Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.18576/jsapl/100102

A General Class of Estimators in the Presence of Non-response and Measurement Error

Kuldeep Kumar Tiwari[1], *Sandeep Bhougal*[2,*] *and Sunil Kumar*[3]

[1]School of Basic and Applied Sciences, Career Point University, Kota 325003, Rajasthan, India
[2]School of Mathematics, Shri Mata Vaishno Devi University, Katra 182320, Jammu and Kashmir, India
[3]Department of Statistics, University of Jammu, 180016, Jammu and Kashmir, India

Received: 2 Aug. 2022, Revised: 17 Sep. 2022, Accepted: 4 Oct. 2022
Published online: 1 Jan. 2023

Abstract: The presence of non-response and measurement error in a study cause bias on the estimate. To reduce this, we have studied the effect of non-response (NR) and measurement error (ME) on the estimation of the population mean of the study variable using auxiliary information by proposing a general class of estimators. The proposed class of estimators studied in the various situations of NR and ME. The expressions of bias and MSE of the estimators are derived and their optimum conditions have been obtained. Various well-known estimators from literature are the members of the proposed estimator. A simulation study is performed which support the theoretical findings in all situations.

Keywords: Measurement error, non-response, bias, mean squared error, estimators, mean

1 Introduction

In a sample survey, a high level of response rate is normally viewed as a good survey. But the participation of respondents in surveys has been deteriorating over time in almost all types of surveys (Leeuw and Heer [1], Goyder [2] and for all survey modes (Hox and Leeuw [3]). In the last few decades survey researchers have more concentrated to counteract the downward trend in response rates (e.g. Dillman [4], Goyder [2], Groves and Couper [5]). The quality of survey data can be dying out to sample composition bias, due to non-response and self-selection of respondents, and response bias from several sources. Increasing the response rate minimizes the impact of selection bias. For example, research has shown that callback and increased fieldwork effort not only bring in more respondents but also can bring in those respondents that are underrepresented such as the elderly, lower educated, and lower-income groups (e.g. Dillman [4]). However, this could be purely decorative. As non-response error is a function of the non-response rate and the difference between respondents and non-respondents on a particular variable of interest (Couper and Leeuw [6]). Non-response error will only be reduced by drawing in those specific respondents that tapered this gap. The effect of non-response error is described in Cochran [7]. Kalton and Karsprzyk [8], Meng [9], Rubin [10], Carpenter and Kenward [11], etc. presents several approaches to handle non-response in sample surveys. To avoid non-response and control it in estimation, the problem of non-response was studied. Hansen and Hurwitz [12] developed the technique to estimate the population mean when non-response occurs in surveys. He simply drew a simple random sample and mailed a questionnaire to sampled units then re-contacted some of the non-responding units by drawing a subsample from the non-responding units in the initial first attempt. Cochran [7] uses Hansen and Hurwitz technique to formulate a ratio estimator of the population mean. Similarly, Rao [13], Okafor and Lee [14], Tabasum and Khan [15], [16], Sodipo and Obisesan [17], Singh and Kumar [18], [19], Singh et al. [20], Chaudhary et al. [21], Khare and Sinha [22], Bhushan and Pandey [23], Unal and Kadilar [24] and Sharma and Kumar [25] considered the problem of estimating population mean in the presence of non-response.

But, even if increasing the response rate does reduce non-response errors, by a convincing special respondent to respond, the question remains whether it decreases the total survey error. Increasing the response rate by callback and with more efforts, only bringing non-respondent to the respondent group may increase another source of error i.e. measurement

* Corresponding author e-mail: sandeep.bhougal@smvdu.ac.in

error (Groves and Couper [5]). Non-response is caused by situational (e.g., time, opportunity, at-home patterns) and motivational (e.g., altruism, low cost compared to benefits, high saliency) factors. Measurement error, on the other hand, is largely cognitive and related to the question-answer process (e.g., poor comprehension of questions, memory, and retrieval difficulties). Measurement errors include observational error, instrument error, respondent error, etc. Many sources of measurement errors like bias in the interviewer, bias in the respondent, or an error occur in recording and processing the data. Many researchers worked on the estimation in the presence of measurement error like Fuller [26], Biemer and Stokes [27], Shalabh [28], Singh and Karpe [29], Kumar et al. [30], Gregoire and Salas [31], Diane and Giordan [32], Shukla et al. [33], Shalabh and Tsai [34] and Tiwari et al. [35].

Measurement error and non-response error may creep into the survey at the same time. If these errors are minute then they can be ignored but if these errors are significant, inferences may lead to adverse consequences. Tiwari et al. [36] studied the combined and separate effects of NR and ME to show their relative effect. Very few studies have been done so far like Jackman [37], Biemer [38], Hox et al. [39], Kumar et al. [40], Singh and Sharma [41], Azeem and Hanif [42], Kumar [43], Kumar and Bhougal [44], Kumar et al. [45], and Singh et al. [46].

Usually the study on the estimation is specific to a particular sampling strategy or method but in real-life any type of situation can be there. So the motive of this paper is to propose and study estimators in a different situations to see their effect. In this paper, we study how NR and ME affect the efficiency of the estimators using auxiliary information.

2 Sampling Procedure and Notations

Let a population of size N and a sample of size n be taken by using the simple random sampling without replacement (SRSWOR) method. Let Y be the study and X be the auxiliary variable. Let $\mu_Y = \frac{1}{N}\sum_{i=1}^{N} y_i$, $\mu_X = \frac{1}{N}\sum_{i=1}^{N} x_i$, $\sigma_Y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i-\mu_Y)^2$ and $\sigma_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\mu_X)^2$ denote the population mean and variance of study variable Y and auxiliary variable X, respectively. Let (x_i, y_i) be the observed and (X_i, Y_i) be the true values on the characteristics (X, Y) associated with the i^{th} unit in the sample.

Let the measurement error present on Y and X are $U_i = y_i - Y_i$ and $V_i = x_i - X_i$.

The usual unbiased estimator for the population mean of the study variable in the presence of measurement error is given as

$$t_0 = \hat{\mu}_Y = \frac{1}{n}\sum_{i=1}^{n} y_i$$

The variance in the presence of measurement error of the usual estimator is given as

$$Var(t_0) = \lambda_2(\sigma_Y^2 + \sigma_U^2) \tag{1}$$

where $\lambda_2 = \frac{1}{n} - \frac{1}{N}$

Let the measurement errors on Y and X be random and uncorrelated with mean zero and variances σ_U^2 and σ_V^2 respectively, with an assumption that the measurement errors for variable Y and X are independent. Let C_y and C_x be the coefficient of variations of variable Y and X respectively for the population and ρ_{yx} be the coefficient of correlation between Y and X.

Now, let the non-response present on the study and auxiliary variables, it is assumed that the population of size N is composed of two mutually exclusive groups, the N_1 respondents and the N_2 non-respondents, though their sizes are unknown. Let $\mu_{Y_1} = \frac{1}{N_1}\sum_{i=1}^{N_1} y_i$ and $\sigma_{Y(1)}^2 = \frac{1}{N_1-1}\sum_{i=1}^{N_1}(y_i-\mu_{Y_1})^2$ denote the mean and variance of the response group. Similarly, let $\mu_{Y_2} = \frac{1}{N_2}\sum_{i=1}^{N_2} y_i$ and $\sigma_{Y(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(y_i-\mu_{Y_2})^2$ denote the mean and variance of the non-response group. The population mean can be written as $\mu_Y = W_1\mu_{Y_1} + W_2\mu_{Y_2}$, where $W_1 = \frac{N_1}{N}$ and $W_2 = \frac{N_2}{N}$. Let $\hat{\mu}_{Y_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} y_i$ and $\hat{\mu}_{Y_{2r}} = \frac{1}{r}\sum_{i=1}^{r} y_i$ denote the means of the n_1 responding units and the r sub-sampled units. Thus, an unbiased estimator of the population mean μ_Y due to Hansen and Hurwitz [12] is given by

$$\hat{\mu}_Y^* = w_1\hat{\mu}_{Y_1} + w_2\hat{\mu}_{Y_{2r}}$$

where $w_1 = \frac{n_1}{n}$ and $w_2 = \frac{n_2}{n}$ are responding and non-responding proportions in the sample.

The variance of $\hat{\mu}_Y^*$ up to the terms of order n^{-1}, is given by

$$Var(\hat{\mu}_Y^*) = \lambda_2\sigma_Y^2 + \theta\sigma_{Y(2)}^2 \tag{2}$$

where $\theta = \frac{W_2(k-1)}{n}$, $C_y = \frac{\sigma_y}{\mu_Y}$ and $C_{y(2)} = \frac{\sigma_{y(2)}}{\mu_Y}$, (see Cochran [7], p. 371).

Similarly one can define for auxiliary variable i.e. $\hat{\mu}_{X_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$ and $\hat{\mu}_{X_{2r}} = \frac{1}{r}\sum_{i=1}^{r} x_i$ denotes the means of responding and r sub-sampled units. Under such a situation, an unbiased estimator for the population mean $\hat{\mu}_X$ of the auxiliary variable as

$$\hat{\mu}_X^* = w_1\hat{\mu}_{X_1} + w_2\hat{\mu}_{X_{2r}}$$

The variance of $\hat{\mu}_X^*$ is

$$Var(\hat{\mu}_X^*) = \lambda_2\sigma_X^2 + \theta\sigma_{X(2)}^2 \tag{3}$$

Many times, the above-mentioned situations occur together i.e. non-response and measurement error present simultaneously. So, let (x_i^*, y_i^*) be the observed values and (X_i^*, Y_i^*) be the true values of (X,Y) respectively associated with the i^{th} sample unit. Let the measurement error associated with the study variable in the presence of non-response be

$$U_i^* = y_i^* - Y_i^*$$

When there is some non-response on the auxiliary variable, let the measurement error associated with the auxiliary variable be

$$V_i^* = x_i^* - X_i^*$$

The measurement errors on Y and X are random with mean zero and variances σ_U^2 and σ_V^2 respectively for the responding units and $\sigma_{U(2)}^2$ and $\sigma_{V(2)}^2$ respectively for the group of non-respondents. Let $\sigma_{X(2)}^2$ and $\sigma_{Y(2)}^2$ be the variances of variables X and Y respectively for the non-respondents and $\rho_{yx(2)}$ be the correlation coefficient between the variables Y and X for the non-respondents of the population. Let $C_{x(2)}$ and $C_{y(2)}$ be the coefficient of variations for variable X and Y respectively for the group of non-respondents.

In situations where the population mean of the auxiliary variable X is not known, a two-phase sampling scheme is adopted. A large sample of size n' is taken from the population at the first phase by the SRSWOR method and the information on the auxiliary variable is obtained. In the second phase, a sub-sample of size n is taken from the first-phase sample using the SRSWOR method and data on the variable of interest are collected. In the first phase, we assume that there is a complete response without measurement error. Let x_{1i} be the observed values and X_{1i} be the true values on an auxiliary characteristic associated with the i^{th} unit in the first-phase sample. Since we have assumed that there are no measurement errors in the first-phase sample, therefore $x_{1i} = X_{1i}$. Let (x_{1i}, y_i) be the observed values and (X_i, Y_i) be the true values on two characteristics (X,Y) respectively associated with the i^{th} unit on the second-phase sample.

We use the following terms to derive the Bias and mean square error (MSE) of the estimators.

Let $\omega_Y^* = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i^* - \mu_Y)$ and $\omega_U^* = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} U_i^*$. Add ω_Y^* and ω_U^* and divide both side by $\sqrt{n}$, we have $\frac{\omega_Y^*+\omega_U^*}{\sqrt{n}} = \frac{1}{n}\sum_{i=1}^{n}[(Y_i - \mu_Y) + U_i^*]$ that is $\frac{\omega_Y^*+\omega_U^*}{\sqrt{n}} = \frac{1}{n}\sum_{i=1}^{n} y_i^* - \mu_Y$

So,

$$\hat{\mu}_Y^* = \mu_Y + \varepsilon_Y^*;\ \text{where } \varepsilon_Y^* = \frac{\omega_Y^* + \omega_U^*}{\sqrt{n}} \tag{4}$$

Similarly, for $\omega_X = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu_X)$ and $\omega_V = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} V_i$

$$\hat{\mu}_X = \mu_X + \varepsilon_X;\ \text{where } \varepsilon_X = \frac{\omega_X + \omega_V}{\sqrt{n}} \tag{5}$$

Again, for $\omega_X^* = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i^* - \mu_X)$ and $\omega_V^* = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} V_i^*$

$$\hat{\mu}_X^* = \mu_X + \varepsilon_X^*;\ \text{where } \varepsilon_X^* = \frac{\omega_X^* + \omega_V^*}{\sqrt{n}} \tag{6}$$

We assumed that non-response doesn't occur on first phase, so for $\omega_{X'} = \frac{1}{\sqrt{n'}}\sum_{i=1}^{n'}(x_i' - \mu_X)$, we have

$$\hat{\mu}_X' = \mu_X + \varepsilon_{X'};\ \text{where } \varepsilon_{X'} = \frac{\omega_{X'}}{\sqrt{n'}} \tag{7}$$

Then, we have

$$E(\varepsilon_Y^*) = E(\varepsilon_X) = E(\varepsilon_X^*) = E(\varepsilon_{X'}) = E(\varepsilon_U) = E(\varepsilon_V) = 0 \tag{8}$$

and

$$E(\varepsilon_Y^{*2}) = \lambda_2\sigma_Y^2 + \theta\sigma_{Y(2)}^2 + \lambda_2\sigma_U^2 + \theta\sigma_{U(2)}^2 = V_1(Say) \tag{9}$$

$$E(\varepsilon_X^2) = \lambda_2\sigma_X^2 + \lambda_2\sigma_V^2 = V_2(Say) \tag{10}$$

$$E(\varepsilon_Y^*\varepsilon_X) = \lambda_2\rho_{yx}\sigma_Y\sigma_X = V_3(Say) \tag{11}$$

$$E(\varepsilon_X^{*2}) = \lambda_2\sigma_X^2 + \theta\sigma_{X(2)}^2 + \lambda_2\sigma_V^2 + \theta\sigma_{V(2)}^2 = V_4(Say) \tag{12}$$

$$E(\varepsilon_Y^*\varepsilon_X^*) = \lambda_2\rho_{yx}\sigma_Y\sigma_X + \theta\rho_{yx(2)}\sigma_{Y(2)}\sigma_{X(2)} = V_5(Say) \tag{13}$$

$$E(\varepsilon_{X'}^2) = \lambda'\sigma_X^2 = V_6(Say) \tag{14}$$

$$E(\varepsilon_X\varepsilon_{X'}) = \lambda'\sigma_X^2 = V_6 \tag{15}$$

$$E(\varepsilon_Y^*\varepsilon_{X'}) = \lambda'\rho_{yx}\sigma_Y\sigma_X = V_3'(Say) \tag{16}$$

where $\lambda' = \frac{1}{n'} - \frac{1}{N}$.

3 Literature survey

In this section, we consider the following estimators

Searl [47]

In simple random sampling, Searl proposed an estimator for estimating the population mean of Y as $t_1 = k\hat{\mu}_Y$, where k is suitable constant.

The mean square error (MSE) of t_1 is

$$MSE(t_1) = (k-1)^2\mu_Y^2 + k^2\lambda_2\mu_Y^2C_y^2$$

The minimum MSE of t_1 for optimum value of $k = \frac{1}{1+\lambda_2C_y^2} = k^o$ is

$$MSE_{min}(t_1) = \frac{\lambda_2\mu_Y^2C_y^2}{1+\lambda_2C_y^2}$$

Cochran [7]

Cochran (1977) proposes the ratio estimator as $t_2 = \hat{\mu}_Y(\frac{\mu_X}{\hat{\mu}_X})$ with mean squared error i.e. MSE of t_2 as

$$MSE(t_2) = \lambda_2\mu_Y^2(C_y^2 + C_x^2 - 2\rho_{yx}C_yC_x)$$

Murthy [48]

Murthy (1964) suggested an product estimator for μ_Y as $t_3 = \hat{\mu}_Y(\frac{\hat{\mu}_X}{\mu_X})$ with MSE of t_3 as

$$MSE(t_3) = \lambda_2\mu_Y^2(C_y^2 + C_x^2 + 2\rho_{yx}C_yC_x)$$

Cochran [7]

The usual regression estimator proposed by Cochran [7] as $t_4 = \hat{\mu}_Y + b(\mu_X - \hat{\mu}_X)$ with

$$MSE(t_4) = \lambda_2 \mu_Y^2 C_y^2 (1 - \rho_{yx}^2)$$

where b is regression coefficient.

Rao [49]

A difference estimator is proposed by Rao [49] as $t_5 = k_1 \hat{\mu}_Y + k_2(\mu_X - \hat{\mu}_X)$, where k_1, k_2 are constant.

The MSE of t_5 is

$$MSE(t_5) = (k_1 - 1)^2 \mu_Y^2 + \lambda_2 \mu_Y^2 C_y^2 k_1^2 + \lambda_2 \mu_X^2 C_x^2 k_2^2 - 2\lambda_2 \mu_Y \mu_X \rho_{yx} C_y C_x k_1 k_2$$

The $MSE(t_5)$ is optimum, when $k_1 = \frac{1}{1+\lambda_2 C_y^2(1-\rho_{yx}^2)} = k_1^o$; $k_2 = \frac{\rho_{yx} R C_y}{C_x[1+\lambda_2 C_y^2(1-\rho_{yx}^2)]} = k_2^o$. The minimum $MSE(t_5)$ is given as

$$MSE_{min}(t_5) = \frac{\lambda_2 \mu_Y^2 C_y^2 (1 - \rho_{yx}^2)}{1 + \lambda_2 C_y^2 (1 - \rho_{yx}^2)}$$

Bahl and Tuteja [50]

The ratio and product type exponential estimators defined by Bahl and Tuteja [50] as $t_6 = \hat{\mu}_Y \exp(\frac{\mu_X - \hat{\mu}_X}{\mu_X + \hat{\mu}_X})$ and $t_7 = \hat{\mu}_Y \exp(\frac{\hat{\mu}_X - \mu_X}{\hat{\mu}_X + \mu_X})$.

The MSE of t_6 and t_7 are given as

$$MSE(t_6) = \lambda_2 \mu_Y^2 (C_y^2 + \frac{1}{4} C_x^2 - \rho_{yx} C_y C_x)$$

$$MSE(t_7) = \lambda_2 \mu_Y^2 (C_y^2 + \frac{1}{4} C_x^2 + \rho_{yx} C_y C_x)$$

Kadilar and Cingi [51]

Kadilar and Cingi [51] proposed a combined ratio cum regression estimator as $t_8 = [\hat{\mu}_Y + b(\mu_X - \hat{\mu}_X)](\frac{\mu_X}{\hat{\mu}_X})$. The MSE of t_8 to the first degree of approximation is

$$MSE(t_8) = \lambda_2 \mu_Y^2 [C_x^2 + C_y^2 (1 - \rho_{yx}^2)]$$

Grover and Kour [52]

Grover and Kour [52] proposes a difference-cum-exponential type estimator as $t_9 = [d_1 \hat{\mu}_Y + d_2(\mu_X - \hat{\mu}_X)] \exp(\frac{\mu_X - \hat{\mu}_X}{\mu_X + \hat{\mu}_X})$, where d_1, d_2 are constant. To the first degree of approximation, one can obtain the optimum MSE of t_9 as

$$MSE_{min}(t_9) = \frac{\lambda_2 \mu_Y^2 [16(1 - \rho^2)(4 - \lambda_2 C_x^2) C_y^2 - \lambda_2 C_x^4]}{64[1 + \lambda_2 C_y^2 (1 - \rho_{yx}^2)]}$$

Some basic estimators of population mean in two-phase sampling from literature are given as

Ratio Estimator

The classical ratio estimator and their MSE in two-phase sampling is given as

$$t_r = \hat{\mu}_Y(\frac{\hat{\mu}'_X}{\hat{\mu}_X})$$

$$MSE(t_r) = \mu_Y^2\left[\left(\frac{1}{n}-\frac{1}{N}\right)C_y^2 + \left(\frac{1}{n}-\frac{1}{n'}\right)(C_x^2 - 2\rho_{yx}C_yC_x)\right]$$

Product Estimator

The classical product estimator in two-phase sampling is given as

$$t_p = \hat{\mu}_Y(\frac{\hat{\mu}_X}{\hat{\mu}'_X})$$

and their MSE is

$$MSE(t_p) = \mu_Y^2\left[\left(\frac{1}{n}-\frac{1}{N}\right)C_y^2 + \left(\frac{1}{n}-\frac{1}{n'}\right)(C_x^2 + 2\rho_{yx}C_yC_x)\right]$$

Regression Estimator

The regression estimator in two-phase sampling is given as

$$t_{reg} = \hat{\mu}_Y + b_{yx}(\hat{\mu}'_X - \hat{\mu}_X)$$

where b_{yx} is regression coefficient. The optimum MSE of t_{reg} is

$$MSE(t_{reg}) = \mu_Y^2C_y^2\left[\left(\frac{1}{n}-\frac{1}{N}\right) - \left(\frac{1}{n}-\frac{1}{n'}\right)\rho_{yx}\right]$$

In this study, we revisited the above $t_1, t_2,..,t_9$ estimators in different situations of non-response and measurement error by defining a general class of estimators for estimating the population mean μ_Y of Y.

4 Proposed class of estimators

To estimate the population mean in the different situations of non-response and measurement error, we define a general class of estimators as

$$T = [k_1\hat{\mu}_Y + k_2(\mu_X - \hat{\mu}_X)]\left(\frac{\mu_X}{\hat{\mu}_X}\right)^{\delta}\left[\exp\left(\frac{\mu_X - \hat{\mu}_X}{\mu_X + \hat{\mu}_X}\right)\right]^{\alpha} \quad (17)$$

where k_1, k_2, δ and α are constant.

For different values of k_1, k_2, δ and α, one can obtain various estimators. Table 1 shows the estimators considered in Section 3 as the member of the general class of estimators.

We study these estimators in various situations of non-response and measurement error in the following sections.

5 Situation-1

When non-response and measurement error are present on the study variable Y with known population mean μ_X of auxiliary variable X.

Table 1: Members of the proposed class of estimator

k_1	k_2	δ	α	Estimator
1	0	0	0	$T^{(1)} = \hat{\mu}_Y$, Usual estimator
k_1	0	0	0	$T^{(2)} = k\hat{\mu}_Y$, Searl [47]
1	0	1	0	$T^{(3)} = \hat{\mu}_Y\left(\frac{\mu_X}{\hat{\mu}_X}\right)$, Cochran [53]
1	0	-1	0	$T^{(4)} = \hat{\mu}_Y\left(\frac{\hat{\mu}_X}{\mu_X}\right)$, Murthy [48]
1	k_2	0	0	$T^{(5)} = \hat{\mu}_Y + k_2(\mu_X - \hat{\mu}_X)$, Cochran [7]
k_1	k_2	0	0	$T^{(6)} = k_1\hat{\mu}_Y + k_2(\mu_X - \hat{\mu}_X)$, Rao [49]
1	0	0	1	$T^{(7)} = \hat{\mu}_Y \exp\left(\frac{\mu_X - \hat{\mu}_X}{\mu_X + \hat{\mu}_X}\right)$, Bahl and Tuteja [50]
1	0	0	-1	$T^{(8)} = \hat{\mu}_Y \exp\left(\frac{\hat{\mu}_X - \mu_X}{\hat{\mu}_X + \mu_X}\right)$, Bahl and Tuteja [50]
1	k_2	1	0	$T^{(9)} = [\hat{\mu}_Y + k_2(\mu_X - \hat{\mu}_X)]\left(\frac{\mu_X}{\hat{\mu}_X}\right)$, Kadilar and Cingi [51]
k_1	k_2	0	1	$T^{(10)} = [k_1\hat{\mu}_Y + k_2(\mu_X - \hat{\mu}_X)]\exp\left(\frac{\mu_X - \hat{\mu}_X}{\mu_X + \hat{\mu}_X}\right)$, Grover and Kaur [52]

5.1 Estimator

Redefine the general class of estimators defined in equation (17) as

$$T_1 = [k_{11}\hat{\mu}_Y^* + k_{21}(\mu_X - \hat{\mu}_X)]\left(\frac{\mu_X}{\hat{\mu}_X}\right)^{\delta_1}\left[\exp\left(\frac{\mu_X - \hat{\mu}_X}{\mu_X + \hat{\mu}_X}\right)\right]^{\alpha_1} \tag{18}$$

where k_{11}, k_{21}, δ_1 and α_1 are constant.

The member estimators can be written as

$1T_1^{(1)} = \hat{\mu}_Y^*$
$2T_1^{(2)} = k_{11}\hat{\mu}_Y^*$
$3T_1^{(3)} = \hat{\mu}_Y^*\left(\frac{\mu_X}{\hat{\mu}_X}\right)$
$4T_1^{(4)} = \hat{\mu}_Y^*\left(\frac{\hat{\mu}_X}{\mu_X}\right)$
$5T_1^{(5)} = \hat{\mu}_Y^* + k_{21}(\mu_X - \hat{\mu}_X)$,
$6T_1^{(6)} = k_{11}\hat{\mu}_Y^* + k_{21}(\mu_X - \hat{\mu}_X)$
$7T_1^{(7)} = \hat{\mu}_Y^* \exp\left(\frac{\mu_X - \hat{\mu}_X}{\mu_X + \hat{\mu}_X}\right)$
$8T_1^{(8)} = \hat{\mu}_Y^* \exp\left(\frac{\hat{\mu}_X - \mu_X}{\hat{\mu}_X + \mu_X}\right)$
$9T_1^{(9)} = [\hat{\mu}_Y^* + k_{21}(\mu_X - \hat{\mu}_X)]\left(\frac{\mu_X}{\hat{\mu}_X}\right)$
$10T_1^{(10)} = [k_{11}\hat{\mu}_Y^* + k_{21}(\mu_X - \hat{\mu}_X)]\exp\left(\frac{\mu_X - \hat{\mu}_X}{\mu_X + \hat{\mu}_X}\right)$

where k_{11}, k_{21} are suitable constant for respective estimator.

5.2 Bias and MSE

To get the bias and MSE of T_1 to the first order of approxmation, express T_1 given in equation (18) in terms of ε's using equations (4) and (5). We have

$$T_1 = (k_{11}\mu_Y + k_{11}\varepsilon_Y^* - k_{21}\varepsilon_X)\left(1 + \frac{\varepsilon_X}{\mu_X}\right)^{-\delta_1}\left[1 + \left(\frac{3}{8}\frac{\varepsilon_X^2}{\mu_X} - \frac{\varepsilon_X}{2\mu_X}\right)\right]^{\alpha_1} \tag{19}$$

Expand and simplify above equation by ignoring the higher terms, we get

$$T_1 = k_{11}\mu_Y + k_{11}\varepsilon_Y^* - [k_{21} + k_{11}R\delta_{\alpha_1}]\varepsilon_X + \left[k_{21}\delta_{\alpha_1} + \frac{Rk_{11}}{2}\delta_{\alpha_1} + \frac{Rk_{11}}{2}\delta_{\alpha_1}^2\right]\frac{\varepsilon_X^2}{\mu_X} - \frac{k_{11}}{\mu_X}\delta_{\alpha_1}\varepsilon_Y^*\varepsilon_X \tag{20}$$

where $\delta_{\alpha_1} = \delta_1 + \frac{\alpha_1}{2}$. Substract μ_Y from both sides, we have

$$T_1 - \mu_Y = (k_{11}-1)\mu_Y + k_{11}\varepsilon_Y^* - [k_{21} + k_{11}R\delta_{\alpha_1}]\,\varepsilon_X + \left[k_{21}\delta_{\alpha_1} + \frac{Rk_{11}}{2}\delta_{\alpha_1} + \frac{Rk_{11}}{2}\delta_{\alpha_1}^2\right]\frac{\varepsilon_X^2}{\mu_X} - \left[k_{21}\delta_{\alpha_1} + \frac{Rk_{11}}{2}\delta_{\alpha_1} + \frac{Rk_{11}}{2}\delta_{\alpha_1}^2\right]\varepsilon_X^2\frac{k_{11}}{\mu_X}\delta_{\alpha_1}\varepsilon_Y^*\varepsilon_X \quad (21)$$

Taking expectation on both sides of equation (21) and using expected values from equation (8), (9), (10) & (11), we get the bias of T_1 as

$$Bias(T_1) = (k_{11}-1)\mu_Y + \left[\delta_{\alpha_1}\left(k_{21} + \frac{Rk_{11}}{2}\right) + \frac{Rk_{11}}{2}\delta_{\alpha_1}^2\right]\frac{V_2}{\mu_X} - \frac{k_{11}}{\mu_X}\delta_{\alpha_1}V_3 \quad (22)$$

Squaring equation (21) on both sides and terminate the higher order terms, we have

$$(T_1 - \mu_Y)^2 = (k_{11}-1)^2\mu_Y^2 + k_{11}^2\varepsilon_Y^{*2} + (k_{21} + k_{11}R\delta_{\alpha_1})^2\varepsilon_X^2 + 2(k_{11}-1)k_{11}\mu_Y\varepsilon_Y^* - 2\mu_Y(k_{11}-1)(k_{21} + k_{11}R\delta_{\alpha_1})\varepsilon_X + 2R(k_{11}-1)\left[k_{21}\delta_{\alpha_1} + \frac{Rk_{11}}{2}\delta_{\alpha_1} + \frac{Rk_{11}}{2}\delta_{\alpha_1}^2\right]\varepsilon_X^2 - 2Rk_{11}\delta_{\alpha_1}(k_{11}-1)\varepsilon_Y^*\varepsilon_X - 2k_{11}(k_{21} + k_{11}R\delta_{\alpha_1})\varepsilon_Y^*\varepsilon_X \quad (23)$$

Taking expectation on both sides of equation (23) and using expected values from equation (8), (9), (10) & (11), we get the MSE of T_1 as

$$MSE(T_1) = \mu_Y^2 + k_{11}\left[2RV_3\delta_{\alpha_1} - 2\mu_Y^2 - R^2V_2\delta_{\alpha_1} - R^2V_2\delta_{\alpha_1}^2\right] + k_{11}^2[\mu_Y^2 + V_1 - 4RV_3\delta_{\alpha_1} + R^2V_2\delta_{\alpha_1} + 2R^2V_2\delta_{\alpha_1}^2] - 2k_{21}RV_2\delta_{\alpha_1} + k_{21}^2V_2 + k_{11}k_{21}[4RV_2\delta_{\alpha_1} - 2V_3] \quad (24)$$

or

$$MSE(T_1) = \mu_Y^2 + k_{11}\varphi_{11} + k_{11}^2\varphi_{21} + k_{21}\varphi_{31} + k_{21}^2\varphi_{41} + k_{11}k_{21}\varphi_{51} \quad (25)$$

where $\varphi_{11} = 2RV_3\delta_{\alpha_1} - 2\mu_Y^2 - R^2V_2\delta_{\alpha_1} - R^2V_2\delta_{\alpha_1}^2$, $\varphi_{21} = \mu_Y^2 + V_1 - 4RV_3\delta_{\alpha_1} + R^2V_2\delta_{\alpha_1} + 2R^2V_2\delta_{\alpha_1}^2$, $\varphi_{31} = -2RV_2\delta_{\alpha_1}$, $\varphi_{41} = V_2$, $\varphi_{51} = 4RV_2\delta_{\alpha_1} - 2V_3$.

For the optimum values of k_{11} and k_{21} which is $k_{11}^o = \frac{\varphi_{31}\varphi_{51} - 2\varphi_{11}\varphi_{41}}{4\varphi_{21}\varphi_{41} - \varphi_{51}^2}$; $k_{21}^o = \frac{\varphi_{11}\varphi_{51} - 2\varphi_{21}\varphi_{31}}{4\varphi_{21}\varphi_{41} - \varphi_{51}^2}$, the minimum MSE of T_1 can be obtained as

$$MSE_{min}(T_1) = \mu_Y^2 - \frac{\varphi_{11}^2\varphi_{41} + \varphi_{21}\varphi_{31}^2 - \varphi_{11}\varphi_{31}\varphi_{51}}{4\varphi_{21}\varphi_{41} - \varphi_{51}^2} \quad (26)$$

or

$$MSE_{min}(T_1) = \mu_Y^2 - \Upsilon_1 \quad (27)$$

where $\Upsilon_1 = \frac{\varphi_{11}^2\varphi_{41} + \varphi_{21}\varphi_{31}^2 - \varphi_{11}\varphi_{31}\varphi_{51}}{4\varphi_{21}\varphi_{41} - \varphi_{51}^2}$.

The bias and MSE of the estimators $T_1^{(i)}; i = 1,2,..,10$ upto the first order of approxmation given in Table 2.

5.3 Efficiency Comparison

An estimator t_1 of population mean μ_Y is said to be more efficient than estimaor t_2 if $MSE(t_1) < MSE(t_2)$. Here we have developed the conditions under which the general class of estimator T_1 is better than the estimators $T_1^{(i)}$, $i = 1,2,..,10$.

- $MSE(T_1) < MSE(T_1^{(1)})$ if $\mu_Y^2 < \Upsilon_1 + V_1$
- $MSE(T_1) < MSE(T_1^{(2)})$ if $\frac{\mu_Y^4}{\mu_Y^2 + V_1} < \Upsilon_1$
- $MSE(T_1) < MSE(T_1^{(3)})$ if $\mu_Y^2 + 2RV_3 < \Upsilon_1 + V_1 + R^2V_2$
- $MSE(T_1) < MSE(T_1^{(4)})$ if $\mu_Y^2 < \Upsilon_1 + V_1 + R^2V_2 + 2RV_3$
- $MSE(T_1) < MSE(T_1^{(5)})$ if $\mu_Y^2 + \frac{V_3^2}{V_2} < \Upsilon_1 + V_1$
- $MSE(T_1) < MSE(T_1^{(6)})$ if $\frac{V_2\mu_Y^4}{V_2\mu_Y^2 + V_1V_2 - V_3^2} < \Upsilon_1$

Table 2: Expressions for bias and MSE of $T_1^{(i)}; i = 1, 2, .., 10$

Estimator	Bias	MSE/MSE_{min} and respective optimum value of constants
$T_1^{(1)}$	0	V_1
$T_1^{(2)}$	$(k_{11}-1)\mu_Y$	$\mu_Y^2 - \frac{\mu_Y^4}{\mu_Y^2+V_1}; k_{11}^o = \frac{\mu_Y^2}{\mu_Y^2+V_1}$
$T_1^{(3)}$	$\frac{RV_2-V_3}{\mu_X}$	$V_1 + R^2V_2 - 2RV_3$
$T_1^{(4)}$	$\frac{V_3}{\mu_X}$	$V_1 + R^2V_2 + 2RV_3$
$T_1^{(5)}$	0	$V_1 - \frac{V_3^2}{V_2}; k_{21}^o = \frac{V_3}{V_2}$
$T_1^{(6)}$	$(k_{11}-1)\mu_Y$	$\mu_Y^2 - \frac{V_2\mu_Y^4}{V_2\mu_Y^2+V_1V_2-V_3^2}; k_{11}^o = \frac{V_2\mu_Y^2}{V_2\mu_Y^2+V_1V_2-V_3^2}, k_{21}^o = \frac{V_3\mu_Y^2}{V_2\mu_Y^2+V_1V_2-V_3^2}$
$T_1^{(7)}$	$\frac{3RV_2-4V_3}{8\mu_X}$	$V_1 + \frac{1}{4}R^2V_2 - RV_3$
$T_1^{(8)}$	$\frac{4V_3-RV_2}{8\mu_X}$	$V_1 + \frac{1}{4}R^2V_2 + RV_3$
$T_1^{(9)}$	$\frac{(R+k_{21})V_2-V_3}{\mu_X}$	$V_1 - \frac{V_3^2}{V_2}; k_{21} = \frac{V_3}{V_2} - R$
$T_1^{(10)}$	$(k_{11}-1)\mu_Y + \frac{(4k_{21}+3Rk_{11})V_2-4k_{11}V_3}{8\mu_X}$	$\mu_Y^2 - \frac{P_{11}^2P_{41}+P_{21}P_{31}^2-P_{11}P_{31}P_{51}}{4P_{21}P_{41}-P_{51}^2}; k_{11}^o = \frac{P_{31}P_{51}-2P_{11}P_{41}}{4P_{21}P_{41}-P_{51}^2}; k_{11}^o = \frac{P_{11}P_{51}-2P_{21}P_{31}}{4P_{21}P_{41}-P_{51}^2}$

where $P_{11} = RV_3 - 2\mu_Y^2 - \frac{3}{4}R^2V_2$, $P_{21} = \mu_Y^2 + V_1 - 2RV_3 + R^2V_2$, $P_{31} = -RV_2$, $P_{41} = V_2$, $P_{51} = 2RV_2 - 2V_3$.

- $MSE(T_1) < MSE(T_1^{(7)})$ if $\mu_Y^2 + RV_3 < \Upsilon_1 + V_1 + \frac{1}{4}R^2V_2$
- $MSE(T_1) < MSE(T_1^{(8)})$ if $\mu_Y^2 < \Upsilon_1 + V_1 + \frac{1}{4}R^2V_2 + RV_3$
- $MSE(T_1) < MSE(T_1^{(9)})$ if $\mu_Y^2 + \frac{V_3^2}{V_2} < \Upsilon_1 + V_1$
- $MSE(T_1) < MSE(T_1^{(10)})$ if $\frac{P_{11}^2P_{41}+P_{21}P_{31}^2-P_{11}P_{31}P_{51}}{4P_{21}P_{41}-P_{51}^2} < \Upsilon_1$

5.4 Simulation

We have executed a simulation study to see the performance of the estimators. We have used R software for simulation. Population size $N = 5000$ and sample size $n = 500$ is taken. The other essential information in the process are $X = rnorm(N, 10, 5)$, $Y = 1 + 3 * X + rnorm(N, 0, 1)$, $y = Y + rnorm(N, 0, 5)$, $x = X + rnorm(N, 0, 5)$, $U = y - Y$, $V = x - X$. For different response rates, the result of the simulation is given in Table 3. For a better approximation, we have averaged the result over 25000 iterations.

The percent relative efficiency (PRE) of estimators with respect to $T_1^{(1)}$ are calculated using

$$PRE(., T_1^{(1)}) = \frac{MSE(T_1^{(1)})}{MSE(.)} \times 100 \tag{28}$$

The MSE of the proposed estimator T_1 depends on δ_{α_1}. When other terms are fixed, we can find the value of δ_{α_1} for which the proposed estimator performs better than other estimators. To get that, we can try different values of δ_{α_1} or plot $MSE(T_1)$ against δ_{α_1} and see where it gets minimum. If $\delta_{\alpha_1} = c$(constant), then we write the particular estimator as $T_1^{\delta_{\alpha_1}=c}$. The terms $\delta_{\alpha_1} = c$ or $\delta_1 + \frac{\alpha_1}{2} = c$ represents that δ_1 and α_1 in the estimator T_1 are taken from anywhere on the line $\delta_1 + \frac{\alpha_1}{2} = c$.

Table 3: PREs of estimators for different values of W_2 and k in Situation 1

W_2	Estimator	PRE of estimators with respect to $T_1^{(1)}$			
		$1/k$			
		1/2	1/3	1/4	1/5
0.1	$T_1^{(1)}$	100	100	100	100
	$T_1^{(2)}$	100.0522	100.0575	100.0627	100.0679
	$T_1^{(3)}$	94.72534	95.18166	95.56531	95.89238
	$T_1^{(4)}$	22.77699	24.49637	26.14086	27.71523
	$T_1^{(5)}$	167.6075	157.9043	150.6369	144.9902
	$T_1^{(6)}$	167.6597	157.9618	150.6995	145.0581
	$T_1^{(7)}$	167.4812	157.8024	150.5518	144.9175
	$T_1^{(8)}$	44.16107	46.52221	48.69176	50.69215
	$T_1^{(9)}$	167.6075	157.9043	150.6369	144.9902
	$T_1^{(10)}$	167.7043	158.0028	150.7380	145.0947
	$T_1^{(\delta_{\alpha_1}=6.6)}$	2127.2342	730.7340	472.3493	363.5797
0.3	$T_1^{(1)}$	100	100	100	100
	$T_1^{(2)}$	100.0627	100.0784	100.0940	100.1097
	$T_1^{(3)}$	95.56566	96.42083	96.99949	97.41708
	$T_1^{(4)}$	26.14244	30.67365	34.68101	38.25040
	$T_1^{(5)}$	150.6306	136.7794	128.8789	123.7724
	$T_1^{(6)}$	150.6933	136.8578	128.9729	123.8821
	$T_1^{(7)}$	150.5456	136.7233	128.8374	123.7396
	$T_1^{(8)}$	48.69381	54.26207	58.74002	62.41934
	$T_1^{(9)}$	150.6306	136.7794	128.8789	123.7724
	$T_1^{(10)}$	150.7318	136.8916	129.0042	123.9117
	$T_1^{(\delta_{\alpha_1}=6.6)}$	472.1990	265.6426	205.6914	177.1525
0.5	$T_1^{(1)}$	100	100	100	100
	$T_1^{(2)}$	100.0731	100.0993	100.1254	100.1515
	$T_1^{(3)}$	96.17496	97.15294	97.73266	98.11623
	$T_1^{(4)}$	29.22643	35.91599	41.45016	46.10447
	$T_1^{(5)}$	140.4696	126.9490	120.2002	116.1546
	$T_1^{(6)}$	140.5427	127.0482	120.3256	116.3061
	$T_1^{(7)}$	140.4062	126.9108	120.1731	116.1337
	$T_1^{(8)}$	52.54547	60.04419	65.49642	69.63932
	$T_1^{(9)}$	140.4696	126.9490	120.2002	116.1546
	$T_1^{(10)}$	140.5777	127.0789	120.3541	116.3334
	$T_1^{(\delta_{\alpha_1}=6.6)}$	303.5731	194.1631	160.4659	144.1040

It is envisaged from Table 3 that for $\delta_{\alpha_1} = 6.6$, the proposed estimator T_1 perform efficiently than other considered estimators $T_1^{(i)}$; $i = 1,2,..,10$ in terms of having high PRE with respect to the usual unbiased estimator $T_1^{(1)}$ for different levels of W_2. Also, it is observed that for different values of k and W_2 the PRE of the estimators decreases but the PRE of $T_1^{(2)}$, $T_1^{(3)}$, $T_1^{(4)}$ and $T_1^{(8)}$ increases.

6 Situation-2

When non-response and measurement error are present on both the study and auxiliary variable with a known population mean μ_X of auxiliary variable X.

6.1 Estimator

Redefine the general class of estimators defined in the equation (17) as

$$T_2 = [k_{12}\hat{\mu}_Y^* + k_{22}(\mu_X - \hat{\mu}_X^*)]\left(\frac{\mu_X}{\hat{\mu}_X^*}\right)^{\delta_2}\left[\exp\left(\frac{\mu_X - \hat{\mu}_X^*}{\mu_X + \hat{\mu}_X^*}\right)\right]^{\alpha_2} \quad (29)$$

where k_{12}, k_{22}, δ_2 and α_2 are constant.

The member estimators can be written as

$1T_2^{(1)} = \hat{\mu}_Y^*$
$2T_2^{(2)} = k_{12}\hat{\mu}_Y^*$
$3T_2^{(3)} = \hat{\mu}_Y^*(\frac{\mu_X}{\hat{\mu}_X^*})$
$4T_2^{(4)} = \hat{\mu}_Y^*(\frac{\hat{\mu}_X^*}{\mu_X})$
$5T_2^{(5)} = \hat{\mu}_Y^* + k_{22}(\mu_X - \hat{\mu}_X^*)$
$6T_2^{(6)} = k_{12}\hat{\mu}_Y^* + k_{22}(\mu_X - \hat{\mu}_X^*)$
$7T_2^{(7)} = \hat{\mu}_Y^* \exp(\frac{\mu_X - \hat{\mu}_X^*}{\mu_X + \hat{\mu}_X^*})$
$8T_2^{(8)} = \hat{\mu}_Y^* \exp(\frac{\hat{\mu}_X^* - \mu_X}{\hat{\mu}_X^* + \mu_X})$
$9T_2^{(9)} = [\hat{\mu}_Y^* + k_{22}(\mu_X - \hat{\mu}_X^*)](\frac{\mu_X}{\hat{\mu}_X^*})$
$10T_2^{(10)} = [k_{12}\hat{\mu}_Y^* + k_{22}(\mu_X - \hat{\mu}_X^*)]\exp\left(\frac{\mu_X - \hat{\mu}_X^*}{\mu_X + \hat{\mu}_X^*}\right)$

where k_{12}, k_{22} are suitable constant for respective estimator.

6.2 Bias and MSE

The bias and MSE of the general class of estimators defined in equation (29) can be derived as

$$Bias(T_2) = (k_{12} - 1)\mu_Y + \left[\delta_{\alpha_2}\left(k_{22} + \frac{Rk_{12}}{2}\right) + \frac{Rk_{12}}{2}\delta_{\alpha_2}^2\right]\frac{V_4}{\mu_X} - \frac{k_{12}}{\mu_X}\delta_{\alpha_2}V_5 \quad (30)$$

$$MSE(T_2) = \mu_Y^2 + k_{12}\left[2RV_5\delta_{\alpha_2} - 2\mu_Y^2 - R^2V_4\delta_{\alpha_2} - R^2V_4\delta_{\alpha_2}^2\right] + k_{12}^2[\mu_Y^2 + V_1 - 4RV_5\delta_{\alpha_2} + R^2V_4\delta_{\alpha_2} + 2R^2V_4\delta_{\alpha_2}^2] - 2k_{22}RV_4\delta_{\alpha_2} + k_{22}^2V_4 + k_{12}k_{22}\left[4RV_4\delta_{\alpha_2} - 2V_5\right] \quad (31)$$

where $\delta_{\alpha_2} = \delta_2 + \frac{\alpha_2}{2}$.

For the optimum values of k_{12} and k_{22} which is $k_{12}^o = \frac{\varphi_{32}\varphi_{52} - 2\varphi_{12}\varphi_{42}}{4\varphi_{22}\varphi_{42} - \varphi_{52}^2}$; $k_{22}^o = \frac{\varphi_{12}\varphi_{52} - 2\varphi_{22}\varphi_{32}}{4\varphi_{22}\varphi_{42} - \varphi_{52}^2}$, the minimum MSE of T_2 can be obtained as

$$MSE_{min}(T_2) = \mu_Y^2 - \frac{\varphi_{12}^2\varphi_{42} + \varphi_{22}\varphi_{32}^2 - \varphi_{12}\varphi_{32}\varphi_{52}}{4\varphi_{22}\varphi_{42} - \varphi_{52}^2} \quad (32)$$

where $\varphi_{12} = 2RV_5\delta_{\alpha_2} - 2\mu_Y^2 - R^2V_4\delta_{\alpha_2} - R^2V_4\delta_{\alpha_2}^2$, $\varphi_{22} = \mu_Y^2 + V_1 - 4RV_5\delta_{\alpha_2} + R^2V_4\delta_{\alpha_2} + 2R^2V_4\delta_{\alpha_2}^2$, $\varphi_{32} = -2RV_4\delta_{\alpha_2}$, $\varphi_{42} = V_4$, $\varphi_{52} = 4RV_4\delta_{\alpha_2} - 2V_5$.

Minimum MSE of T_2 can also be written as

$$MSE_{min}(T_2) = \mu_Y^2 - \Upsilon_2 \quad (33)$$

where $\Upsilon_2 = \frac{\varphi_{12}^2\varphi_{42} + \varphi_{22}\varphi_{32}^2 - \varphi_{12}\varphi_{22}\varphi_{32}}{4\varphi_{22}\varphi_{42} - \varphi_{52}^2}$.

The bias and MSE of estimators $T_2^{(i)}$; $i = 1, 2, .., 10$ upto the first order of approxmation given in Table 4.

Table 4: Expressions for bias and MSE of $T_2^{(i)}$; $i = 1, 2, .., 10$

Estimator	Bias	MSE/MSE_{min} and respective optimum value of constants
$T_2^{(1)}$	0	V_1
$T_2^{(2)}$	$(k_{12}-1)\mu_Y$	$\mu_Y^2 - \frac{\mu_Y^4}{\mu_Y^2+V_1}$; $k_{12}^o = \frac{\mu_Y^2}{\mu_Y^2+V_1}$
$T_2^{(3)}$	$\frac{RV_4-V_5}{\mu_X}$	$V_1 + R^2V_4 - 2RV_5$
$T_2^{(4)}$	$\frac{V_5}{\mu_X}$	$V_1 + R^2V_4 + 2RV_5$
$T_2^{(5)}$	0	$V_1 - \frac{V_5^2}{V_4}$; $k_{22}^o = \frac{V_5}{V_4}$
$T_2^{(6)}$	$(k_{12}-1)\mu_Y$	$\mu_Y^2 - \frac{V_4\mu_Y^4}{V_4\mu_Y^2+V_1V_4-V_5^2}$; $k_{12}^o = \frac{V_4\mu_Y^2}{V_4\mu_Y^2+V_1V_4-V_5^2}$, $k_{22}^o = \frac{V_5\mu_Y^2}{V_4\mu_Y^2+V_1V_4-V_5^2}$
$T_2^{(7)}$	$\frac{3RV_4-4V_5}{8\mu_X}$	$V_1 + \frac{1}{4}R^2V_4 - RV_5$
$T_2^{(8)}$	$\frac{4V_5-RV_4}{8\mu_X}$	$V_1 + \frac{1}{4}R^2V_4 + RV_5$
$T_2^{(9)}$	$\frac{(R+k_{22})V_4-V_5}{\mu_X}$	$V_1 - \frac{V_5^2}{V_4}$; $k_{22} = \frac{V_5}{V_4} - R$
$T_2^{(10)}$	$(k_{12}-1)\mu_Y + \frac{(4k_{22}+3Rk_{12})V_4-4k_{12}V_5}{8\mu_X}$	$\mu_Y^2 - \frac{P_{12}^2P_{42}+P_{22}P_{32}^2-P_{12}P_{32}P_{52}}{4P_{22}P_{42}-P_{52}^2}$; $k_{12}^o = \frac{P_{32}P_{52}-2P_{12}P_{42}}{4P_{22}P_{42}-P_{52}^2}$; $k_{12}^o = \frac{P_{12}P_{52}-2P_{22}P_{32}}{4P_{22}P_{42}-P_{52}^2}$

where $P_{12} = RV_5 - 2\mu_Y^2 - \frac{3}{4}R^2V_4$, $P_{22} = \mu_Y^2 + V_1 - 2RV_5 + R^2V_4$, $P_{32} = -RV_4$, $P_{42} = V_4$, $P_{52} = 2RV_4 - 2V_5$.

6.3 Efficiency Comparison

An estimator t_1 of population mean μ_Y is said to be more efficient than estimaor t_2 if $MSE(t_1) < MSE(t_2)$. Here we have developed the conditions under which the general class of estimator T_2 works better than the estimators $T_2^{(i)}$, $i = 1, 2, .., 10$.

- $MSE(T_2) < MSE(T_2^{(1)})$ if $\mu_Y^2 < \Upsilon_2 + V_1$
- $MSE(T_2) < MSE(T_2^{(2)})$ if $\frac{\mu_Y^4}{\mu_Y^2+V_1} < \Upsilon_2$
- $MSE(T_2) < MSE(T_2^{(3)})$ if $\mu_Y^2 + 2RV_5 < \Upsilon_2 + V_1 + R^2V_4$
- $MSE(T_2) < MSE(T_2^{(4)})$ if $\mu_Y^2 < \Upsilon_2 + V_1 + R^2V_4 + 2RV_5$
- $MSE(T_2) < MSE(T_2^{(5)})$ if $\mu_Y^2 + \frac{V_5^2}{V_4} < \Upsilon_2 + V_1$
- $MSE(T_2) < MSE(T_2^{(6)})$ if $\frac{V_4\mu_Y^4}{V_4\mu_Y^2+V_1V_4-V_5^2} < \Upsilon_2$
- $MSE(T_2) < MSE(T_2^{(7)})$ if $\mu_Y^2 + RV_5 < \Upsilon_2 + V_1 + \frac{1}{4}R^2V_4$
- $MSE(T_2) < MSE(T_2^{(8)})$ if $\mu_Y^2 < \Upsilon_2 + V_1 + \frac{1}{4}R^2V_4 + RV_5$
- $MSE(T_2) < MSE(T_2^{(9)})$ if $\mu_Y^2 + \frac{V_5^2}{V_4} < \Upsilon_2 + V_1$
- $MSE(T_2) < MSE(T_2^{(10)})$ if $\frac{P_{12}^2P_{42}+P_{22}P_{32}^2-P_{12}P_{32}P_{52}}{4P_{22}P_{42}-P_{52}^2} < \Upsilon_2$

6.4 Simulation

The same data used for simulation as in the Situation 1.

The percent relative efficiency (PRE) of estimators with respect to $T_2^{(1)}$ are calculated using

$$PRE(., T_2^{(1)}) = \frac{MSE(T_2^{(1)})}{MSE(.)} \times 100 \tag{34}$$

The result of simulation is given in Table 5.

It is noted from Table 5 that the PRE of proposed estimator T_2 when $\delta_{\alpha_2} = 4.2$ is maximum as compared to the PREs of other considered estimators. For different values of W_2 and k, the PREs of the estimators showed an increasing trend.

Table 5: PREs of estimators for different values of W_2 and k in Situation 2

W_2	Estimator	PRE of estimators with respect to $T_2^{(1)}$			
		$1/k$			
		$1/2$	$1/3$	$1/4$	$1/5$
0.1	$T_2^{(1)}$	100	100	100	100
	$T_2^{(2)}$	100.0522	100.0575	100.0627	100.0679
	$T_2^{(3)}$	94.17814	94.18191	94.18505	94.18770
	$T_2^{(4)}$	20.97770	20.97788	20.97803	20.97815
	$T_2^{(5)}$	181.2213	181.2246	181.2274	181.2297
	$T_2^{(6)}$	181.2735	181.2821	181.2901	181.2976
	$T_2^{(7)}$	181.0575	181.0611	181.0640	181.0665
	$T_2^{(8)}$	41.58202	41.58219	41.58234	41.58246
	$T_2^{(9)}$	181.2213	181.2246	181.2274	181.2297
	$T_2^{(10)}$	181.3287	181.3427	181.3563	181.3694
	$T_2^{(\delta_{\alpha_2}=4.2)}$	212.7201	216.6192	220.6895	224.9426
0.3	$T_2^{(1)}$	100	100	100	100
	$T_2^{(2)}$	100.0627	100.0784	100.0940	100.1097
	$T_2^{(3)}$	94.18596	94.19342	94.19839	94.20194
	$T_2^{(4)}$	20.97786	20.97809	20.97825	20.97836
	$T_2^{(5)}$	181.2301	181.2379	181.2431	181.2467
	$T_2^{(6)}$	181.2928	181.3162	181.3371	181.3565
	$T_2^{(7)}$	181.0668	181.0750	181.0805	181.0844
	$T_2^{(8)}$	41.58207	41.58223	41.58234	41.58241
	$T_2^{(9)}$	181.2301	181.2379	181.2431	181.2467
	$T_2^{(10)}$	181.3590	181.3990	181.4364	181.4724
	$T_2^{(\delta_{\alpha_2}=4.2)}$	220.6984	234.0678	249.4709	267.4126
0.5	$T_2^{(1)}$	100	100	100	100
	$T_2^{(2)}$	100.0731	100.0993	100.1254	100.1515
	$T_2^{(3)}$	94.18420	94.18926	94.19220	94.19413
	$T_2^{(4)}$	20.97777	20.97791	20.97799	20.97805
	$T_2^{(5)}$	181.2286	181.2340	181.2372	181.2393
	$T_2^{(6)}$	181.3018	181.3333	181.3626	181.3908
	$T_2^{(7)}$	181.0652	181.0709	181.0742	181.0764
	$T_2^{(8)}$	41.58198	41.58207	41.58212	41.58215
	$T_2^{(9)}$	181.2286	181.2340	181.2372	181.2393
	$T_2^{(10)}$	181.3790	181.4382	181.4951	181.5509
	$T_2^{(\delta_{\alpha_2}=4.2)}$	229.3999	255.1317	288.5715	333.7979

7 Situation-3

When non-response and measurement error present only on study variable with unknown μ_X. Following are the estimators obtained:

7.1 Estimator

Redefine the general class of estimators defined in equation (17) as

$$T_3 = [k_{13}\hat{\mu}_Y^* + k_{23}(\hat{\mu}_X' - \hat{\mu}_X)] \left(\frac{\hat{\mu}_X'}{\hat{\mu}_X}\right)^{\delta_3} \left[\exp\left(\frac{\hat{\mu}_X' - \hat{\mu}_X}{\hat{\mu}_X' + \hat{\mu}_X}\right)\right]^{\alpha_3} \quad (35)$$

where k_{13}, k_{23}, δ_3 and α_3 are constant.

The member estimators can be written as

$1 T_3^{(1)} = \hat{\mu}_Y^*$
$2 T_3^{(2)} = k_{13}\hat{\mu}_Y^*$
$3 T_3^{(3)} = \hat{\mu}_Y^*(\frac{\hat{\mu}_X'}{\hat{\mu}_X})$
$4 T_3^{(4)} = \hat{\mu}_Y^*(\frac{\hat{\mu}_X}{\hat{\mu}_X'})$
$5 T_3^{(5)} = \hat{\mu}_Y^* + k_{23}(\hat{\mu}_X' - \hat{\mu}_X),$
$6 T_3^{(6)} = k_{13}\hat{\mu}_Y^* + k_{23}(\hat{\mu}_X' - \hat{\mu}_X)$
$7 T_3^{(7)} = \hat{\mu}_Y^* \exp(\frac{\hat{\mu}_X' - \hat{\mu}_X}{\hat{\mu}_X' + \hat{\mu}_X})$
$8 T_3^{(8)} = \hat{\mu}_Y^* \exp(\frac{\hat{\mu}_X - \hat{\mu}_X'}{\hat{\mu}_X + \hat{\mu}_X'})$
$9 T_3^{(9)} = [\hat{\mu}_Y^* + k_{23}(\hat{\mu}_X' - \hat{\mu}_X)](\frac{\hat{\mu}_X'}{\hat{\mu}_X})$
$10 T_3^{(10)} = [k_{13}\hat{\mu}_Y^* + k_{23}(\hat{\mu}_X' - \hat{\mu}_X)] \exp\left(\frac{\hat{\mu}_X' - \hat{\mu}_X}{\hat{\mu}_X' + \hat{\mu}_X}\right)$

where k_{13}, k_{23} are suitable constant for respective estimator.

7.2 Bias and MSE

The bias and MSE of the general class of estimators defined in equation (35) can be derived as

$$Bias(T_3) = (k_{13} - 1)\mu_Y + \left[\delta_{\alpha_3}\left(k_{23} + \frac{Rk_{13}}{2}\right) + \frac{Rk_{13}}{2}\delta_{\alpha_3}^2\right] \frac{(V_2 - V_6)}{\mu_X} - \frac{k_{13}}{\mu_X}\delta_{\alpha_3}(V_3 - V_3') \quad (36)$$

$$\begin{aligned} MSE(T_3) = \mu_Y^2 &+ k_{13}\left[2R(V_3 - V_3')\delta_{\alpha_3} - 2\mu_Y^2 - R^2(V_2 - V_6)\delta_{\alpha_3}^2 - R^2(V_2 - V_6)\delta_{\alpha_3}\right] \\ &+ k_{13}^2\left[\mu_Y^2 + V_1 + R^2(V_2 - V_6)\delta_{\alpha_3} - 4R(V_3 - V_3')\delta_{\alpha_3} + 2R^2(V_2 - V_6)\delta_{\alpha_3}^2\right] \\ &- 2k_{23}R(V_2 - V_6)\delta_{\alpha_3} + k_{23}^2(V_2 - V_6) + k_{13}k_{23}\left[4R(V_2 - V_6)\delta_{\alpha_3} - 2(V_3 - V_3')\right] \end{aligned} \quad (37)$$

where $\delta_{\alpha_3} = \delta_3 + \frac{\alpha_3}{2}$.

For the optimum values of k_{13} and k_{23} which is $k_{13}^o = \frac{\varphi_{33}\varphi_{53} - 2\varphi_{13}\varphi_{43}}{4\varphi_{23}\varphi_{43} - \varphi_{53}^2}$; $k_{23}^o = \frac{\varphi_{13}\varphi_{53} - 2\varphi_{23}\varphi_{33}}{4\varphi_{23}\varphi_{43} - \varphi_{53}^2}$, the minimum MSE of T_3 can be obtained as

$$MSE_{min}(T_3) = \mu_Y^2 - \frac{\varphi_{13}^2\varphi_{43} + \varphi_{23}\varphi_{33}^2 - \varphi_{13}\varphi_{33}\varphi_{53}}{4\varphi_{23}\varphi_{43} - \varphi_{53}^2} \quad (38)$$

where $\varphi_{13} = 2R(V_3 - V_3')\delta_{\alpha_3} - 2\mu_Y^2 - R^2(V_2 - V_6)\delta_{\alpha_3} - R^2(V_2 - V_6)\delta_{\alpha_3}^2$, $\varphi_{23} = \mu_Y^2 + V_1 - 4R(V_3 - V_3')\delta_{\alpha_3} + R^2(V_2 - V_6)\delta_{\alpha_3} + 2R^2(V_2 - V_6)\delta_{\alpha_3}^2$, $\varphi_{33} = -2R(V_2 - V_6)\delta_{\alpha_3}$, $\varphi_{43} = (V_2 - V_6)$, $\varphi_{53} = 4R(V_2 - V_6)\delta_{\alpha_3} - 2(V_3 - V_3')$.

Minimum MSE of T_3 can also be written as

$$MSE_{min}(T_3) = \mu_Y^2 - \Upsilon_3 \quad (39)$$

where $\Upsilon_3 = \frac{\varphi_{13}^2\varphi_{43} + \varphi_{23}\varphi_{33}^2 - \varphi_{13}\varphi_{33}\varphi_{53}}{4\varphi_{23}\varphi_{43} - \varphi_{53}^2}$.

The bias and MSE of the estimators $T_3^{(i)}$; $i = 1, 2, .., 10$ upto the first order of approxmation given in Table 5.

Table 6: Expressions for bias and MSE of $T_3^{(i)}$; $i = 1,2,..,10$

Estimator	Bias	MSE/MSE_{min} and respective optimum value of constants
$T_3^{(1)}$	0	V_1
$T_3^{(2)}$	$(k_{13}-1)\mu_Y$	$\mu_Y^2 - \frac{\mu_Y^4}{\mu_Y^2+V_1}$; $k_{13}^o = \frac{\mu_Y^2}{\mu_Y^2+V_1}$
$T_3^{(3)}$	$\frac{R(V_2-V_6)-(V_3-V_3')}{\mu_X}$	$V_1 + R^2(V_2-V_6) - 2R(V_3-V_3')$
$T_3^{(4)}$	$\frac{(V_3-V_3')}{\mu_X}$	$V_1 + R^2(V_2-V_6) + 2R(V_3-V_3')$
$T_3^{(5)}$	0	$V_1 - \frac{(V_3-V_3')^2}{(V_2-V_6)}$; $k_{23}^o = \frac{(V_3-V_3')}{(V_2-V_6)}$
$T_3^{(6)}$	$(k_{13}-1)\mu_Y$	$\mu_Y^2 - \frac{(V_2-V_6)\mu_Y^4}{(V_2-V_6)(\mu_Y^2+V_1)-(V_3-V_3')^2}$; $k_{13}^o = \frac{(V_2-V_6)\mu_Y^2}{(V_2-V_6)(\mu_Y^2+V_1)-(V_3-V_3')^2}$, $k_{23}^o = \frac{(V_3-V_3')\mu_Y^2}{(V_2-V_6)(\mu_Y^2+V_1)-(V_3-V_3')^2}$
$T_3^{(7)}$	$\frac{3R(V_2-V_6)-4(V_3-V_3')}{8\mu_X}$	$V_1 + \frac{1}{4}R^2(V_2-V_6) - R(V_3-V_3')$
$T_3^{(8)}$	$\frac{4(V_3-V_3')-R(V_2-V_6)}{8\mu_X}$	$V_1 + \frac{1}{4}R^2(V_2-V_6) + R(V_3-V_3')$
$T_3^{(9)}$	$\frac{(R+k_{23})(V_2-V_6)-(V_3-V_3')}{\mu_X}$	$V_1 - \frac{(V_3-V_3')^2}{(V_2-V_6)}$; $k_{23} = \frac{(V_3-V_3')}{(V_2-V_6)} - R$
$T_3^{(10)}$	$(k_{13}-1)\mu_Y + \frac{(4k_{23}+3Rk_{13})(V_2-V_6)-4k_{13}(V_3-V_3')}{8\mu_X}$	$\mu_Y^2 - \frac{P_{13}^2P_{43}+P_{23}P_{33}^2-P_{13}P_{33}P_{53}}{4P_{23}P_{43}-P_{53}^2}$; $k_{13}^o = \frac{P_{33}P_{53}-2P_{13}P_{43}}{4P_{23}P_{43}-P_{53}^2}$; $k_{13}^o = \frac{P_{13}P_{53}-2P_{23}P_{33}}{4P_{23}P_{43}-P_{53}^2}$

where $P_{13} = R(V_3-V_3') - 2\mu_Y^2 - \frac{3}{4}R^2(V_2-V_6)$, $P_{23} = \mu_Y^2 + V_1 - 2R(V_3-V_3') + R^2(V_2-V_6)$, $P_{33} = -R(V_2-V_6)$, $P_{43} = (V_2-V_6)$, $P_{53} = 2R(V_2-V_6) - 2(V_3-V_3')$.

7.3 Efficiency Comparison

An estimator t_1 of population mean μ_Y is said to be more efficient than estimaor t_2 if $MSE(t_1) < MSE(t_2)$. Here we have developed the conditions under which the proposed general class of estimator T_3 is better than the estimators $T_3^{(i)}$, $i = 1,2,..,10$.

- $MSE(T_3) < MSE(T_3^{(1)})$ if $\mu_Y^2 < \Upsilon_3 + V_1$
- $MSE(T_3) < MSE(T_3^{(2)})$ if $\frac{\mu_Y^4}{\mu_Y^2+V_1} < \Upsilon_3$
- $MSE(T_3) < MSE(T_3^{(3)})$ if $\mu_Y^2 + 2R(V_3-V_3') < \Upsilon_3 + V_1 + R^2(V_2-V_6)$
- $MSE(T_3) < MSE(T_3^{(4)})$ if $\mu_Y^2 < \Upsilon_3 + V_1 + R^2(V_2-V_6) + 2R(V_3-V_3')$
- $MSE(T_3) < MSE(T_3^{(5)})$ if $\mu_Y^2 + \frac{(V_3-V_3')^2}{(V_2-V_6)} < \Upsilon_3 + V_1$
- $MSE(T_3) < MSE(T_3^{(6)})$ if $\frac{(V_2-V_6)\mu_Y^4}{(V_2-V_6)\mu_Y^2+V_1(V_2-V_6)-(V_3-V_3')^2} < \Upsilon_3$
- $MSE(T_3) < MSE(T_3^{(7)})$ if $\mu_Y^2 + R(V_3-V_3') < \Upsilon_3 + V_1 + \frac{1}{4}R^2(V_2-V_6)$
- $MSE(T_3) < MSE(T_3^{(8)})$ if $\mu_Y^2 < \Upsilon_3 + V_1 + \frac{1}{4}R^2(V_2-V_6) + R(V_3-V_3')$
- $MSE(T_3) < MSE(T_3^{(9)})$ if $\mu_Y^2 + \frac{(V_3-V_3')^2}{(V_2-V_6)} < \Upsilon_3 + V_1$
- $MSE(T_3) < MSE(T_3^{(10)})$ if $\frac{P_{13}^2P_{43}+P_{23}P_{33}^2-P_{13}P_{33}P_{53}}{4P_{23}P_{43}-P_{53}^2} < \Upsilon_3$

7.4 Simulation

The data used to perform simulation are: $N = 5000$, $n = 500$ $n' = 1000$, $X = rnorm(N,10,5)$, $Y = 1+3*X+rnorm(N,0,1)$, $y = Y + rnorm(N,0,5)$, $x = X + rnorm(N,0,5)$, $U = y - Y$, $V = x - X$. For different response rate, the result of the simulation is given in Table 7. For a better approximation, we have averaged the result over 25000 iterations.

The percent relative efficiency (PRE) of estimators with respect to $T_3^{(1)}$ are calculated using

$$PRE(.,T_3^{(1)}) = \frac{MSE(T_3^{(1)})}{MSE(.)} \times 100 \tag{40}$$

Table 7: PREs of estimators for different values of W_2 and k in Situation 3

W_2	Estimator	PRE of estimators with respect to $T_3^{(1)}$			
		$1/k$			
		$1/2$	$1/3$	$1/4$	$1/5$
0.1	$T_3^{(1)}$	100	100	100	100
	$T_3^{(2)}$	100.0522	100.0575	100.0627	100.0679
	$T_3^{(3)}$	70.72865	72.66183	74.35548	75.85153
	$T_3^{(4)}$	30.61407	32.67473	34.61653	36.44945
	$T_3^{(5)}$	119.0567	117.0296	115.3924	114.0423
	$T_3^{(6)}$	119.1089	117.0871	115.4550	114.1102
	$T_3^{(7)}$	114.6938	113.1822	111.9526	110.9329
	$T_3^{(8)}$	55.61132	57.94931	60.05335	61.95686
	$T_3^{(9)}$	119.0567	117.0296	115.3924	114.0423
	$T_3^{(10)}$	119.1319	117.1094	115.4769	114.1316
	$T_3^{(\delta_{\alpha_3}=8.1)}$	3568.4635	831.1187	507.0318	381.2519
0.3	$T_3^{(1)}$	100	100	100	100
	$T_3^{(2)}$	100.0627	100.0784	100.0940	100.1097
	$T_3^{(3)}$	74.35704	78.37690	81.30722	83.53811
	$T_3^{(4)}$	34.61838	39.82651	44.26613	48.09564
	$T_3^{(5)}$	115.3909	111.9448	109.7596	108.2503
	$T_3^{(6)}$	115.4536	112.0232	109.8537	108.3600
	$T_3^{(7)}$	111.9515	109.3379	107.6622	106.4965
	$T_3^{(8)}$	60.05531	65.26997	69.28034	72.46040
	$T_3^{(9)}$	115.3909	111.9448	109.7596	108.2503
	$T_3^{(10)}$	115.4754	112.0440	109.8739	108.3798
	$T_3^{(\delta_{\alpha_3}=8.1)}$	506.8538	272.8734	208.6849	178.6818
0.5	$T_3^{(1)}$	100	100	100	100
	$T_3^{(2)}$	100.0731	100.0993	100.1254	100.1515
	$T_3^{(3)}$	77.18481	82.11526	85.29341	87.51246
	$T_3^{(4)}$	38.18531	45.60413	51.43299	56.13356
	$T_3^{(5)}$	112.9082	109.1986	107.1452	105.8413
	$T_3^{(6)}$	112.9813	109.2979	107.2706	105.9928
	$T_3^{(7)}$	110.0720	107.2297	105.6385	104.6214
	$T_3^{(8)}$	63.69005	70.41912	75.04403	78.41829
	$T_3^{(9)}$	112.9082	109.1986	107.1452	105.8413
	$T_3^{(10)}$	113.0024	109.3179	107.2901	106.0119
	$T_3^{(\delta_{\alpha_3}=8.1)}$	314.3192	196.5229	161.3007	144.3721

From Table 7, it is clear that the PRE of the proposed estimator T_3 at $\delta_{\alpha_3} = 8.1$ is maximum among the other estimators with respect to $T_3^1 = \hat{\mu}_Y^*$. For increasing values of W_2 and k, the estimators T_3^2, T_3^3, T_3^4 and T_3^8 increases while the estimators T_3^5, T_3^6, T_3^7, T_3^9, T_3^{10} and $T_3^{(\delta_{\alpha_3}=8.1)}$ decreases.

8 Situation-4

When non-response and measurement error present in study as well as auxiliary variable with unknown μ_X. Following are the estimators:

8.1 Estimator

Redefine the general class of estimators defined in equation (17) as

$$T_4 = [k_{14}\hat{\mu}_Y^* + k_{24}(\hat{\mu}_X' - \hat{\mu}_X^*)] \left(\frac{\hat{\mu}_X'}{\hat{\mu}_X^*}\right)^{\delta_4} \left[\exp\left(\frac{\hat{\mu}_X' - \hat{\mu}_X^*}{\hat{\mu}_X' + \hat{\mu}_X^*}\right)\right]^{\alpha_4} \tag{41}$$

where k_{14}, k_{24}, δ_4 and α_4 are constant.

The member estimators can be written as

$1T_4^{(1)} = \hat{\mu}_Y^*$
$2T_4^{(2)} = k_{14}\hat{\mu}_Y^*$
$3T_4^{(3)} = \hat{\mu}_Y^*(\frac{\hat{\mu}_X'}{\hat{\mu}_X^*})$
$4T_4^{(4)} = \hat{\mu}_Y^*(\frac{\hat{\mu}_X^*}{\hat{\mu}_X'})$
$5T_4^{(5)} = \hat{\mu}_Y^* + k_{24}(\hat{\mu}_X' - \hat{\mu}_X^*)$,
$6T_4^{(6)} = k_{14}\hat{\mu}_Y^* + k_{24}(\hat{\mu}_X' - \hat{\mu}_X^*)$
$7T_4^{(7)} = \hat{\mu}_Y^* \exp(\frac{\hat{\mu}_X' - \hat{\mu}_X^*}{\hat{\mu}_X' + \hat{\mu}_X^*})$
$8T_4^{(8)} = \hat{\mu}_Y^* \exp(\frac{\hat{\mu}_X^* - \hat{\mu}_X'}{\hat{\mu}_X^* + \hat{\mu}_X'})$
$9T_4^{(9)} = [\hat{\mu}_Y^* + k_{24}(\hat{\mu}_X' - \hat{\mu}_X^*)](\frac{\hat{\mu}_X'}{\hat{\mu}_X^*})$
$10T_4^{(10)} = [k_{14}\hat{\mu}_Y^* + k_{24}(\hat{\mu}_X' - \hat{\mu}_X^*)] \exp(\frac{\hat{\mu}_X' - \hat{\mu}_X^*}{\hat{\mu}_X' + \hat{\mu}_X^*})$

where k_{14}, k_{24} are suitable constant for respective estimator.

8.2 Bias and MSE

The bias and MSE of the general class of estimators defined in equation (41) can be derived as

$$Bias(T_4) = (k_{14} - 1)\mu_Y + \left[\delta_{\alpha_4}\left(k_{24} + \frac{Rk_{14}}{2}\right) + \frac{Rk_{14}}{2}\delta_{\alpha_4}^2\right]\frac{(V_4 - V_6)}{\mu_X} - \frac{k_{14}}{\mu_X}\delta_{\alpha_4}(V_5 - V_3') \tag{42}$$

$$\begin{aligned} MSE(T_4) = \mu_Y^2 &+ k_{14}\left[2R(V_5 - V_3')\delta_{\alpha_4} - 2\mu_Y^2 - R^2(V_4 - V_6)\delta_{\alpha_4}^2 - R^2(V_4 - V_6)\delta_{\alpha_4}\right] \\ &+ k_{14}^2\left[R^2(V_4 - V_6)\delta_{\alpha_4} - 4R(V_5 - V_3')\delta_{\alpha_4} + 2R^2(V_4 - V_6)\delta_{\alpha_4}^2 + \mu_Y^2 + V_1\right] \\ &- 2k_{24}R(V_4 - V_6)\delta_{\alpha_4} + k_{24}^2(V_4 - V_6) + k_{14}k_{24}\left[4R(V_4 - V_6)\delta_{\alpha_4} - 2(V_5 - V_3')\right] \end{aligned} \tag{43}$$

where $\delta_{\alpha_4} = \delta_4 + \frac{\alpha_4}{2}$.

For the optimum values of k_{14} and k_{24} which is $k_{14}^o = \frac{\varphi_{34}\varphi_{54} - 2\varphi_{14}\varphi_{44}}{4\varphi_{24}\varphi_{44} - \varphi_{54}^2}$; $k_{24}^o = \frac{\varphi_{14}\varphi_{54} - 2\varphi_{24}\varphi_{34}}{4\varphi_{24}\varphi_{44} - \varphi_{54}^2}$, the minimum MSE of T_4 can be obtained as

$$MSE_{min}(T_4) = \mu_Y^2 - \frac{\varphi_{14}^2\varphi_{44} + \varphi_{24}\varphi_{34}^2 - \varphi_{14}\varphi_{34}\varphi_{54}}{4\varphi_{24}\varphi_{44} - \varphi_{54}^2} \tag{44}$$

where $\varphi_{14} = 2R(V_5 - V_3')\delta_{\alpha_4} - 2\mu_Y^2 - R^2(V_4 - V_6)\delta_{\alpha_4} - R^2(V_4 - V_6)\delta_{\alpha_4}^2$, $\varphi_{24} = \mu_Y^2 + V_1 - 4R(V_5 - V_3')\delta_{\alpha_4} + R^2(V_4 - V_6)\delta_{\alpha_4} + 2R^2(V_4 - V_6)\delta_{\alpha_4}^2$, $\varphi_{34} = -2R(V_4 - V_6)\delta_{\alpha_4}$, $\varphi_{44} = (V_4 - V_6)$, $\varphi_{54} = 4R(V_4 - V_6)\delta_{\alpha_4} - 2(V_5 - V_3')$.

Minimum MSE of T_4 can also be written as

$$MSE_{min}(T_4) = \mu_Y^2 - \Upsilon_4 \tag{45}$$

where $\Upsilon_4 = \frac{\varphi_{14}^2\varphi_{44} + \varphi_{24}\varphi_{34}^2 - \varphi_{14}\varphi_{34}\varphi_{54}}{4\varphi_{24}\varphi_{44} - \varphi_{54}^2}$.

The bias and MSE of the estimators $T_4^{(i)}$; $i = 1, 2, .., 10$ upto the first order of approxmation given in Table 6.

Table 8: Expressions for bias and MSE of $T_4^{(i)}$; $i = 1, 2, .., 10$

Estimator	Bias	MSE/MSE_{min} and respective optimum value of constants
$T_4^{(1)}$	0	V_1
$T_4^{(2)}$	$(k_{14}-1)\mu_Y$	$\mu_Y^2 - \frac{\mu_Y^4}{\mu_Y^2+V_1}$; $k_{14}^o = \frac{\mu_Y^2}{\mu_Y^2+V_1}$
$T_4^{(3)}$	$\frac{R(V_4-V_6)-(V_5-V_3')}{\mu_X}$	$V_1 + R^2(V_4-V_6) - 2R(V_5-V_3')$
$T_4^{(4)}$	$\frac{(V_5-V_3')}{\mu_X}$	$V_1 + R^2(V_4-V_6) + 2R(V_5-V_3')$
$T_4^{(5)}$	0	$V_1 - \frac{(V_5-V_3')^2}{(V_4-V_6)}$; $k_{24}^o = \frac{(V_5-V_3')}{(V_4-V_6)}$
$T_4^{(6)}$	$(k_{14}-1)\mu_Y$	$\mu_Y^2 - \frac{(V_4-V_6)\mu_Y^4}{(V_4-V_6)(\mu_Y^2+V_1)-(V_5-V_3')^2}$; $k_{14}^o = \frac{(V_4-V_6)\mu_Y^2}{(V_4-V_6)(\mu_Y^2+V_1)-(V_5-V_3')^2}$, $k_{24}^o = \frac{(V_5-V_3')\mu_Y^2}{(V_4-V_6)(\mu_Y^2+V_1)-(V_5-V_3')^2}$
$T_4^{(7)}$	$\frac{3R(V_4-V_6)-4(V_5-V_3')}{8\mu_X}$	$V_1 + \frac{1}{4}R^2(V_4-V_6) - R(V_5-V_3')$
$T_4^{(8)}$	$\frac{4(V_5-V_3')-R(V_4-V_6)}{8\mu_X}$	$V_1 + \frac{1}{4}R^2(V_4-V_6) + R(V_5-V_3')$
$T_4^{(9)}$	$\frac{(R+k_{24})(V_4-V_6)-(V_5-V_3')}{\mu_X}$	$V_1 - \frac{(V_5-V_3')^2}{(V_4-V_6)}$; $k_{24} = \frac{(V_5-V_3')}{(V_4-V_6)} - R$
$T_4^{(10)}$	$(k_{14}-1)\mu_Y + \frac{(4k_{24}+3Rk_{14})(V_4-V_6)-4k_{14}(V_5-V_3')}{8\mu_X}$	$\mu_Y^2 - \frac{P_{14}^2P_{44}+P_{24}P_{34}^2-P_{14}P_{34}P_{54}}{4P_{24}P_{44}-P_{54}^2}$; $k_{14}^o = \frac{P_{34}P_{54}-2P_{14}P_{44}}{4P_{24}P_{44}-P_{54}^2}$; $k_{14}^o = \frac{P_{14}P_{54}-2P_{24}P_{34}}{4P_{24}P_{44}-P_{54}^2}$

where $P_{14} = R(V_5-V_3') - 2\mu_Y^2 - \frac{3}{4}R^2(V_4-V_6)$, $P_{24} = \mu_Y^2 + V_1 - 2R(V_5-V_3') + R^2(V_4-V_6)$, $P_{34} = -R(V_4-V_6)$, $P_{44} = (V_4-V_6)$, $P_{54} = 2R(V_4-V_6) - 2(V_5-V_3')$.

8.3 Efficiency Comparison

An estimator t_1 of population mean μ_Y is said to be more efficient than estimaor t_2 if $MSE(t_1) < MSE(t_2)$. Here we have developed the conditions under which the proposed general class of estimator T_4 is better than the estimators $T_4^{(i)}$, $i = 1, 2, .., 10$.

- $MSE(T_4) < MSE(T_4^{(1)})$ if $\mu_Y^2 < \Upsilon_4 + V_1$
- $MSE(T_4) < MSE(T_4^{(2)})$ if $\frac{\mu_Y^4}{\mu_Y^2+V_1} < \Upsilon_4$
- $MSE(T_4) < MSE(T_4^{(3)})$ if $\mu_Y^2 + 2R(V_5-V_3') < \Upsilon_4 + V_1 + R^2(V_4-V_6)$
- $MSE(T_4) < MSE(T_4^{(4)})$ if $\mu_Y^2 < \Upsilon_4 + V_1 + R^2(V_4-V_6) + 2R(V_5-V_3')$
- $MSE(T_4) < MSE(T_4^{(5)})$ if $\mu_Y^2 + \frac{(V_5-V_3')^2}{(V_4-V_6)} < \Upsilon_4 + V_1$
- $MSE(T_4) < MSE(T_4^{(6)})$ if $\frac{(V_4-V_6)\mu_Y^4}{(V_4-V_6)\mu_Y^2+V_1(V_4-V_6)-(V_5-V_3')^2} < \Upsilon_4$
- $MSE(T_4) < MSE(T_4^{(7)})$ if $\mu_Y^2 + R(V_5-V_3') < \Upsilon_4 + V_1 + \frac{1}{4}R^2(V_4-V_6)$
- $MSE(T_4) < MSE(T_4^{(8)})$ if $\mu_Y^2 < \Upsilon_4 + V_1 + \frac{1}{4}R^2(V_4-V_6) + R(V_5-V_3')$
- $MSE(T_4) < MSE(T_4^{(9)})$ if $\mu_Y^2 + \frac{(V_5-V_3')^2}{(V_4-V_6)} < \Upsilon_4 + V_1$
- $MSE(T_4) < MSE(T_4^{(10)})$ if $\frac{P_{14}^2P_{44}+P_{24}P_{34}^2-P_{14}P_{34}P_{54}}{4P_{24}P_{44}-P_{54}^2} < \Upsilon_4$

8.4 Simulation

The same data used for simulation as in the Situation 3.

The percent relative efficiency (PRE) of estimators with respect to $T_4^{(1)}$ are calculated using

$$PRE(., T_4^{(1)}) = \frac{MSE(T_4^{(1)})}{MSE(.)} \times 100 \quad (46)$$

The result of simulation is given in Table 9.

It is envisaged from Table 9 that the proposed estimator T_4 at $\delta_{\alpha_4} = -3.4$ is the maximum among the other considered estimators. The estimators $T_4^{(4)}$ and $T_4^{(8)}$ showed decreasing trend with the increase in the value of W_2 and k, while other estimators increased, respectively.

From the simulation study on all four situations, it is clear that the proposed estimator T performs efficiently in terms of having maximum PRE among the other considered estimators.

Table 9: PREs of estimators for different values of W_2 and k in Situation 4

W_2	Estimator	PRE of estimators with respect to $T_4^{(1)}$			
		$1/k$			
		$1/2$	$1/3$	$1/4$	$1/5$
0.1	$T_4^{(1)}$	100	100	100	100
	$T_4^{(2)}$	100.0522	100.0575	100.0627	100.0679
	$T_4^{(3)}$	70.42313	72.07774	73.51722	74.78095
	$T_4^{(4)}$	27.44957	26.70114	26.10791	25.62614
	$T_4^{(5)}$	125.2628	128.5061	131.4118	134.0253
	$T_4^{(6)}$	125.3150	128.5636	131.4745	134.0932
	$T_4^{(7)}$	120.9022	124.6686	127.9914	130.9446
	$T_4^{(8)}$	51.58248	50.47929	49.59535	48.87121
	$T_4^{(9)}$	125.2628	128.5061	131.4118	134.0253
	$T_4^{(10)}$	125.3431	128.5961	131.5117	134.1351
	$T_4^{(\delta_{\alpha_4}=-3.4)}$	135.5507	141.2855	146.9188	152.4949
0.3	$T_4^{(1)}$	100	100	100	100
	$T_4^{(2)}$	100.0627	100.0784	100.0940	100.1097
	$T_4^{(3)}$	73.51909	76.89875	79.32988	81.16266
	$T_4^{(4)}$	26.10714	24.89023	24.14011	23.63141
	$T_4^{(5)}$	131.4157	138.5330	143.9006	148.0818
	$T_4^{(6)}$	131.4784	138.6113	143.9946	148.1916
	$T_4^{(7)}$	127.9959	135.9713	141.8643	146.3961
	$T_4^{(8)}$	49.59420	47.75404	46.60133	45.81147
	$T_4^{(9)}$	131.4157	138.5330	143.9006	148.0818
	$T_4^{(10)}$	131.5156	138.6629	144.0611	148.2733
	$T_4^{(\delta_{\alpha_4}=-3.4)}$	146.9262	163.6520	180.8812	199.5004
0.5	$T_4^{(1)}$	100	100	100	100
	$T_4^{(2)}$	100.0731	100.0993	100.1254	100.1515
	$T_4^{(3)}$	75.89734	79.98798	82.58436	84.37877
	$T_4^{(4)}$	25.22579	23.94965	23.26317	22.83436
	$T_4^{(5)}$	136.3887	145.4004	151.4199	155.7157
	$T_4^{(6)}$	136.4618	145.4997	151.5453	155.8672
	$T_4^{(7)}$	133.5891	143.4943	149.9810	154.5586
	$T_4^{(8)}$	48.26539	46.30676	45.23597	44.56088
	$T_4^{(9)}$	136.3887	145.4004	151.4199	155.7157
	$T_4^{(10)}$	136.5085	145.5712	151.6425	155.9907
	$T_4^{(\delta_{\alpha_4}=-3.4)}$	158.0662	186.8893	220.4046	263.3491

9 Conclusion

In the present study, we have suggested a general class of estimators for estimating the population mean of the study variable by using the auxiliary variable in four different situations viz Situation 1 and 3: When non-response and measurement errors are present only on the study variable with known and unknown μ_X, respectively; Situation 2 and 4: When non-response and measurement errors are present on both the study as well as auxiliary variable with known and unknown μ_X, respectively. Some members of the proposed estimators in all situations have been obtained which are the well established estimators like Searl's $(T^{(2)})$, Cochran's $(T^{(3)})$, Murthy's $(T^{(4)})$, Cochran's $(T^{(5)})$, Rao's $(T^{(6)})$, Bahl and Tuteja's $(T^{(7)}$ and $T^{(8)})$, Kadilar and Cingi's $(T^{(9)})$ and Grover and Kour's $(T^{(10)})$ estimators. The expressions of the bias and MSE of the proposed estimators have been obtained along with all other estimators in all situations. Also, the conditions have been obtained under which the proposed estimators are efficient as compared to the other considered estimators. Further, the theoretical results have been verified through a simulation study. The simulation results show that the proposed class of estimators perform efficiently as compared to the usual estimator, Searl estimator, Cochran's estimator, Murthy's estimator, Rao's estimator, Bahl and Tuteja' estimator, Kadilar and Cingi and Grover and Kour's estimator in terms of having maximum PRE with respect to usual unbiased estimator in all the situations and for different values to W_2 and k.

Overall, we recommend our proposed class of estimators which are efficient as compared to the well-known existing estimators in different situations in the simultaneous presence of non-response and measurement error on both the study as well as auxiliary variables.

References

[1] E. D. Leeuw and W. D. Heer, Trends in household survey nonresponse: A longitudinal and international comparison, In R. M. Groves, D. A. Dillman, J. L. Eltinge, and R. J. A. Little (Eds.), Survey Nonresponse, New York, pp. 41–54. Wiley, 2002.
[2] J. Goyder, The Silent Minority: Nonrespondents on Sample Surveys (1st ed.), New York: Routledge, 1987.
[3] J. J. Hox and E. D. Leeuw (1994). A comparison of nonresponse in mail, telephone, and face-to-face surveys, Quality and Quantity, **28**, 329–344, 1994.
[4] D. A. Dillman, Mail and telephone surveys: The total design method, Volume 19, New York: Wiley, 1978.
[5] R. M. Groves and M. P. Couper, Nonresponse in Household Interview Surveys, Volume 19, New York: Wiley, 1998.
[6] M. Couper and E. D. Leeuw, Nonresponse in cross-cultural and cross-national surveys, In J. Harkness, F. J. R. Van de Vijver, and P. P. Mohler (Eds.), Cross-Cultural Survey Methods, New York, Wiley, 2003.
[7] W. G. Cochran, Sampling techniques, Wiley, New York, 1977.
[8] G. Kalton and D. Kasprzyk, The treatment of missing data, Survey Methodology, **12**, 1–16, 1986.
[9] X. L. Meng, Multiple-imputation inferences with uncongenial sources of input, Statistical Science, **9(4)**, 538-558, 1994.
[10] D. B. Rubin, Multiple imputation after 18+ years, Journal of the American Statistical Association, **91**, 473–490, 1996.
[11] J. R. Carpenter and M. G. Kenward, Missing Data in Randomised Controlled Trials - A Practical Guide, Health Technology Assessment Methodology Programme, Birmingham, 2007.
[12] M. H. Hansen and W. N. Hurwitz, The problem of non-response in sample surveys, J Amer Statist Assoc, **41**, 517, 1946.
[13] P. Rao, Ratio estimation with sub sampling the non-respondents, Survey Methodology, **12**, 217–230, 1986.
[14] F. C. Okafor and H. Lee, Double sampling for ratio and regression estimation with sub-sampling the non-respondents, Survey Methodology, **26**, 183–188, 2000.
[15] R. Tabasum and I. Khan, Double sampling for ratio estimation with non-response, J Ind Statist Assoc, **58**, 300–306, 2004.
[16] R. Tabasum and I. Khan, Double sampling ratio estimation for population mean in presence of non-response, Assam Statist Rev, **20**, 73–83, 2006.
[17] A. Sodipo and K. Obisesan, Estimation of the population mean using difference cum ratio estimator with full response on the auxiliary character, Res. J. App. Sci., **2**, 769–772, 2007.
[18] H. Singh and S. Kumar, A general procedure of estimating the population mean in the presence of non-response under double sampling using auxiliary information, SORT, **33(1)**, 71–84, 2008.
[19] H. Singh and S. Kumar, Estimation of population product in presence of non-response in successive sampling, Statist Papers, **51**, 559–582, 2010.
[20] R. Singh, S. Malik, and M. Khoshnevisan, An alternative estimator for estimating the finite population mean in presence of measurement errors with the view to financial modeling, Science Journal of Applied Mathematics and Statistics, **2(6)**, 107–111, 2014.
[21] M. K. Chaudhary, A. Prajapati, R. Singh, and F. Smarandache, Two-phase sampling in estimation of population mean in the presence of non-response. In R. Singh and F. Smarandache (Eds.), The efficient use of supplementary information in finite population sampling, USA, 28–41, Education Publishing, 2014.
[22] B. B. Khare and R. R. Sinha, Estimation of product of two population means by multi-auxiliary characters under double sampling the non-respondent, Statistics in Transition New Series, **20(3)**, 81–95, 2019.

[23] S. Bhushan and A. P. Pandey, An improved estimation procedure of population mean using bivariate auxiliary information under non-response, Communications for Statistical Applications and Methods, **26(4)**, 347–357, 2019.
[24] C. Ünal and C. Kadilar, Improved family of estimators using exponential function for the population mean in the presence of non-response, Communications in Statistics - Theory and Methods, **50(1)**, 237–248, 2019.
[25] V. Sharma and S. Kumar, