2}_\alpha\left(\frac{2(b-1)(1-A)^2}{\left\{1-(1-A)^2\right\}}-1\right)+A''_\alpha(1-A)+\frac{2b\,A'^{\,2}_\alpha(1-A)^2\left\{1-(1-A)^2\right\}^{b-1}}{\left[1-\left\{1-(1-A)^2\right\}^b\right]}\right],$$

$$U(\alpha\beta)=\frac{-\theta}{\alpha\,\beta}\sum_{i=1}^{r}Z_i^{\alpha}-\frac{(\alpha+1)\theta}{\beta}\sum_{i=1}^{r}\left[Z_i^{2\alpha}\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}\right]-\sum_{i=1}^{r}\left[\frac{Z''_{i\alpha\beta}}{\left(1-Z_i\right)}\right]-\sum_{i=1}^{r}\left[\frac{Z'_{i\alpha}Z'_{i\beta}}{\left(1-Z_i\right)^2}\right]$$
$$-\theta\left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{r}\left\{\alpha Z_i^{\alpha-1}Z'_{i\beta}\ln\left(\frac{x_{(i)}}{\beta}\right)-\left(\frac{Z_i^{\alpha}}{\beta}\right)\right\}+2(b-1)\sum_{i=1}^{r}\left[\frac{Z''_{i\alpha\beta}\left(1-Z_i\right)-\left(Z'_{i\alpha}Z'_{i\beta}\right)}{\left[1-\left(1-Z_i\right)^2\right]}\right]$$
$$-4(b-1)\sum_{i=1}^{r}\left(\frac{Z'_{i\alpha}Z'_{i\beta}\left(1-Z_i\right)^2}{\left[1-\left(1-Z_i\right)^2\right]^2}\right)-2b(n-r)\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^b\right)^{-1}\times$$
$$\left[A'_\alpha A'_\beta\left(\frac{2(b-1)(1-A)^2}{\left\{1-(1-A)^2\right\}}-1\right)+A''_{\alpha\beta}(1-A)+\frac{2b\,A'_\alpha A'_\beta(1-A)^2\left\{1-(1-A)^2\right\}^{b-1}}{\left[1-\left\{1-(1-A)^2\right\}^b\right]}\right],$$

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$$U(\alpha\theta)=(\alpha+1)\sum_{i=1}^{r}\ln\left(\frac{x_{(i)}}{\beta}\right)Z_i^{2\alpha}\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}-\sum_{i=1}^{r}\left[Z_i^{\alpha}\ln\left(\frac{x_{(i)}}{\beta}\right)\right]-\sum_{i=1}^{r}\left[\frac{Z''_{i\alpha\theta}}{(1-Z_i)}\right]$$
$$-\sum_{i=1}^{r}\left[\frac{Z'_{i\alpha}Z'_{i\theta}}{(1-Z_i)^2}\right]-\theta(\alpha+1)\sum_{i=1}^{r}\left\{Z_i^{\alpha-1}Z'_{i\theta}\ln\left(\frac{x_{(i)}}{\beta}\right)\right\}+2(b-1)\sum_{i=1}^{r}\left[\frac{Z''_{i\alpha\theta}(1-Z_i)-(Z'_{i\alpha}Z'_{i\theta})}{\left[1-(1-Z_i)^2\right]}\right]$$
$$-4(b-1)\sum_{i=1}^{r}\left(\frac{Z'_{i\alpha}Z'_{i\theta}(1-Z_i)^2}{\left[1-(1-Z_i)^2\right]^2}\right)-2b(n-r)\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^{b}\right)^{-1}\times$$
$$\left[A'_{\alpha}A'_{\theta}\left(\frac{2(b-1)(1-A)^2}{\left\{1-(1-A)^2\right\}}-1\right)+A''_{\alpha\theta}(1-A)+\frac{2b\,A'_{\alpha}A'_{\theta}(1-A)^2\left\{1-(1-A)^2\right\}^{b-1}}{\left[1-\left\{1-(1-A)^2\right\}^{b}\right]}\right],$$

