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Statistical Inference of Modified Frechet–Exponential Distribution with Applications to Real-Life Data

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Abstract: This paper introduces a new lifetime model, referred to as modified Frechet–Exponential distribution (MFED), is developed on the basis of the modified Frechet method. Numerous statistical properties of the suggested model are derived and discussed including ordinary and incomplete moments, quantile, mode, the moment generating functions, reliability and order statistics. The observed Fisher's information matrix is provided, and the model parameters are estimated using the maximum likelihood technique. The suggested model is very adaptable and has the capacity to simulate datasets with monotonic and nonmonotonic failure rates. The proposed model is applied on three real datasets for checking its performance in comparison with available well-known models. The motivation of this work that the suggested model has shown good performance in comparison with the available versions of the Exponential distribution used in the literature.

Keywords: Exponential distribution, Fréchet distribution, maximum likelihood estimation, Modified Frechet technique, Moments

1 Introduction

The exponential distribution (ED) is a popular lifetime model and has a wide range of applications including reliability analysis and applied statistics but its inability to properly model real life phenomena like wind speed, sea waves and earthquakes whose failure rate is not constant has led to many modifications and generalizations of the exponential distribution for get more flexible models such as Exponentiated Odd Lomax Exponential [1], Lomax exponential [2], Odd Lomax Inverse Exponential [3] and other models of Exponential distribution to deal with this data. The source and other information about the Exponential distribution can be found in [4,5]. A random variable X is said to have the Exponential (E) distribution with parameter $\lambda > 0$ if it's probability density function (pdf) is given by

$$f(x) = \lambda e^{-\lambda x}, \ \ x > 0, \tag{1.1}$$

while the cumulative distribution function (cdf) is given by

$$F(x) = 1 - e^{-\lambda x}, \ \ x > 0. \tag{1.2}$$

The Fréchet distribution was proposed to model extreme events such as foods, earthquakes, horse racing, wind speed, precipitation, sea waves, river discharges and more by Fréchet [6]. For more information on the Fréchet distribution and its applications, see [7]. Some extensions of the Fréchet distribution are available in the literature, such as the Exponentiated Fréchet (EFr) [8], transmuted Fréchet (TFr) [9], Beta Fréchet (BFr) [10], transmuted Exponentiated Fréchet (TEFr) [11], gamma extended Fréchet (GEFr) [12], Kumaraswamy Fréchet (Kw-Fr) [13], Marshall-Olkin Fréchet [14], transmuted MarshallOlkin Fréchet [15] and Weibull Fréchet (WFr) [16]. A random variable X is said to have the Fréchet (Fr) distribution with parameters $\theta > 0$ as a scale parameter and $\beta > 0$ as a shape parameter if it's probability density function (pdf) is given by

$$g(x,\theta,\beta) = \beta\theta^{\beta}x^{-\beta-1}e^{-\left(\frac{\theta}{x}\right)^{\beta}}, \ \ x > 0, \tag{1.3}$$

while the cumulative distribution function (cdf) is given by

$$G(x,\theta,\beta) = e^{-\left(\frac{\theta}{x}\right)^{\beta}}, \ \ x > 0. \tag{1.4}$$

The purpose of this paper is to provide another extension of the Fréchet model called the modified Frechet-Exponential (MFE) distribution. The main

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feature of this model is the additional parameter will be introduced in Eq. (1.2) to give greater flexibility in the form of the generated distribution. Using the modified Frechet-G (MF-G) family introduced by Alamgir et al. [17], we construct the new two-parameters MFE model. We give a comprehensive description of some mathematical properties of the new distribution with the hope that it will attract wider applications in real life phenomena like sea waves, river discharges, wind speed in addition to reliability, clinical studies and other areas of research. The CDF and PDF of the modified Frechet-G (MF-G) family are specified by the expressions as follows:

$$G_{MF}(x) = \frac{e^{-(F(x))^{\alpha}} - 1}{e^{-1} - 1}, \quad x > 0, \tag{1.5}$$

$$g_{MF}(x) = \frac{\alpha f(x)(F(x))^{(\alpha-1)} e^{-(F(x))^{\alpha}}}{1 - e^{-1}}, \quad x > 0. \tag{1.6}$$

In Eqs. (1.5) and (1.6), $F(x)$ and $f(x)$ denote the CDF and PDF of the input model, respectively. This technique is used to introduce modified Frechet–Exponential distribution (MFED). The basic purpose of producing MFED has a more flexible distribution to model data in comparison to other versions of Exponential distribution.

We applied our model to three practical datasets. Dataset I is taken from [18] representing the daily mean wind speed data for March, taken in 2015 from the Turkish Meteorological Services for Sinop, Turkey, Dataset II is taken from Bjerkedal [19] who gave various doses of tubercle bacilli to groups of 72 guinea pigs and recorded their survival times (in days) and Dataset III is taken from the work of Gross and Clark [20] which represents the relief times (in hours) of 20 patients who received an analgesic. This model provided a satisfactory fit to the datasets in comparison with various versions of Exponential distribution.

This paper is organized as follows. We derive the cumulative, density, survival and hazard functions of the modified Frechet–Exponential (MFE) distribution in Section 2. In Section 3, we present some statistical properties including, quantile function, median, mode, rth moment, skewness, kurtosis and the moment generating function. The distribution of the order statistics is expressed in Section 4. The shannon entropy is inferred in Section 5. The mean residual life function is given in Section 6. The stress-strength parameter is obtained in Section 7. The maximum likelihood estimation of the parameters is determined in Section 8. A simulation study is introduced in Section 9. Real data sets are analyzed in Section 10 and the results are compared with existing distributions. Finally, we introduce the conclusions in Section 11.

2 The Modified Frechet–Exponential Distribution

In this section, we study the two parameters modified Frechet–Exponential distribution. Substituting from Eqs. (1.1) and (1.2) into Eq. (1.5), the cumulative distribution function of the modified Frechet–Exponential distribution (MFED) is given by

$$F_{MFED}(x) = \frac{e^{-\left(1-e^{-\lambda x}\right)^{\alpha}} - 1}{e^{-1} - 1}, x > 0, \lambda, \alpha > 0. \tag{2.1}$$

Substituting from Eqs. (1.1) and (1.2) in Eq. (1.6), the pdf corresponding to Eq. (2.1) is given by

$$f_{MFED}(x) = \frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}, \tag{2.2}$$

$x > 0$ and $\lambda, \alpha > 0$.

The survival function, hazard rate function and reversed-hazard rate function of X $\sim$ MFED (α, λ) are given by

$$S_{MFED}(x) = \frac{e^{-1} - e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}}{e^{-1} - 1}, x > 0, \tag{2.3}$$

$$h_{MFED}(x) = \frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{e^{-\left(1-e^{-\lambda x}\right)^{\alpha}} - e^{-1}} \tag{2.4}$$

and

$$r_{MFED}(x) = \frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}}, \tag{2.5}$$

respectively, $x > 0$ and $\lambda, \alpha > 0$.

Figures (1-3) display the cdf, pdf, survival, hazard rate and reversed hazard rate functions of the MFE (α, λ) distribution for some parameters values.

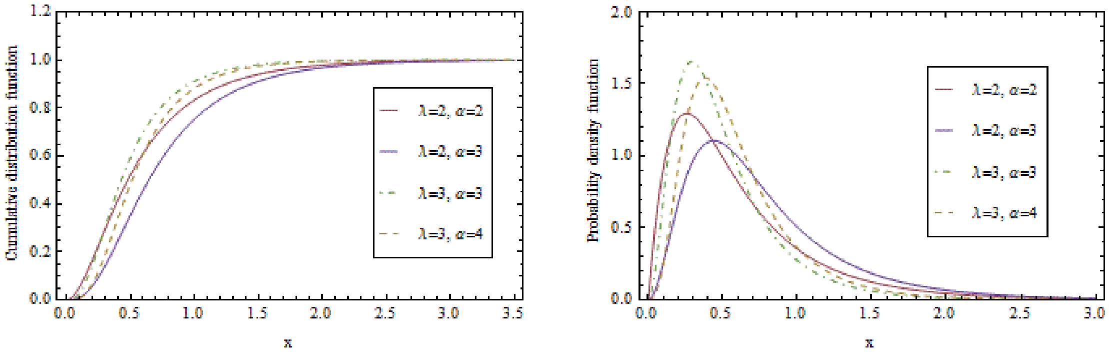

Figure 1: Plots of the cdf and pdf of the MFE distribution for different values of parameters.

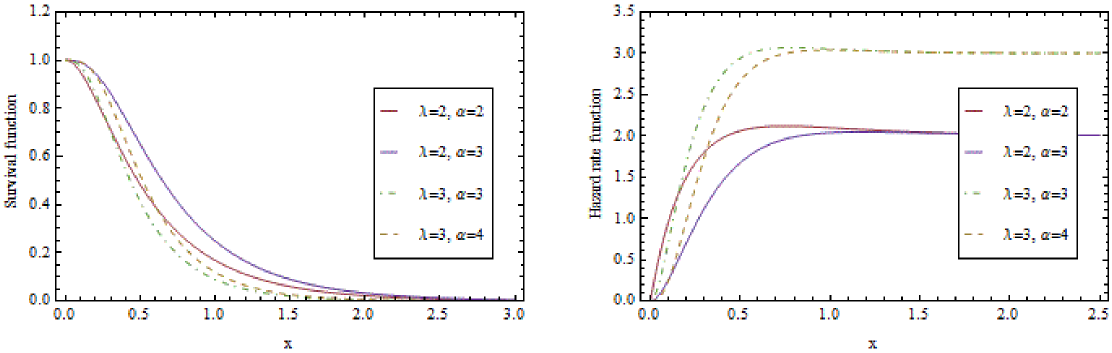

Figure 2: Plots of the survival and hazard rate functions of the MFE distribution for different values of parameters.

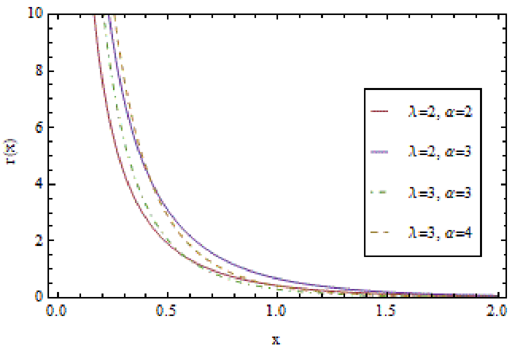

Figure 3: The reversed hazard rate function of MFE distribution for different values of parameters.

3 Statistical Properties

In this section, we will study some statistical properties for the MFE distribution, specially quantile function, median, mode, moments, skewness, kurtosis and the moment generating function.

3.1 Quantile and median

If X $\sim$ MFED(α, λ) , then the quantile x_p of the MFED is given by

$$F(x_p) = u, \quad 0 < u < 1. \tag{3.1}$$

From Eq. (2.2), x_p can be obtained as follows

$$x_p = \frac{1}{\lambda}\left[-\log\left\{1-\left(-\log\left(u\left(e^{-1}-1\right)+1\right)\right)^{\frac{1}{\alpha}}\right\}\right]. \tag{3.2}$$

By putting $u = 0.5$ in Eq. (3.2), we get the median of MFE distribution as follows

$$Median = \frac{1}{\lambda}\left[-\log\left\{1-\left(-\log\left(0.5\left(e^{-1}-1\right)+1\right)\right)^{\frac{1}{\alpha}}\right\}\right]. \tag{3.3}$$

3.2 Mode

In this subsection, The mode of the MFE distribution can be obtained by differentiating its probability density function pdf with respect to x and equate it to zero.

The mode is the solution the following equation

$$f'(x) = 0. \tag{3.4}$$

By substitution PDF from Eq. (2.2) in Eq. (3.4), we have

$$\frac{\partial}{\partial x}\left(\frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}\right) = 0$$

and

$$\alpha\lambda^2 \frac{e^{\left(-x\lambda-\left(1-e^{-x\lambda}\right)^{\alpha}\right)}\left(1-e^{-x\lambda}\right)^{\alpha}}{\left(e^{-x\lambda}-1\right)^2\left(e^{-1}-1\right)} \times \left(\alpha e^{-x\lambda}\left(1-e^{-x\lambda}\right)^{\alpha} - \alpha e^{-x\lambda}+1\right) = 0 \tag{3.5}$$

3.3 Skewness and kurtosis

Variability analysis Skewness and Kurtosis on the shape parameter α can be investigated based on quantile measures. The short comings of the classical Kurtosis measure are widely acknowledged. The Bowely's skewness [21] based on quartiles is given by

$$S_k = \frac{p_{(0.75)} - 2p_{(0.5)} + p_{(0.25)}}{p_{(0.75)} - p_{(0.25)}}, \tag{3.6}$$

and the Moors' Kurtosis [22] is based on octiles

$$K_u = \frac{p_{(0.875)} - p_{(0.625)} - p_{(0.375)} + p_{(0.125)}}{p_{(0.75)} - p_{(0.25)}}, \tag{3.7}$$

where p(.) represents quantile function.

3.4 Moments

If X has MFE (α, λ) distribution, then the rth moments of random variable X is given by

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r \frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}dx, \tag{3.8}$$

where $e^{-\left(1-e^{-\lambda x}\right)^{\alpha}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(1-e^{-\lambda x}\right)^{n\alpha}$, by using exponential expansion.

$$\mu_r' = \sum_{n=0}^{\infty}\frac{(-1)^n\alpha\lambda}{n!\left(1-e^{-1}\right)}\int_0^{\infty} x^r e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{(n+1)\alpha-1}dx. \tag{3.9}$$

Using binomial expansion of $\left(1-e^{-\lambda x}\right)^{(n+1)\alpha-1} = \sum_{m=0}^{\infty}\binom{(n+1)\alpha-1}{m}(-1)^m e^{-m\lambda x}$, we obtain

$$\mu_r' = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{(n+1)\alpha-1}{m} \times \frac{(-1)^{n+m}\alpha\lambda}{n!\left(1-e^{-1}\right)}\int_0^{\infty} x^r e^{-(m+1)\lambda x}dx. \tag{3.10}$$

Put $z = (m+1)\lambda x$ in Eq. (3.10), after simplification, it will take the following form:

$$\mu_r' = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{(n+1)\alpha-1}{m}\frac{(-1)^{n+m}\alpha\lambda}{n!(1-e^{-1})} \times \int_0^{\infty}\frac{1}{(m+1)^{r+1}\lambda^{r+1}}z^r e^{-z}dz = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{(n+1)\alpha-1}{m} \times \frac{(-1)^{n+m}\alpha}{n!(1-e^{-1})(m+1)^{r+1}\lambda^r}\Gamma(r+1), \tag{3.11}$$

where $\Gamma(.)$ is the ordinary gamma function.

3.5 Moment generating function

If a random variable X has MFED$(x;\alpha,\lambda)$, then the MGF of X is given by

$$M_x(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx}\frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}dx. \tag{3.12}$$

The Taylor series yields the following simplified expression:

$$M_x(t) = E\left(e^{tx}\right) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r\frac{e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}dx. \tag{3.13}$$

Using Eq. (3.11) in Eq. (3.13), we obtain

$$M_x(t) = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{(n+1)\alpha-1}{m} \times \frac{(-1)^{n+m}\alpha t^r}{r!n!(1-e^{-1})(m+1)^{r+1}\lambda^r}\Gamma(r+1). \tag{3.14}$$

4 Order Statistics

Let $X_{1:n}, X_{2:n}, \ldots.., X_{n:n}$ denote the order statistics obtained from a random sample $X_1, X_2, \ldots., X_n$ which taken from the MFE distribution, then the pdf of $X_{i:n}$ is given as follows

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)[F(x)]^{i-1}[1-F(x)]^{(n-i)}. \tag{4.1}$$

Substitute $f(x)$ and $F(x)$ of MFE in Eq. (4.1), we obtain distribution of order statistic as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \times \left(\frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}\right) \times \left[\frac{e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}-1}{e^{-1}-1}\right]^{i-1} \times \left[1-\frac{e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}-1}{e^{-1}-1}\right]^{(n-i)} = \frac{n!}{(i-1)!(n-i)!(1-e^{-1})^n} \times \left[e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}-1\right]^{i-1} \times \alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)} \times \left[e^{-1}-e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}\right]^{(n-i)}. \tag{4.2}$$

By putting $i=1$ in Eq. (4.2), we have expression for first-order statistic as given below:

$$f_{1:n}(x) = \frac{n}{(1-e^{-1})^n}\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}} \times \left(1-e^{-\lambda x}\right)^{(\alpha-1)}\left[e^{-1}-e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}\right]^{(n-1)}. \tag{4.3}$$

By substituting $i=n$ in Eq.(4.2), the expression for the nth order statistic will take the form

$$f_{n:n}(x) = \frac{n}{(1-e^{-1})^n}\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}} \times \left(1-e^{-\lambda x}\right)^{(\alpha-1)}\left[e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}-1\right]^{n-1}. \tag{4.4}$$

For median's distribution insert $i=n/2$ in Eq. (4.2), we have

$$f_{i:n}(x) = \frac{n!}{((n/2)-1)!(n-(n/2))!(1-e^{-1})^n} \times \left[e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}-1\right]^{(n/2)-1} \times \alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)} \times \left[e^{-1}-e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}\right]^{(n-(n/2))}. \tag{4.5}$$

5 Shannon Entropy

The Shannon entropy of MFED is given as follows:

$$\begin{aligned}
S.E_x &= E\left[-\log f(x)\right] \\
&= E\left[-\log\left(\frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}\right)\right] \\
&= -\log\left[E\left(\frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}\right)\right] \\
&= -\log\left[\begin{array}{c}\frac{\alpha^2\lambda^2}{\left(1-e^{-1}\right)^2}\int_0^\infty e^{-2\lambda x}e^{-2\left(1-e^{-\lambda x}\right)^{\alpha}} \\ \times\left(1-e^{-\lambda x}\right)^{2\alpha-2}dx\end{array}\right], \qquad (5.1)
\end{aligned}$$

substituting $1-e^{-\lambda x}=y$ in Eq. (5.1), the expression will be

$$S.E_x = -\log\left[\frac{\alpha^2\lambda}{\left(1-e^{-1}\right)^2}\int_0^1 (1-y)\, e^{-2y^{\alpha}} y^{2\alpha-2}dy\right]. \qquad (5.2)$$

Putting $2y^{\alpha}=z$ in Eq. (5.2), after simplification, it will take the following form

$$S.E_x = -\log\left[\frac{2^{\frac{1}{\alpha}-2}\alpha\lambda}{\left(1-e^{-1}\right)^2}\int_0^2 \left(2^{1/\alpha}-z^{1/\alpha}\right)e^{-z}z^{1-\frac{1}{\alpha}}dz\right]. \qquad (5.3)$$

Using series $e^{-z}=\sum_{k=0}^{\infty}(-z)^k/k!$ in Eq. (5.3), we obtain the final expression as given below

$$\begin{aligned}
S.E_x &= -\log\left[\begin{array}{c}\frac{\alpha\lambda}{4\left(1-e^{-1}\right)^2}\int_0^2\sum_{k=0}^{\infty}\frac{(-z)^k}{k!} \\ \times\left(2^{1/\alpha}-z^{1/\alpha}\right)z^{1-\frac{1}{\alpha}}dz\end{array}\right] \\
&= -\log\left[\begin{array}{c}\frac{\alpha\lambda}{4\left(1-e^{-1}\right)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} \\ \times\left(\frac{2^{k+2}}{(k+2)(\alpha(k+2)-1)}\right)\end{array}\right]. \qquad (5.4)
\end{aligned}$$

6 Mean Residual Life Function

If X follows MFED. Then, $\mu(t)$ of MFRD has the following expression

$$\mu(t) = \frac{1}{S(t)}\left(E(t)-\int_0^t xf(x)\,dx\right)-t, \qquad t\geq 0, \quad (6.1)$$

where

$$\int_0^t xf(x)\,dx = \int_0^t x\frac{\alpha\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha}}\left(1-e^{-\lambda x}\right)^{(\alpha-1)}}{1-e^{-1}}dx. \qquad (6.2)$$

Putting, $1-e^{-\lambda x}=y$, the above expression will take the form given below

$$\int_0^t xf(x)\,dx = \frac{-\alpha}{\lambda\left(1-e^{-1}\right)} \times\int_0^{1-e^{-\lambda t}}\left(\log(1-y)\right)e^{-y^{\alpha}}y^{\alpha-1}dy, \qquad (6.3)$$

substitute $\log(1-y)=\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}y^{n+1}}{n+1}$ in Eq. (6.3), we obtain

$$\int_0^t xf(x)\,dx = \frac{-\alpha}{\lambda\left(1-e^{-1}\right)} \times\int_0^{1-e^{-\lambda t}}\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}y^{n+1}}{n+1}e^{-y^{\alpha}}y^{\alpha-1}dy, \qquad (6.4)$$

substitute $y^{\alpha}=z$ in Eq. (6.4), we obtain

$$\begin{aligned}
\int_0^t xf(x)\,dx &= \frac{1}{\lambda\left(1-e^{-1}\right)}\sum_{n=0}^{\infty}\frac{(-1)^{2n+2}}{n+1} \times\int_0^{\left(1-e^{-\lambda t}\right)^{\alpha}} z^{\frac{n+1}{\alpha}}e^{-z}dz \\
&= \frac{1}{\lambda\left(1-e^{-1}\right)}\sum_{n=0}^{\infty}\frac{(-1)^{2n+2}}{n+1} \times\gamma\left(\frac{n+1}{\alpha}+1,\left(1-e^{-\lambda t}\right)^{\alpha}\right) \\
&= \frac{1}{\lambda\left(1-e^{-1}\right)} \times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{2n+2}\left(\left(1-e^{-\lambda t}\right)^{\alpha}\right)^m}{(n+1)\Gamma\left(\frac{n+1}{\alpha}+m+2\right)} \\
&\quad \times\Gamma\left(\frac{n+1}{\alpha}+1\right)\left(\left(1-e^{-\lambda t}\right)^{\alpha}\right)^{\left(\frac{n+1}{\alpha}+1\right)} \times e^{-\left(1-e^{-\lambda t}\right)^{\alpha}}, \qquad (6.5)
\end{aligned}$$

where $\gamma(.,.)$ is the lower incomplete gamma function and $\Gamma(.)$ is the ordinary gamma function.

$$E(t) = \int_0^\infty tf(t)\,dt = \int_0^\infty t\frac{\alpha\lambda e^{-\lambda t-\left(1-e^{-\lambda t}\right)^{\alpha}}\left(1-e^{-\lambda t}\right)^{(\alpha-1)}}{1-e^{-1}}.$$

Using Eq. (3.11) and putting $r=1$, we obtain

$$E(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{(n+1)\alpha-1}{m} \times\frac{(-1)^{n+m}\alpha}{n!\left(1-e^{-1}\right)(m+1)^2\lambda}\Gamma(2), \qquad (6.6)$$

substituting Eqs. (2.3), (6.5), and (6.6) in Eq. (6.1), we obtain

$$\mu(t)=\frac{\left(e^{-1}-1\right)}{\lambda\left(1-e^{-1}\right)\left(e^{-1}-e^{-\left(1-e^{-\lambda x}\right)^{\alpha}}\right)}\times\left(\begin{array}{c}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{2n+2}\left(\left(1-e^{-\lambda t}\right)^{\alpha}\right)^{m}}{(n+1)\Gamma\left(\frac{n+1}{\alpha}+m+2\right)}\\ \times\Gamma\left(\frac{n+1}{\alpha}+1\right).\left(\left(1-e^{-\lambda t}\right)^{\alpha}\right)^{\left(\frac{n+1}{\alpha}+1\right)}\\ \times e^{-\left(1-e^{-\lambda t}\right)^{\alpha}}\\ \times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{(n+1)\alpha-1}{m}\\ \times\frac{(-1)^{n+m}\alpha}{n!(m+1)^{2}}\Gamma(2)\end{array}\right)-t. \tag{6.7}$$

7 Stress-Strength Parameter (SSP)

Let X_1 and X_2 be two independently and identically distributed random variables such that $X_1 \sim \text{MFED}(\alpha_1,\lambda)$ and $X_2 \sim \text{MFED}(\alpha_2,\lambda)$. Then, the stress-strength parameter is defined by

$$R=\int_{-\infty}^{+\infty} f_1(x)\,F_2(x)\,dx. \tag{7.1}$$

Utilizing Eqs. (2.1) and (2.2) in Eq. (7.1), the stress-strength parameter is given as

$$\begin{aligned}R&=\int_0^{\infty}\left(\frac{\alpha_1\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha_1}}\left(1-e^{-\lambda x}\right)^{(\alpha_1-1)}}{1-e^{-1}}\right)\\&\times\left(\frac{e^{-\left(1-e^{-\lambda x}\right)^{\alpha_2}}-1}{e^{-1}-1}\right)dx\\&=\int_0^{\infty}\left(\begin{array}{c}\frac{\alpha_1\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha_1}}\left(1-e^{-\lambda x}\right)^{(\alpha_1-1)}e^{-\left(1-e^{-\lambda x}\right)^{\alpha_2}}}{\left(1-e^{-1}\right)\left(e^{-1}-1\right)}\\-\frac{\alpha_1\lambda e^{-\lambda x-\left(1-e^{-\lambda x}\right)^{\alpha_1}}\left(1-e^{-\lambda x}\right)^{(\alpha_1-1)}}{\left(1-e^{-1}\right)\left(e^{-1}-1\right)}\end{array}\right)dx,\end{aligned} \tag{7.2}$$

substitute $1-e^{-\lambda x}=y$ in Eq. (7.2) and simplify, we obtain

$$\begin{aligned}R&=\frac{\alpha_1}{\left(1-e^{-1}\right)\left(e^{-1}-1\right)}\\&\times\int_0^1 e^{-y^{\alpha_1}}y^{(\alpha_1-1)}e^{-y^{\alpha_2}}dy\\&-\frac{1}{\left(e^{-1}-1\right)}.\end{aligned} \tag{7.3}$$

Using $e^{-y^{\alpha_2}}=\sum_{n=0}^{\infty}\frac{(-1)^n y^{n\alpha_2}}{n!}$ in Eq. (7.3) and simplify, we get

$$\begin{aligned}R&=\frac{\alpha_1}{\left(1-e^{-1}\right)\left(e^{-1}-1\right)}\\&\times\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 y^{(\alpha_1+n\alpha_2-1)}e^{-y^{\alpha_1}}dy\\&-\frac{1}{\left(e^{-1}-1\right)}.\end{aligned} \tag{7.4}$$

Now, putting $y^{\alpha_1}=z$ in Eq. (7.4), after simplification, we get the following form

$$\begin{aligned}R&=\frac{1}{\left(1-e^{-1}\right)\left(e^{-1}-1\right)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 z^{\frac{n\alpha_2}{\alpha_1}}e^{-z}dz\\&-\frac{1}{\left(e^{-1}-1\right)}\\&=\frac{1}{\left(1-e^{-1}\right)\left(e^{-1}-1\right)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma\left(\frac{n\alpha_2}{\alpha_1}+1,1\right)\\&-\frac{1}{\left(e^{-1}-1\right)}\\&=\frac{1}{\left(1-e^{-1}\right)\left(e^{-1}-1\right)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{n!\Gamma\left(\frac{n\alpha_2}{\alpha_1}+m+2\right)}\\&\times\Gamma\left(\frac{n\alpha_2}{\alpha_1}+1\right)e^{-1}-\frac{1}{\left(e^{-1}-1\right)}.\end{aligned} \tag{7.5}$$

8 Parameters Estimation

In this section, the method of maximum likelihood estimation is used to obtain point and interval estimation of the MFED's unknown parameters.

8.1 Maximum likelihood estimation

Let $x_1,x_2,\ldots,x_n$ denote a random sample of size n selected from the MFE distribution. The Likelihood function is given as

$$\begin{aligned}L(\alpha,\lambda)&=\left(\frac{\alpha\lambda}{1-e^{-1}}\right)^n e^{-\sum_{i=1}^n\lambda x_i}e^{-\sum_{i=1}^n\left(1-e^{-\lambda x}\right)^{\alpha}}\\&\times\prod_{i=1}^n\left(1-e^{-\lambda x_i}\right)^{(\alpha-1)}.\end{aligned} \tag{8.1}$$

Taking natural logarithm of Eq. (8.1), we obtain

$$\begin{aligned}\ell&=\log L(\alpha,\lambda)=n\log(\alpha\lambda)-n\log\left(1-e^{-1}\right)\\&-\sum_{i=1}^n\lambda x_i-\sum_{i=1}^n\left(1-e^{-\lambda x_i}\right)^{\alpha}\\&+(\alpha-1)\sum_{i=1}^n\log\left(1-e^{-\lambda x_i}\right).\end{aligned} \tag{8.2}$$

The maximum likelihood estimation of the parameters (α,λ) are obtained by differentiating the log-likelihood function ℓ with respect to the parameters α and λ and setting the result to zero, we have the following normal equations.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{\alpha}\log\left(1-e^{-\lambda x_i}\right) + \sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right) = 0. \tag{8.3}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \alpha\sum_{i=1}^{n} x_i e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{(\alpha-1)} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)} = 0. \tag{8.4}$$

The MLEs can be obtained by solving the Eqs. (8.3) and (8.4), numerically for α and λ.

8.2 Asymptotic confidence bounds

In this subsection, we use the variance covariance matrix I^{-1} see [23] to estimate the asymptotic confidence intervals of these parameters when $\lambda,\alpha > 0$ because the MLEs of the unknown parameters cannot be computed in closed forms, where I^{-1} is the inverse of the observed information matrix which is defined as follows

$$I^{-1} = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} & -\frac{\partial^2 \ell}{\partial \lambda^2} \end{pmatrix}^{-1} = \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\alpha}) & var(\hat{\lambda}) \end{pmatrix}. \tag{8.5}$$

The second derivative of Eqs. (8.3) and (8.4) with respect to α and λ yields Eqs. (8.6) and (8.7) given as

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \sum_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{\alpha}\left(\log\left(1-e^{-\lambda x_i}\right)\right)^2 \tag{8.6}$$

and

$$\frac{\partial^2 \ell}{\partial \lambda^2} = \frac{-n}{\lambda^2} + \alpha\sum_{i=1}^{n} x_i^2 e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{(\alpha-1)} - \alpha(\alpha-1)\sum_{i=1}^{n} x_i^2 e^{-2\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{(\alpha-2)} + (\alpha-1)\frac{\sum_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)x_i^2 e^{-\lambda x_i} - \sum_{i=1}^{n} x_i e^{-2\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)^2}. \tag{8.7}$$

Differentiating Eq. (8.3) with respect to λ, we obtain

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = -\alpha\sum_{i=1}^{n} x_i e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\log\left(1-e^{-\lambda x_i}\right) - \sum_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{\alpha}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)} + \sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)}. \tag{8.8}$$

We can derive the (1 - δ)100% confidence intervals of the parameters λ,α by using variance matrix as in the following forms

$$\hat{\alpha} \pm Z_{\delta/2}\sqrt{var(\hat{\alpha})},\ \ \hat{\lambda} \pm Z_{\delta/2}\sqrt{var(\hat{\lambda})}$$

where Z_δ denote the upper percentile of the standard normal distribution.

9 Simulations

In order to conduct a simulation study to investigate the behaviour of MLEs, 1000 samples are generated from the MFE distribution using the following expression of the quantile function

$$x_p = \frac{1}{\lambda}\left[-\log\left\{1-\left(-\log\left(u\left(e^{-1}-1\right)+1\right)\right)^{\frac{1}{\alpha}}\right\}\right], \tag{9.1}$$

where u follows uniform distribution over [0, 1]. MSE of MFED is computed by the following expression

$$MSE = \frac{1}{W}\sum_{i=1}^{W}\left(\hat{b}_i - b_i\right)^2, \tag{9.2}$$

where $b = (\alpha,\lambda)$, and $W = 1000$ simulations. Simulation results were obtained for various values of α and λ. Using the Monte Carlo simulation method. Various sample sizes (n = 50, 100, and 200) have been considered. Table 1 displays the average estimates of the parameters, mean square errors, and biases, showing that the property of consistency is proven by the fact that as sample size increases, the estimated parameter values approach the assumed parameter values quite closely.

Table **1:** Parameters estimates, MSEs and biases.

Parameter	n	$\hat{\alpha}$	$\hat{\lambda}$	MSE$(\hat{\alpha})$	MSE$\left(\hat{\lambda}\right)$	Bias$(\hat{\alpha})$	Bias$\left(\hat{\lambda}\right)$
$\alpha=2$	50	2.15622	2.10109	0.262797	0.154579	0.15622	0.101086
$\lambda=2$	100	2.06723	2.03787	0.0965705	0.0650739	0.0672321	0.0378668
	200	2.03152	2.02068	0.0400649	0.0322064	0.031519	0.0206803
$\alpha=3$	50	3.27521	2.09021	0.766525	0.128042	0.275211	0.0902082
$\lambda=2$	100	3.11654	2.03374	0.269606	0.0543291	0.11654	0.0337424
	200	3.05475	2.01823	0.110469	0.0268141	0.0547477	0.0182267
$\alpha=4$	50	4.41144	3.12672	1.655	0.258108	0.411439	0.126717
$\lambda=3$	100	4.17186	3.04725	0.562971	0.109937	0.171863	0.0472487
	200	4.08023	3.02517	0.228801	0.0541353	0.0802296	0.0251739
$\alpha=5$	50	5.56289	3.12149	3.02057	0.239814	0.562887	0.121486
$\lambda=3$	100	5.23293	3.04525	0.999035	0.102351	0.232933	0.0452481
	200	5.10843	3.0239	0.403382	0.0503066	0.108427	0.0238983

10 Application

In this section, three practical datasets are used to assess the performance of MFE$(x;\alpha,\lambda)$ model. Different submodels of Exponential distribution are considered for comparison such as Inverse Exponential (IE) [24], Exponentiated Exponential (EE) [25], Marshall-olkin Exponential (MOE), Kumaraswamy Exponential $(K_W - E)$ [26], Weibull-Exponential (WE) [27] and Exponential (E) distributions using Kolmogorov Smirnov (K-S) statistic and P-value, as well as the negative of the log-likelihood functions (-ℓ), Akaike information criterion(AIC) [28], Akaike Information Citerion with correction (AICC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) [29] and consistent Akaike's information criteria (CAIC) values.

10.1 Dataset I

Dataset I which represents the daily mean wind speed data for March, taken in 2015 from the Turkish Meteorological Services for Sinop, Turkey, is taken from [18]. The data points are given below: 2.8, 1.8, 3.2, 5.0, 2.4, 4.8, 2.9, 2.9, 2.3, 3.2, 2.3, 2.0, 1.9, 3.3, 4.4, 6.7, 4.3, 1.9, 2.2, 3.3, 2.1, 4.0, 2.0, 3.1, 3.8, 3.1, 3.2, 3.4, 2.8, 2.1, 3.1.

Table 2 gives MLEs of parameters of the MFE, K-S Statistics and P-value. The values of -ℓ, AIC, AICC, BIC, CAIC, and HQIC are in Table 3.

We find that the MFE distribution with two parameters provides a better fit than the different submodels of Exponential distribution. It has the smallest K-S, AIC, AICC, BIC, CAIC and HQIC values among those considered in this paper.

Substituting the MLE's of the unknown parameters α and λ for dataset I into the inverse of the observed information matrix, we get estimation of the variance covariance matrix as the following

$$I_{\circ}^{-1} = \begin{pmatrix} 125.454 & 1.92706 \\ 1.92706 & 0.0334179 \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the unknown parameters α and λ are [2.08,45.934] and [0.7597,1.476], respectively.

To show that the likelihood equation have unique solution, we plot the profiles of the log-likelihood function of α and λ in Figure 4.

Figure 5 gives the cdf and pdf for the EE, $Kw-E$, WE, MOE, IE, E and MFE which are used to fit data I after substituting each distribution's unknown parameters with its MLE.

Figure 6 provides plots of PDF, CDF, PP-plots and QQ-plots for dataset I which shows that the MFE distribution fits the dataset I.

Table 2: MLEs, K–S and P-value for dataset I.

Model	MLEs of the parameters			K-S	P-vaue
$MFE(\alpha,\lambda)$	24.0073	1.11801		0.106072	0.840531
$EE(\alpha,\lambda)$	27.7412	1.26946		0.107863	0.826089
$Kw-E(a,b,\lambda)$	240.73	0.36836	2.57063	0.13035	0.621456
$WE(\alpha,\beta,\lambda)$	$1.65529*10^6$	2.83568	0.00184598	0.160922	0.359461
$MOE(\alpha)$	19.3602			0.206869	0.121737
$IE(\alpha)$	2.80764			0.397405	0.0000628408
$E(\lambda)$	0.321911			0.439788	$5.62052*10^{-6}$

Table 3: -ℓ, AIC, AICC, BIC, CAIC and HQIC for dataset I.

Model	-ℓ	AIC	AICC	BIC	CAIC	HQIC
$MFE(\alpha,\lambda)$	41.6624	87.3249	87.7534	90.1928	87.7534	88.2598
$EE(\alpha,\lambda)$	41.7245	87.4491	87.8777	90.3171	87.8777	88.384
$Kw-E(a,b,\lambda)$	40.9643	87.9287	88.8175	92.2306	88.8175	89.331
$WE(\alpha,\beta,\lambda)$	46.1526	98.3051	99.194	102.607	99.194	99.7075
$MOE(\alpha)$	49.0402	100.08	100.218	101.514	100.218	100.548
$IE(\alpha)$	65.9913	133.983	134.121	135.417	134.121	134.45
$E(\lambda)$	66.1379	134.276	134.414	135.71	134.414	134.743

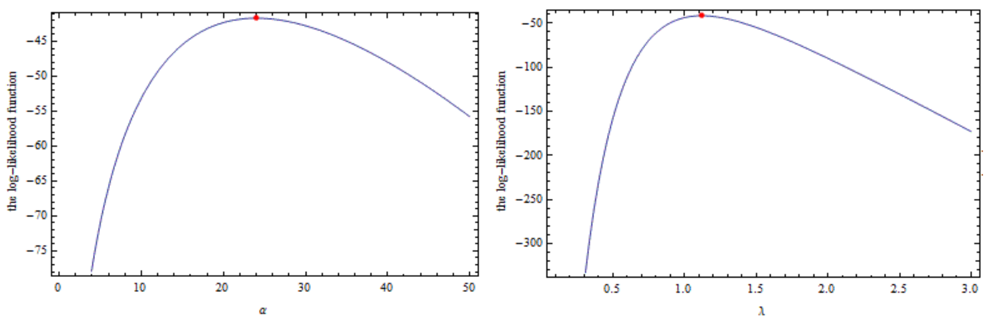

Figure 4: the profile of the log-likelihood function of α and λ.

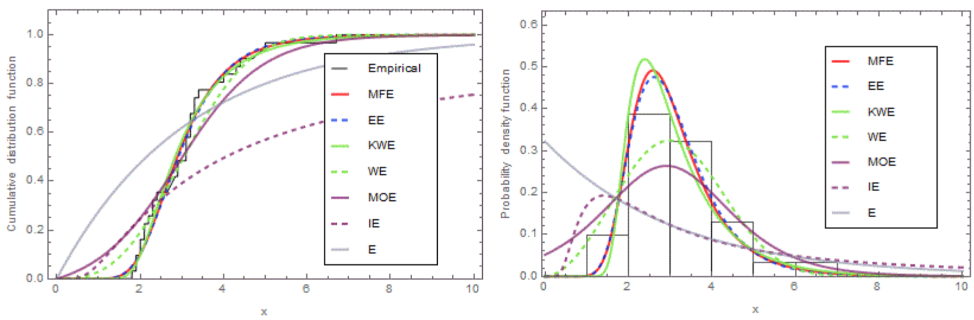

Figure 5: Fitted cdf with the empirical distribution and fitted pdf with histogram for dataset I.

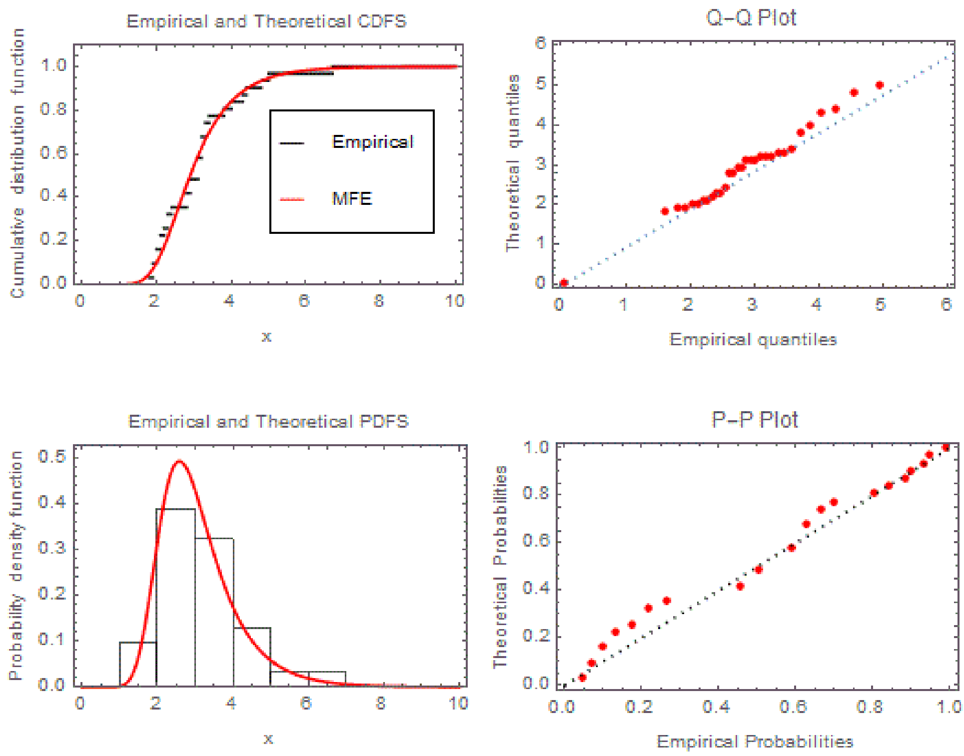

Figure 6: Fitted cdf, fitted pdf, QQ-plots and PP-plots of MFED for dataset I.

10.2 Dataset II

Dataset II which represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, is taken from Bjerkedal [19]. The data points are given below: 0.1, 0.33, 1.08, 1.08, 1.08, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 2.54, 2.78, 2.93, 3.27, 3.42, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.09, 1.12, 1.13, 1.15,1.36, 1.39, 1.44, 1.83, 1.95, 1.96, 1.97, 2.02, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 2.13, 1.46,1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53,2.54, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 4 gives MLEs of parameters of the MFE, K-S Statistics and P-value. The values of -ℓ, AIC, AICC, BIC, CAIC, and HQIC are in Table 5.

We find that the MFE distribution with two parameters provides a better fit than the different submodels of Exponential distribution. It has the smallest K-S, AIC, AICC, BIC, CAIC and HQIC values among those considered in this paper.

Substituting the MLE's of the unknown parameters α and λ for dataset II into the inverse of the observed information matrix, we get estimation of the variance covariance matrix as the following

$$I_{\circ}^{-1} = \begin{pmatrix} 0.406659 & 0.0606906 \\ 0.0606906 & 0.0132538 \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the unknown parameters α and λ are [0,4.734] and [0,1.122], respectively.

To show that the likelihood equation have unique solution, we plot the profiles of the log-likelihood function of α and λ in Figure 7.

Figure 8 gives the cdf and pdf for the EE, $Kw-E$, WE, MOE, IE, E and MFE which are used to fit data II after substituting each distribution's unknown parameters with its MLE.

Figure 9 provides plots of PDF, CDF, PP-plots and QQ-plots for dataset II which shows that the MFE distribution fits the dataset II.

Table 4: MLEs, K–S, and P-value for dataset II.

Model	MLEs of the parameters			K-S	P-vaue
$MFE(\alpha,\lambda)$	3.48421	0.896249		0.0871817	0.613003
$EE(\alpha,\lambda)$	3.30367	1.03703		0.0893901	0.581617
$Kw-E(a,b,\lambda)$	3.30401	0.999711	1.03726	0.0893923	0.581585
$WE(\alpha,\beta,\lambda)$	24025.7	1.62952	0.000987213	0.122193	0.214319
$MOE(\alpha)$	4.21441			0.15258	0.0627444
$IE(\alpha)$	1.1502			0.211399	0.00265859
$E(\lambda)$	0.540378			0.280626	0.0000165512

Table 5: $-\ell$, AIC, AICC, BIC, CAIC and HQIC for dataset II.

Model	$-\ell$	AIC	AICC	BIC	CAIC	HQIC
$MFE(\alpha,\lambda)$	99.3531	202.706	202.88	207.26	202.88	204.519
$EE(\alpha,\lambda)$	99.7196	203.439	203.613	207.993	203.613	205.252
$Kw-E(a,b,\lambda)$	99.7196	205.439	205.792	212.269	205.792	208.158
$WE(\alpha,\beta,\lambda)$	102.739	211.478	211.831	218.308	211.831	214.197
$MOE(\alpha)$	105.273	212.545	212.603	214.822	212.603	213.452
$IE(\alpha)$	123.154	248.309	248.366	250.585	248.366	249.215
$E(\lambda)$	116.315	234.63	234.687	236.907	234.687	235.536

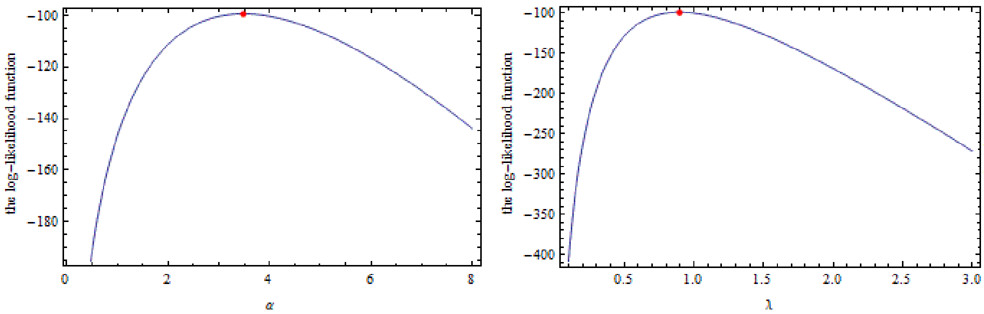

Figure 7: the profile of the log-likelihood function of α and λ.

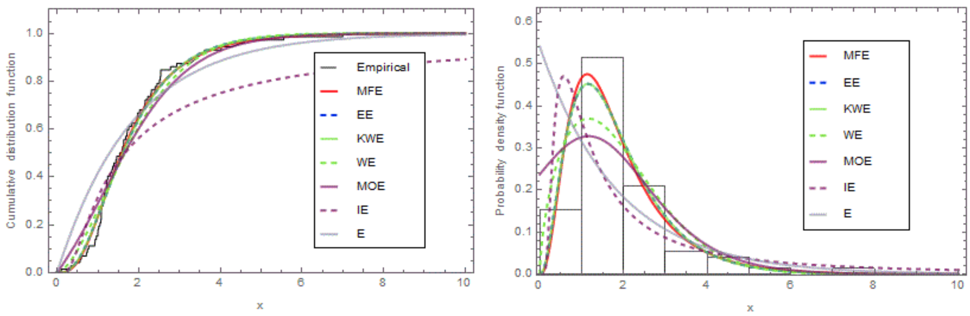

Figure 8: Fitted cdf with the empirical distribution and fitted pdf with histogram for dataset II.

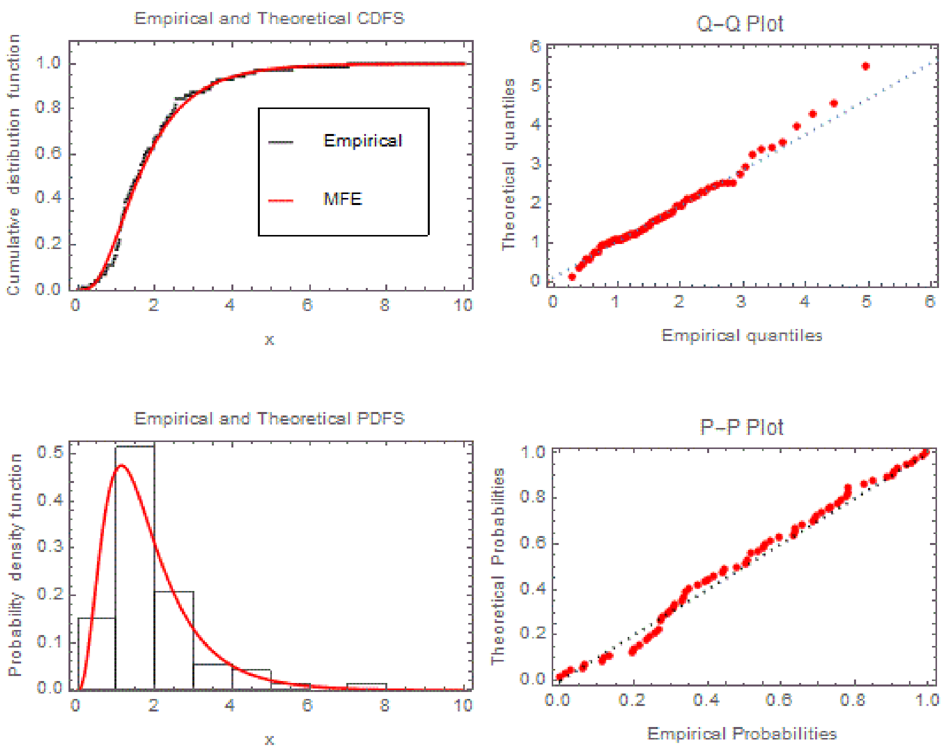

Figure 9: Fitted cdf, fitted pdf, QQ-plots and PP-plots of MFED for dataset II.

10.3 Dataset III

Dataset III which represents the relief times (in hours) of 20 patients who received an analgesic, is taken from the work of Gross and Clark [20]. The data points are given below: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

Table 6 gives MLEs of parameters of the MFE, K-S Statistics and P-value. The values of AIC, AICC, BIC, CAIC, and HQIC are in Table 7.

We find that the MFE distribution with two parameters provides a better fit than the different submodels of Exponential distribution. It has the smallest K-S, AIC, AICC, BIC, CAIC and HQIC values among those considered in this paper.

Substituting the MLE's of the unknown parameters α and λ for dataset III into the inverse of the observed information matrix, we get estimation of the variance covariance matrix as the following

$$I_{\circ}^{-1} = \begin{pmatrix} 426.496 & 7.97448 \\ 7.97448 & 0.164501 \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the unknown parameters α and λ are [8.044,72.911] and [0,2.226], respectively.

To show that the likelihood equation have unique solution, we plot the profiles of the log-likelihood function of α and λ in Figure 10.

Figure 11 gives the cdf and pdf for the EE, WE, MOE, IE, E and MFE which are used to fit data III after substituting each distribution's unknown parameters with its MLE.

Figure 12 provides plots of PDF, CDF, PP-plots and QQ-plots for dataset III which shows that the MFE distribution fits the dataset III.

Table 6: MLEs, K–S, and P-value for dataset III.

Model	MLEs of the parameters			K-S	P-vaue
$MFE(\alpha,\lambda)$	32.4333	2.00038		0.120612	0.899975
$EE(\alpha,\lambda)$	36.6832	2.23524		0.134321	0.817352
$WE(\alpha,\beta,\lambda)$	$3.66805*10^6$	2.68471	0.0016895	0.175318	0.514469
$MOE(\alpha)$	5.25823			0.275965	0.0773872
$IE(\alpha)$	1.72473			0.387245	0.00328244
$E(\lambda)$	0.526316			0.439512	0.000491546

Table 7: $-\ell$, AIC, AICC, BIC, CAIC and HQIC for dataset III.

Model	$-\ell$	AIC	AICC	BIC	CAIC	HQIC
$MFE(\alpha,\lambda)$	15.9512	35.9024	36.6082	37.8938	36.6082	36.2911
$EE(\alpha,\lambda)$	16.2606	36.5212	37.2271	38.5127	37.2271	36.91
$WE(\alpha,\beta,\lambda)$	20.6213	47.2426	48.7426	50.2298	48.7426	47.8257
$MOE(\alpha)$	26.4381	54.8763	55.0985	55.872	55.0985	55.0707
$IE(\alpha)$	32.6687	67.3373	67.5596	68.3331	67.5596	67.5317
$E(\lambda)$	32.8371	67.6742	67.8964	68.6699	67.8964	67.8685

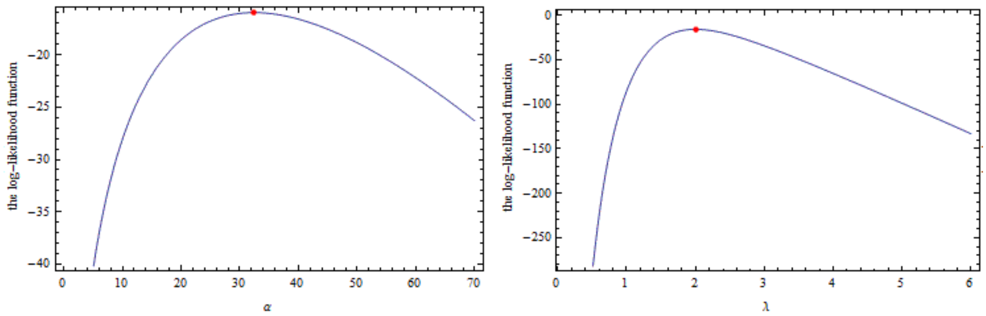

Figure 10: the profile of the log-likelihood function of α and λ.

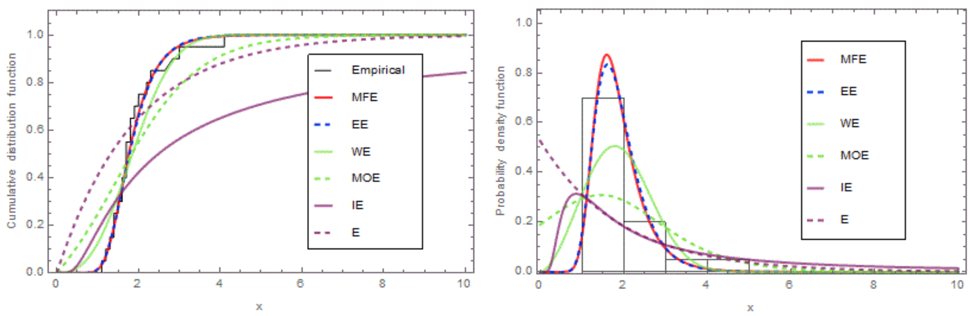

Figure 11: Fitted cdf with the empirical distribution and fitted pdf with histogram for dataset III.

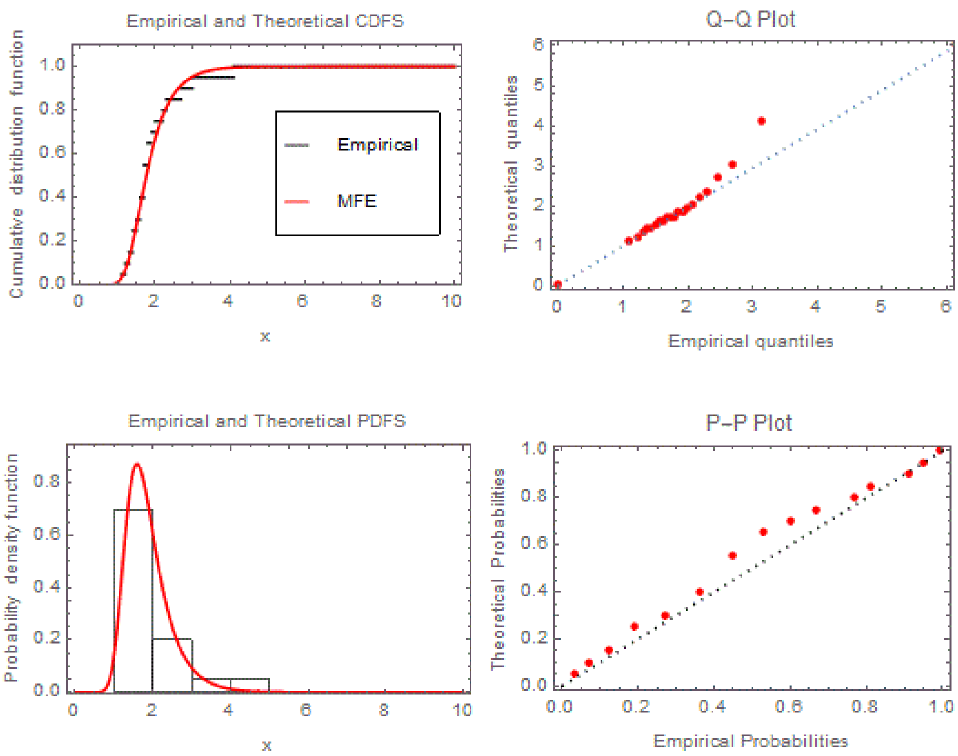

Figure 12: Fitted cdf, fitted pdf, QQ-plots and PP-plots of MFED for dataset III.

11 Conclusion

In this article, we suggeste a new lifetime model called modified Frechet–Exponential distribution (MFED) using the modified Frechet technique. We were able to determine the moments, MGF, and median in closed form, as well as the Quantile function, Mean Residuals' Life Function, Order Statistics, and Shannon Entropy for the proposed distribution. The MLE technique was used to estimate the model's parameters. A simulation study was performed by generating data from MFED, and the maximum likelihood estimates of the unknown parameter were obtained. The simulation results showed consistency of the parameter estimates of the MFE model. The suggested model was also fitted to three real datasets to show its usefulness. MFED model provided a satisfactory fit to the datasets in comparison with other distributions considered here for modeling real-life datasets, especially the exponentiated Exponential distribution. Perhaps this is because the cumulative distribution function of the modified Frechet–Exponential distribution can be expressed by the cumulative distribution function of the exponentiated Exponential distribution as follows $F_{MFE}(x) = \frac{e^{-F_{EE}(x)}-1}{e^{-1}-1}$, where $F_{EE}(x)$ is the cumulative distribution function of the exponentiated Exponential distribution. We believe in the importance of this distribution, and based on this premise that we believe in it, we think it will be the motivation to continue conducting more research on this distribution, we may apply one of the copulas family such as Farlie-Gumbel-Morgenstern (FGM) copulas and estimate some life-time parameters under progressive type-II censored data.

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