Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/120405

Recurrence Relations for Moment Generating Functions of the Generalized Compound Rayleigh Distribution

M. Maswadah and M. Seham *

Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt

Received: 3 Oct. 2022, Revised: 22 Nov. 2022, Accepted: 3 Dec. 2022
Published online: 1 Apr. 2023

Abstract: In this paper, we consider the three-parameter generalized compound Rayleigh distribution. The recurrence relations for moment generating functions based on the generalized order statistics are obtained. The results presented here are a generalization of the recurrence relations for Single and product moment generating functions, based on the ordinary order statistics and the upper record statistics for some lifetime distributions in the literature such as Beta-Prime, Lomax, Burr XII, and Compound Rayleigh distributions which are special cases from the generalized compound Rayleigh distribution.

Keywords: Generalized order statistics; record values; single and product moments; Beta-Prime distribution; Lomax distribution; Burr distribution.

1 Introduction

Consider the Generalized Compound Rayleigh (GCR) distribution with probability density function (pdf) $f(x)$

$$f(x) = \alpha\lambda\beta^{\lambda}x^{\alpha-1}(\beta+x^{\alpha})^{-(\lambda+1)} \ ,x \geq 0 \ , \tag{1}$$

and cumulative distribution function (cdf) F(x)

$$F(x) = 1-\beta^{\lambda}(\beta+x^{\alpha})^{-\lambda} \ , \quad x \geq 0, \tag{2}$$

where $\alpha > 0,\ \beta > 0$ and $\lambda > 0$. Thus, from (1) and (2) we have the relation

$$f(x)(\beta+x^{\alpha}) = \alpha\lambda x^{\alpha-1}(1-F). \tag{3}$$

The three-parameter Generalized Compound Rayleigh distribution GCR (α,β,λ), includes among others four well-known distributions the Beta-Prime distribution $(\alpha=\beta=1)$, Lomax distribution $(\alpha=1)$, Burr XII distribution $\beta=1$, and the Compound Rayleigh distribution $(\alpha=2)$. Recently, the GCR model has been used extensively as life-testing model for its flexible hazard rate, [1] studied this distribution from a Poisson process perspective and [2] studied the Bayesian estimators for the parameter of this model. This paper establishes recurrence relations between moment generating functions (MGF) based on the Generalized order statistics (GOS), which can be used to generate the moments of record values and the ordinary order statistics as special cases in a simple recursive process. It worthwhile to mention that, the results presented here are a generalization of the recurrence relations for moments and moment generating functions of the above distributions in the literature. The concept of the generalized order statistics (GOS) is introduced by [3] as unified approach to ordinary OS, record values and k-record values, which can be outlined as: The random variables $X(1,n,m,k),....,X(n,n,m,k)$ be GOS from an absolutely continuous $F(x)$ and $f(x)$, with noting that $(X(0,n,m,k)=0,k\geq 1)$, then their joint (pdf) can be written in the form:

$$f(x_1,x_2,....,x_n) = C\prod_{i=1}^{n-1} f(x_i)\left[1-F(x_i)\right]^{m}\left[1-F(x_n)\right]^{k-1} f(x_n), \tag{4}$$

on the cone $F^{-1}(0) < x_1 < \cdots < x_n < F^{-1}(1)$ of R^n, where $C=\prod_{i=1}^{n}\gamma_i$, $\gamma_i = k+n-i+M_i$, $M_i = \sum_{j=i}^{n-1} m_j$, $,\gamma_n = k > 0$, and $\widetilde{m} = (m_1,m_2,...,m_{n-1}) \in R^{n-1}$.

* Corresponding author e-mail: seham_mohamed@sci.aswu.edu.eg

- If $m=0$ and $k=1$, then (4) is the joint pdf of the ordinary OS..
- If $m=-1$ and $k=1$, then (4) is the joint pdf of the first n upper record values $Y_{U(1)} < Y_{U(2)} < .... < Y_{U(n)} < .....$
- If $m=-1$ and $k \neq 1$, then (4) is the joint (pdf) of the k-record values.