$$U(\beta\beta)=\frac{r\theta}{\beta^2}+\frac{(\alpha+1)\theta}{\beta^2}\sum_{i=1}^{r}Z_i^{\alpha}-\frac{(\alpha+1)\theta\alpha}{\beta}\sum_{i=1}^{r}\left[Z_i^{\alpha-1}Z'_{i\beta}\right]-\sum_{i=1}^{r}\left[\frac{Z''_{i\beta}(1-Z_i)+(Z'_{i\beta})^2}{(1-Z_i)^2}\right]$$
$$+2(b-1)\sum_{i=1}^{r}\left[\frac{Z''_{i\beta}(1-Z_i)-(Z'_{i\beta})^2}{\left[1-(1-Z_i)^2\right]}\right]-4(b-1)\sum_{i=1}^{r}\left(\frac{Z'_{i\beta}(1-Z_i)}{\left[1-(1-Z_i)^2\right]}\right)^2$$
$$-2b(n-r)\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^{b}\right)^{-1}\times$$
$$\left[A'^{\,2}_{\beta}\left(\frac{2(b-1)(1-A)^2}{\left\{1-(1-A)^2\right\}}-1\right)+A''_{\beta}(1-A)+\frac{2b\,A'^{\,2}_{\beta}(1-A)^2\left\{1-(1-A)^2\right\}^{b-1}}{\left[1-\left\{1-(1-A)^2\right\}^{b}\right]}\right],$$

$$U(\beta\theta)=\frac{-r}{\beta}-\frac{(\alpha+1)}{\beta}\sum_{i=1}^{r}Z_i^{\alpha}-\frac{(\alpha+1)\theta\alpha}{\beta}\sum_{i=1}^{r}Z_i^{\alpha-1}Z'_{i\theta}-\sum_{i=1}^{r}\left[\frac{Z''_{i\beta\theta}(1-Z_i)+(Z'_{i\beta}Z'_{i\theta})}{(1-Z_i)^2}\right]$$
$$+2(b-1)\sum_{i=1}^{r}\left[\frac{Z''_{i\beta\theta}(1-Z_i)-(Z'_{i\beta}Z'_{i\theta})}{\left[1-(1-Z_i)^2\right]}\right]-4(b-1)\sum_{i=1}^{r}\left(\frac{Z'_{i\beta}Z'_{i\theta}(1-Z_i)^2}{\left[1-(1-Z_i)^2\right]^2}\right)$$
$$-2b(n-r)\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^{b}\right)^{-1}\times$$
$$\left[A'_{\beta}A'_{\theta}\left(\frac{2(b-1)(1-A)^2}{\left\{1-(1-A)^2\right\}}-1\right)+A''_{\beta\theta}(1-A)+\frac{2b\,A'_{\beta}A'_{\theta}(1-A)^2\left\{1-(1-A)^2\right\}^{b-1}}{\left[1-\left\{1-(1-A)^2\right\}^{b}\right]}\right],$$

and

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$$U(\theta\theta)=\frac{-r}{\theta^2}-\alpha(\alpha+1)\sum_{i=1}^{r}\left[Z_i^{\alpha-1}Z'_{i\theta}\ln\left(\frac{x_{(i)}}{\beta}\right)\right]-\sum_{i=1}^{r}\left[\frac{Z''_{i\theta}(1-Z_i)+(Z'_{i\theta})^2}{(1-Z_i)^2}\right]$$

$$+2(b-1)\sum_{i=1}^{r}\left[\frac{Z''_{i\theta}(1-Z_i)-(Z'_{i\theta})^2}{\left[1-(1-Z_i)^2\right]}\right]-4(b-1)\sum_{i=1}^{r}\left(\frac{Z'_{i\theta}(1-Z_i)}{\left[1-(1-Z_i)^2\right]}\right)^2$$

$$-2b(n-r)\left\{1-(1-A)^2\right\}^{b-1}\left(1-\left\{1-(1-A)^2\right\}^b\right)^{-1}\times$$

$$\left[A'^{2}_{\theta}\left(\frac{2(b-1)(1-A)^2}{\left\{1-(1-A)^2\right\}}-1\right)+A''_{\theta}(1-A)+\frac{2b\,A'^{2}_{\theta}(1-A)^2\left\{1-(1-A)^2\right\}^{b-1}}{\left[1-\left\{1-(1-A)^2\right\}^b\right]}\right],$$

where $A=W$, for type I censored and for type II censored $A=Z_r$ and

$$Z'_{i\alpha}=\left(\frac{1}{\alpha}\right)Z_i\ln\left(\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)-\left(\frac{1}{\alpha}\right)Z_i^{\alpha+1}\left[\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}+\theta\ln\left(\frac{x_{(i)}}{\beta}\right)\right],$$

$$Z'_{i\beta}=\frac{\theta}{\beta}\left(Z_i^{\alpha+1}-Z_i\right),$$

$$Z'_{i\theta}=\ln\left(\frac{x_{(i)}}{\beta}\right)\left(Z_i-Z_i^{\alpha+1}\right),$$

$$Z''_{i\alpha}=\left(\frac{1}{\alpha^2}\right)\ln\left(\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)\left[Z'_{i\alpha}-\left(\frac{2}{\alpha}\right)Z_i\right]+\left(\frac{\theta}{\alpha}\right)Z_i^{\alpha+1}\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}\ln\left(\frac{x_{(i)}}{\beta}\right)$$

$$+\left\{\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}+\theta\ln\left(\frac{x_{(i)}}{\beta}\right)\right\}\left[\left(\frac{2}{\alpha^2}\right)Z_i^{\alpha+1}-\left(1+\frac{1}{\alpha}\right)Z_i^{\alpha}Z'_{i\alpha}-\left(\frac{1}{\alpha}\right)Z_i^{\alpha+1}\ln Z_i\right],$$

$$Z''_{i\alpha\beta}=\left(\frac{1}{\alpha^2}\right)Z'_{i\beta}\ln\left(\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)-\left(1+\frac{1}{\alpha}\right)Z_i^{\alpha}Z'_{i\beta}\left[\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}+\theta\ln\left(\frac{x_{(i)}}{\beta}\right)\right]$$

$$+\left(\frac{\theta}{\beta}\right)Z_i^{\alpha+1}\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta},$$

$$Z''_{i\alpha\theta}=\left(\frac{1}{\alpha^2}\right)Z'_{i\theta}\ln\left(\alpha+\left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)-\left(1+\frac{1}{\alpha}\right)Z_i^{\alpha}Z'_{i\theta}\left[\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}+\theta\ln\left(\frac{x_{(i)}}{\beta}\right)\right]$$

$$+Z_i^{\alpha+1}\left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta}\ln\left(\frac{x_{(i)}}{\beta}\right),$$

$$Z''_{i\beta}=\left(\frac{\theta}{\beta^2}\right)Z_i\left[1-Z_i^{\alpha}\right]+\left(\frac{\theta}{\beta}\right)Z'_{i\beta}\left[(\alpha+1)Z_i^{\alpha}-1\right],$$

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$$Z''_{i\beta\theta} = \left(\frac{1}{\beta}\right)\left[Z_i^{\alpha+1} + \theta(\alpha+1)Z'_{i\theta} Z_i^{\alpha} - \theta Z'_{i\theta} - Z_i\right],$$

$$Z''_{i\theta} = \ln\left(\frac{x_{(i)}}{\beta}\right) Z'_{i\theta}\left[1-(\alpha+1)Z_i^{\alpha}\right],$$

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References

[1] P. W. Mielke, Another family of distributions for describing and analyzing precipitation data. *Journal Applied Meteorology*. 12, 275-280, (1973).

[2] P. W. Mielke, and E. S. Johnson, Three parameter Kappa distribution maximum likelihood estimation and likelihood ratio tests. *Monthely Weather Review*. 101, 701-707, (1973).

[3] J. S. Park, S. C. Seo, and T. Y. Kim, *A Kappa distribution with a hydrological application*. Stochastic Environmental Research. Spring Berlin/ Heidelberg. (2008).

[4] S. K. Ashour, E. A. Elsherpieny, and Y. Y. Abdelall, Parameter estimation for three-parameter Kappa distribution under type II censored samples. *Journal of Applied Sciences Research*. 5(10). 1762-1766, (2009).

[5] S. Hussain, *Extension, Properties and Application of Kappa distribution*. M. Phil, Thesis, Government College University, Faisalabad, Pakistan. (2015).

[6] M. Javed, T. Nawaz, and M. Irfan, The Marshall-Olkin Kappa distribution: Properties and Applications. *Journal of King Saud University-Science.* 1-8, (2018).

[7] T. Nawaz, S. Hussain, T. Ahmed, F. Naz, and M. Abid, Kumaraswamy generalized Kappa distribution with application to stream flow data. *Journal of King Saud University-Science*. 320(1), 172-182, (2020).

[8] A. Al-Shomrani, and A. Al-Arfaj, A new flexible odd kappa-g family of distributions: theory and properties. *Advances and applications in mathematical sciences*. 210(1), 3443-3514, (2021).

[9] A. Al-Shomrani, O. Arif, A. Shawky, S. Hanif, and M. Q. Shahbaz, Topp-leone family of distributions: some properties and application. *Pakistan Journal of Statistics and Operation Reseaarch*. 12(3), 443-451, (2016).

[10] A. C. Cohen, Maximum likelihood Estimation in the Weibull Distribution Based on Complete and Censored Samples. *Technometrics*. 7. 579-588, (1965).

[11] T. Bjerkedal, Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Hygiene*. 72(1). 130-148, (1960).

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