From (4) the pdf of the r-th GOS $X(r,n,m,k)$ can be derived as

$$f_r(x) = \frac{c_{r-1}}{(r-1)!}(1-F(x))^{\gamma_r-1} f(x) g_m^{r-1}(F(x)), \tag{5}$$

and the joint (pdf) of $X(r,n,m,k)$ and $X(s,n,m,k)$, $r<s$ is given by

$$f_{r,s}(x,y) = C_{r,s}(1-F(x))^m f(x) g_m^{r-1}(F(x)) \ \ [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1-F(y)]^{\gamma_s-1} f(y), \tag{6}$$

for $x<y$. Here $C_{r-1} = \prod_{j=1}^{r-1}\gamma_j$, $r=1,2,...,n$, $C_{r,s} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}$, for $x \in [0,1]$, and $g_m(x) = h_m(x) - h_m(0)$,

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -log(1-x), & m=-1 \end{cases}.$$

2 Relations for Marginal Moment Generating Functions

Theorem2.1 :

For $n,k \in N$, $m \in Z$, $2 \leq r \leq n$ $and\, j=0,1,2,\ldots$

$$M_{r:n}^{j}(t) = \frac{\alpha\lambda\gamma_r}{\alpha\lambda\gamma_r - j} M_{r-1:n}^{j}(t) + \frac{j\beta}{\alpha\lambda\gamma_r - j} M_{r:n}^{j-\alpha}(t) + \frac{t}{\alpha\lambda\gamma_r - j}\left(M_{r:n}^{j+1}(t) + \beta M_{r:n}^{j-\alpha+1}(t)\right), \tag{7}$$

For $n,k \in N$, $m \in Z$, $2 \leq r \leq n-1$ $and\, j=0,1,2,\ldots,$

$$M_{r:n}^{j}(t) = \frac{\alpha\lambda\gamma_1}{\alpha\lambda\gamma_1 - j} M_{r-1:n-1}^{j}(t)+) + \frac{j\beta}{\alpha\lambda\gamma_1 - j} M_{r:n}^{j-\alpha}(t) + \frac{t}{\alpha\lambda\gamma_1 - j}\left(M_{r:n}^{j+1}(t) + \beta M_{r:n}^{j-\alpha+1}(t)\right). \tag{8}$$

Here $M_{r:n}^{j}(t)$ is the jth derivative of the MGF $M_{r:n}(t)$ of the rth GOS $X(r,n,m,k)$.

Proof :

For $2 \leq r \leq n$, the MGF of the r-th GOS $X(r,n,m,k)$ is given by

$$M_{r:n}(t) = \frac{C_{r-1}}{(r-1)!}\int_0^\infty e^{tx} g_m^{r-1}(F(x))[1-F(x)]^{\gamma_r-1} f(x)dx.$$

Integrating the right-hand side by parts with treating $[1-F(x)]^{\gamma_r-1}$ as the part of integration and the rest of the integrand as the part of differentiation, we get

$$\begin{aligned} M_{r:n}(t) = & \frac{(r-1)C_{r-1}}{(r-1)!\gamma_r}\int_0^\infty e^{tx} g_m^{r-2}(F(x))[1-F(x)]^{m+\gamma_r} f(x)dx \\ & + \frac{tC_{r-1}}{(r-1)!\gamma_r}\int_0^\infty e^{tx} g_m^{r-1}(F(x))[1-F(x)]^{\gamma_r} dx. \end{aligned}$$

Upon using $\gamma_{r-1} = \gamma_r + i(m+1)$ and $C_{r-1} = \gamma_r C_{r-2}$, we obtain

$$M_{r:n}^{j}(t) = M_{r-1:n}^{j}(t) + \frac{C_{r-1}}{(r-1)!\gamma_r}\int_0^\infty te^{tx}\, g_m^{r-1}(F(x))[1-F(x)]^{\gamma_r} dx.$$

Differentiating both sides of the last identity j times, we obtain

$$M_{r:n}^{j}(t) = M_{r-1:n}^{j}(t) + \frac{C_{r-1}}{(r-1)!\gamma_r}\int_0^\infty (jx^{j-1} + tx^j)e^{tx}\, g_m^{r-1}(F(x))[1-F(x)]^{\gamma_r} dx.$$

Using (3) we obtain

$$M_{r:n}^{j}(t) = M_{r-1:n}^{j}(t) + \frac{j\beta C_{r-1}}{\alpha\lambda\gamma_r(r-1)!}\int_0^\infty x^{j-\alpha} e^{tx} g_m^{r-1}(F(x))[1-F(x)]^{\gamma_r-1} f(x)dx$$

$$+\frac{jC_{r-1}}{\alpha\lambda\gamma_r(r-1)!}\int_0^\infty x^j e^{tx} g_m^{r-1}(F(x))\left[1-F(x)\right]^{\gamma_r-1} f(x)dx$$

$$+\frac{t\beta C_{r-1}}{\alpha\lambda\gamma_r(r-1)!}\int_0^\infty x^{j-\alpha+1} e^{tx} g_m^{r-1}(F(x))\left[1-F(x)\right]^{\gamma_r-1} f(x)dx$$

$$+\frac{tC_{r-1}}{\alpha\lambda\gamma_r(r-1)!}\int_0^\infty x^{j+1} e^{tx} g_m^{r-1}(F(x))\left[1-F(x)\right]^{\gamma_r-1} f(x)dx,$$

Thus, we obtain the recurrence relation

$$M_{r:n}^j(t)=M_{r-1:n}^j(t)+\frac{j\beta}{\alpha\lambda\gamma_r}M_{r:n}^{j-\alpha}(t)+\frac{j}{\alpha\lambda\gamma_r}M_{r:n}^j\,(t)+\frac{t}{\alpha\lambda\gamma_r}\left(\beta M_{r:n}^{j-\alpha+1}(t)+M_{r:n}^{j+1}(t)\right),$$

rearranging this relation, (7) can be derived. Upon substituting in (7) by the well-known recurrence relation $\gamma_r M_{r-1:n}^j(t)=\gamma_1 M_{r-1:n-1}^j(t)-(\gamma_1-\gamma_r)M_{r:n}^j(t)$, [3], we can derive (8) □

Corollary2.1.

For $2\le r\le n$, by repeatedly applying the recurrence relation in (7) we obtain

$$M_{r:n}^j(t)=\prod_{i=m}^r\frac{\alpha\lambda\gamma_i}{\alpha\lambda\gamma_i-j}M_{m-1:n}^j(t)+\sum_{P=0}^{r-m}\frac{\prod_{i,m+p+1,r}}{\alpha\lambda\gamma_{m+P}-j}\left[j\beta M_{r:n}^{j-\alpha}+t\left(M_{m+p:n}^{j+1}(t)+\beta M_{m+p:n}^{j-\alpha+1}(t)\right)\right] \tag{9}$$

$$\prod_{i,m+p+1,r}=\prod_{i=m+p+1}^r\frac{\alpha\lambda\gamma_i}{\alpha\lambda\gamma_i-j},and\prod_{i,r+1,r}=1.$$

For $2\le r\le n-1$ and $1\le m\le r-1$ applying the recurrence relation in (8) repeatedly we obtain

$$M_{r:n}^j(t)=\left(\frac{\alpha\lambda\gamma_1}{\alpha\lambda\gamma_1-j}\right)^{r-m}M_{m:n-1}^j(t)+\frac{1}{\alpha\lambda\gamma_1-j}\sum_{p=0}^{r-m-1}\left(\frac{\alpha\lambda\gamma_1}{\alpha\lambda\gamma_1-j}\right)^p\left[j\beta M_{r-p:n}^{j-\alpha}(t)+t\left(M_{r-p:n}^{j+1}(t)+\beta M_{r-p:n}^{j-\alpha+1}(t)\right)\right]. \tag{10}$$

For $r=m$, the recurrence relation (9) reduces to (7) and for $m=r-1$, the recurrence relation (10)reduces to (8).

Corollary2.2.

Putting $t=0$ in (7) and (8) we deduce recurrence relations for the moments of the GCR distribution based on GOS as:
For $\mathbf{n},k$ in N,m in Z, $2\le r\le n$ and $j=0,1,2,...$

$$EX^j(r,n,m,k)=\frac{\alpha\lambda\gamma_r}{\alpha\lambda\gamma_r-j}EX^j(r-1,n,m,k)+\frac{j\beta}{\alpha\lambda\gamma_r-j}EX^{j-\alpha}(r,n,m,k), \tag{11}$$

and for $n,k\in N,m\in Z,\ 2\le r\le n-1$ and $j=0,1,2,...$

$$EX^j(r,n,m,k)=\frac{\alpha\lambda\gamma_1}{\alpha\lambda\gamma_1-j}EX^j(r-1,n-1,m,k)+\frac{j\beta}{\alpha\lambda\gamma_1-j}EX^{j-\alpha}(r,n,m,k). \tag{12}$$

Corollary2.3.

Putting $t=0$ in (9) and (10) we deduce recurrence relations for moments of GOS as:
For $n,k\in N,m\in Z$, $2\le r\le n$ and $j=0,1,2,...$i

$$EX^j(r,n,m,k)=\prod_{i=s+1}^r\frac{\alpha\lambda\gamma_i}{\alpha\lambda\gamma_i-j}EX^j(s,n,m,k)+\sum_{P=s+1}^r\frac{j\beta}{\alpha\lambda\gamma_P-j}\prod_{i=P+1}^r\frac{\alpha\lambda\gamma_i}{\alpha\lambda\gamma_i-j}EX^{j-\alpha}(p,n,m,k).$$

For $2\le r\le n-1$ and $r-s\le n-1$

$$EX^j(r,n,m,k)=\left(\frac{\alpha\lambda\gamma_1}{\alpha\lambda\gamma_1-j}\right)^{r-s}EX^j(s,n-1,m,k).$$

Corollary2.4.

The recurrence relations for single moments of order statistics for the GCR distribution
have an analogous form to (11) and (12) with $\gamma_r=n-r+1$.

Corollary2.5.

The recurrence relation for single moments of k-th record values from the sequence of $Y_1^{(k)} < Y_2^{(k)} < .... < Y_n^{(k)} < ....$ of k-th upper record values from the GCR distribution has the form

$$E(Y_n^{(k)})^j = \frac{\alpha\lambda k}{\alpha\lambda k - j} E(Y_{n-1}^{(k)})^j + \frac{j\beta}{\alpha\lambda k - j} E(Y_n^{(k)})^{j-\alpha}.$$

Corollary2.6.

The recurrence relation for single moments of the GCR distribution based on n upper record values for $2 \leq n$ *and* $j = 0,1,2,...$ has the form:

$$E(Y_{U(n)}^j) = \frac{\alpha\lambda}{\alpha\lambda - j} E(Y_{U(n-1)}^j) + \frac{j\beta}{\alpha\lambda - j} E(Y_{U(n)}^{j-\alpha}).$$

3 Relations for the Joint Moment Generating Functions

Theorem3.1.

For $n,k \in N, m \in Z, 1 \leq r < s-2 \leq n$ and $i,j = 0,1,2,...$ .Then

$$M_{r,s:n}^{i,j}(t_1,t_2) = \frac{\alpha\lambda\gamma_s}{\alpha\lambda\gamma_s - j} M_{r,s-1:n}^{i,j}(t_1,t_2) + \frac{1}{\alpha\lambda\gamma_s - j}\left[j\beta M_{r,s:n}^{i,j-\alpha}(t_1,t_2) + t_2 L_{12}\right]. \quad (13)$$

$$L_{12} = M_{r,s:n}^{i,j+1}(t_1,t_2) + \beta M_{r,s:n}^{i,j-\alpha+1}(t_1,t_2)$$

Proof :

Form (6), for $n,k \in N, m \in Z, 1 \leq r < s-2 \leq n$ and $i,j = 0,1,2,...$ .Then we have

$$M_{r,s:n}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \iint_{0<x<y<\infty} e^{(t_1 x + t_2 y)} [1-F(x)]^m g^{r-1}(F(x)) f(x)$$

$$\times [h(F(y)) - h(F(x))]^{s-r-1} [1-F(y)]^{\gamma_s - 1} f(y) dy dx$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty e^{t_1 x} [1-F(x)]^m g^{r-1}(F(x)) f(x) I(x) dx,$$

where

$$I(x) = \int_x^\infty e^{t_2 y} [1-F(y)]^{\gamma_s - 1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} f(y) dy.$$

Integrating $I(x)$ by parts with treating $[1-F(y)]^{\gamma_s - 1}$ as the part of integration and the rest of the integration as the part of differentiation, we obtain

$$I(x) = \frac{(s-r-1)}{\gamma_s} \int_x^\infty e^{t_2 y} [1-F(y)]^{\gamma_s + m} [h_m(F(y)) - h_m(F(x)]^{s-r-2} f(y) dy$$

$$+ \frac{t_2}{\gamma_s} \int_x^\infty e^{t_2 y} [1-F(y)]^{\gamma_s} [h_m(F(y)) - h_m(F(x))]^{s-r-1} dy.$$

Substituting $I(x)$ into $M_{r,s:n}(t_1,t_2)$, using $\gamma_{r-i} = \gamma_{r+i}(m+1)$ and $C_{r-1} = \gamma_r C_{r-2}$ we obtain

$$M_{r,s}(t_1,t_2) = M_{r,s-1}(t_1,t_2) + \frac{C_{s-1} t_2}{(r-1)!(s-r-1)!\gamma_s}$$

$$\times \int_{-\infty}^\infty \int_x^\infty e^{t_1 x + t_2 y} g_m^{r-1}(F(x)) [1-F(x)]^m G(y) f(x) dy dx$$

where $G(y) = [h(F(y)) - h(F(x))]^{s-r-1} [1-F(y)]^{\gamma_s}$.

Taking the i-th derivative with respect to t_1 and then j-th derivative with respect to t_2 of the last equation we obtain

$$M_{r,s}^{i,j}(t_1,t_2) = M_{r,s-1}^{i,j}(t_1,t_2) + \frac{C_{s-1}}{\gamma_s (r-1)!(s-r-1)!}$$
$$\times \int_{-\infty}^\infty \int_x^\infty x^i (j y^{j-1} + t_2 y^j) e^{t_1 x + t_2 y} g_m^{r-1}(F(x)) [1-F(x)]^m G_1(y) f(x) dy dx,$$

where $G_1(y) = [h(F(y)) - h(F(x))]^{s-r-1}[1-F(y)]^{\gamma_{s-1}}$, thus using (3) we get

$$M^{i,j}_{r,s:n}(t_1,t_2) = M^{i,j}_{r,s-1:n}(t_1,t_2) + \frac{j\beta}{\alpha\lambda\gamma_s}M^{i,j-\alpha}_{r,s:n}(t_1,t_2)\frac{j}{\alpha\lambda\gamma_s}M^{i,j}_{r,s:n}(t_1,t_2) + \left(\frac{t_2}{\alpha\lambda\gamma_s}\right)L_{12}$$

rearranging the last identity we obtain the recurrence relation (13). □

Corollary3.1.

For $2 \le r \le n$, by repeatedly applying the recurrence relation in (13) we obtain

$$M^{i,j}_{r,s:n}(t_1,t_2) = \prod_{i=m}^{s}\frac{\alpha\lambda\gamma_i}{\alpha\lambda\gamma_i - j}M^{i,j}_{r,m-1:n}(t_1,t_2) + \sum_{P=0}^{s-m}\frac{\prod_{i,m+p+1,s}}{\alpha\lambda\gamma_{m+P}-j}\left[j\beta M^{i,j-\alpha}_{r,m+p:n}(t_1,t_2) + t_2L_{13}\right],$$

$$L_{13} = M^{i,j+1}_{r,m+p:n}(t_1,t_2) + \beta M^{i,j-\alpha+1}_{r,m+p:n}(t_1,t_2)$$

where

$$\prod_{i,m+p+1,s} = \prod_{i=m+p+1}^{s}\frac{\alpha\lambda\gamma_i}{\alpha\lambda\gamma_i - j}, and \prod_{i,s+1,s} = 1.$$

Corollary3.2.

Putting t=0 in (13) we deduce recurrence relations for moments of GOS as:
For $n,k \in N, m \in Z, 1 \le r < s-2 \le n \ \ and\ i,j = 0,1,2,...$

$$EX^i(r,n,m,k)X^j(s,n,m,k) = \frac{\alpha\lambda\gamma_s}{\alpha\lambda\gamma_s - j}X^i(r,n,m,k)X^j(s-1,n,m,k)$$

$$+\frac{j\beta}{\alpha\lambda\gamma_s - j}EX^i(r,n,m,k)X^{j-\alpha}(s,n,m,k).$$

Corollary3.3.

The recurrence relations for product moments of the first n upper record values from the GCR distribution can be derived from (13) as:
For $n,k \in N, m \in Z, 1 \le r < s-2 \le n \ \ and\ i,j = 0,1,2,...$ we have

$$EY^i_{U(r)}Y^j_{U(s)} = \frac{\alpha\lambda}{\alpha\lambda - j}EY^i_{U(r)}Y^j_{U(s-1)} + \frac{j\beta}{\alpha\lambda - j}EY^i_{U(r)}Y^{j-\alpha}_{U(s)}.$$

4 Applications

In this section, we will summarize the applications of theorems 1 and 2 for Compound Rayleigh, Lomax, Burr XII and Beta-Prime distributions which are special cases from the GCR distribution.

4.1 Compound Rayleigh distribution

The recurrence relations for single moments based on generalized order statistics have the forms:
For $n,k \in N, m \in Z,\ 2 \le r \le n\ and\ j = 0,1,2,...$

$$EX^j(r,n,m,k) = \frac{2\lambda\gamma_r}{2\lambda\gamma_r - j}EX^j(r-1,n,m,k) + \frac{j\beta}{2\lambda\gamma_r - j}EX^{j-2}(r,n,m,k),$$

and For $n,k \in N, m \in Z,\ 2 \le r \le n-1\ and\ j = 0,1,2,...$

$$EX^j(r,n,m,k) = \frac{2\lambda\gamma_1}{2\lambda\gamma_1 - j}EX^j(r-1,n-1,m,k) + \frac{j\beta}{2\lambda\gamma_1 - j}EX^{j-2}(r,n,m,k).$$

The recurrence relation for single moments of the first n upper record values for $2 \le n\ and\ j = 0,1,2,...$ has the form:

$$E(Y^j_{U(n)}) = \frac{2\lambda}{2\lambda - j}E(Y^j_{U(n-1)}) + \frac{j\beta}{2\lambda - j}E(Y^{j-2}_{U(n)}).$$

The recurrence relation for product moments based on generalized order statistics has the form: For $n,k \in N, m \in Z, 1 \le r < s-2 \le n\ and\ i,j = 0,1,2,....$Then

$$EX^i(r,n,m,k)X^j(s,n,m,k) = \frac{2\lambda\gamma_s}{2\lambda\gamma_s - j}EX^i(r,n,m,k)X^j(s-1,n,m,k)$$
$$+\frac{j\beta}{2\lambda\gamma_s - j}EX^i(r,n,m,k)X^{j-2}(s,n,m,k).$$

4.2 Lomax distribution

The recurrence relations for single moments based on the generalized order statistics have the forms:
For $n,k \in N, m \in Z,\ 2 \le r \le n$ *and* $j = 0,1,2,...$

$$EX^j(r,n,m,k) = \frac{\lambda\gamma_r}{\lambda\gamma_r - j} EX^j(r-1,n,m,k) + \frac{j\beta}{\lambda\gamma_r - j} EX^{j-1}(r,n,m,k),$$

and for $n,k \in N, m \in Z,\ 2 \le r \le n-1$ *and* $j = 0,1,2,...$

$$EX^j(r,n,m,k) = \frac{\lambda\gamma_1}{\lambda\gamma_1 - j} EX^j(r-1,n-1,m,k) + \frac{j\beta}{\lambda\gamma_1 - j} EX^{j-1}(r,n,m,k).$$

For the ordinary order statistics the recurrence relations have an analogous forms with $\gamma_r = n-r+1$, see,[4]. The recurrence relation for single moments of the first n upper record values for $2 \le n$ and $j = 0,1,2,...$ has the form:

$$E(Y^j_{U(n)}) = \frac{\lambda}{\lambda - j} E(Y^j_{U(n-1)}) + \frac{j\beta}{\lambda - j} E(Y^{j-1}_{U(n)}).$$

The recurrence relation for product moments based on generalized order statistics has the form:

$$EX^i(r,n,m,k)X^j(s,n,m,k) = \frac{\lambda\gamma_s}{\lambda\gamma_s - j} EX^i(r,n,m,k)X^j(s-1,n,n,k) + \frac{j\beta}{\lambda\gamma_s - j} EX^i(r,n,m,k)X^{j-1}(s,n,m,k).$$

For the ordinary order statistics the recurrence relation for the product moments has an analogous relation with $\gamma_s = n-s+1$, see,[4]. The recurrence relation for the product moments of the first n upper record values can be derived from (3.1) as:
For $1 \le r < s-2 \le n$ and $i,j = 0,1,2,...$

$$EY^i_{U(r)}Y^j_{U(s)} = \frac{\lambda}{\lambda - j} EY^i_{U(r)}Y^j_{U(s-1)} + \frac{j\beta}{\lambda - j} EY^i_{U(r)}Y^{j-1}_{U(s)},$$

see, [5].

4.3 Burr distribution

The recurrence relations for single moments based on generalized order statistics have the forms:
For $n,k \in N, m \in Z,\ 2 \le r \le n$ *and* $j = 0,1,2,...$

$$EX^j(r,n,m,k) = \frac{\alpha\lambda\gamma_r}{\alpha\lambda\gamma_1 - j} EX^j(r-1,n,m,k) + \frac{j}{\alpha\lambda\gamma_1 - j} EX^{j-\alpha}(r,n,m,k),$$

and for $n,k \in N, m \in Z,\ 2 \le r \le n-1$ *and* $j = 0,1,2,...$

$$EX^j(r,n,m,k) = \frac{\alpha\lambda\gamma_1}{\alpha\lambda\gamma_1 - j} EX^j(r-1,n-1,m,k) + \frac{j}{\alpha\lambda\gamma_1 - j} EX^{j-\alpha}(r,n,m,k),$$

see, [6]. The recurrence relation for single moments of the first n upper record values has the form:
For $2 \le n$ *and* $j = 0,1,2,...$

$$E(Y^j_{U(n)}) = \frac{\alpha\lambda}{\alpha\lambda - j} E(Y^j_{U(n-1)}) + \frac{j}{\alpha\lambda - j} E(Y^{j-\alpha}_{U(n)}).$$

see, [6]. The recurrence relation for product moments based on generalized order statistics has the form:

$$EX^i(r,n,m,k)X^j(s,n,m,k) = \frac{j}{\alpha\lambda\gamma_s - j} EX^i(r,n,m,k)X^{j-\alpha}(s,n,m,k) + \frac{\alpha\lambda\gamma_r}{\alpha\lambda\gamma_s - j} EX^i(r,n,m,k)X^j(s-1,n,n,k),$$

see,[6]. The recurrence relation for single moments of k-th record values has the form

$$E(Y^{(k)}_n)^j = \frac{\alpha\lambda k}{\alpha\lambda k - j} E(Y^{(k)}_{n-1})^j + \frac{j}{\alpha\lambda k - j} E(Y^{(k)}_n)^{j-\alpha},$$

see, [7]. The recurrence relations for the product moments of the k-th record values can be derived from (13) as:
and for $n,k \in N, m \in Z, 1 \le r < s-2 \le n$ and $i,j = 0,1,2,...$

$$E(Y^{(k)}_r)^i(Y^{(k)}_s)^j = \frac{\alpha\lambda k}{\alpha\lambda k - j} E(Y^{(k)}_r)^i(Y^{(k)}_{s-1})^j + \frac{j}{\alpha\lambda k - j} E(Y^{(k)}_r)^i(Y^{(k)}_s)^{j-\alpha},$$

see,[7].

4.4 Beta- Prime distribution

The recurrence relations for single moments based on generalized order statistics have the forms: For $n,k \in N, m \in Z,\ 2 \le r \le n\ and\ j = 0,1,2,...$

$$EX^j(r,n,m,k) = \frac{\lambda\gamma_r}{\lambda\gamma_r - j} EX^j(r-1,n,m,k) + \frac{j}{\lambda\gamma_r - j} EX^{j-1}(r,n,m,k),$$

and for $n,k \in N, m \in Z,\ 2 \le r \le n-1\ and\ j = 0,1,2,...$

$$EX^j(r,n,m,k) = \frac{\lambda\gamma_1}{\lambda\gamma_1 - j} EX^j(r-1,n-1,m,k) + \frac{j}{\lambda\gamma_1 - j} EX^{j-1}(r,n,m,k),$$

The recurrence relation for single moments of the first n upper record values for $2 \le n$ and $j = 0,1,2,...$ has the form:

$$E(Y^j_{U(n)}) = \frac{\lambda}{\lambda - j} E(Y^j_{U(n-1)}) + \frac{j}{\lambda - j} E(Y^{j-1}_{U(n)}).$$

The recurrence relation for product moments based on the generalized order statistics has the form:

$$EX^i(r,n,m,k)X^j(s,n,m,k) = \frac{j}{\lambda\gamma_s - j} EX^i(r,n,m,k)X^{j-1}(s,n,m,k)$$
$$+ \frac{\lambda\gamma_r}{\lambda\gamma_s - j} EX^i(r,n,m,k)X^j(s-1,n,n,k),$$

5 Conclusion

This paper discussed the recurrence relations for single and product moments for the GCR distribution based on the moment and joint moment-generating functions based on the generalized order statistics. These relations can be used to study the characteristics of special distributions from this distribution such as the means, standard deviations, skewness, and kurtoses with reliable and easy way to apply, especially for researchers in social sciences and psychology.

References

[1] L. J. Bain, & M. Engelhardt, Statistical analysis of a Weibull process with left- censored data, Kluwer Academical Publisher, Dordrecht, (1992).
[2] A. Bekker, J.J.J. Roux, and P.J. Mostert, A Generalization of the Compound Rayleigh Distribution using a Bayesian Method on Cancer Survival Times. Commun. Statist. - Theory Meth., **29**(**7**), 1419-1433 (2000).
[3] U. Kamps, A Concept of generalized order statistics. J. Statistical Planning & Inference, **48**, 1-23 (1995).
[4] N. Balakrishnan, H. J. Malik, and S.E. Ahmed, Recurrence Relations and Identities for Moments of Order Statistics, II: Specific Continuous Distributions. Commun, Statist. - Theory Meth., **17**(**8**), 2657-2694(1988).
[5] N. Balakrishnan and M. Ahsanullah, Relations for Single and Product Moments of Record Values from Lomax Distribution. Sankhya B, **56**, 140-146 (1994).
[6] P. Pawlas and D. Szynal, Recurrence Relations for Single and Product Moments of Generalized Order Statistics from Pareto, Generalized Pareto and Burr Distributions. Commun. Statist. – Theory Meth., **30**(**4**), 739-749 (2001).
[7] P. Pawlas, and D. Szynal, Recurrence Relations for Single and Product Moments of k-th record values from Pareto, Generalized Pareto and Burr Distributions. Commun. Statist. – Theory Meth., **28**(**7**), 1699-1709 (1999).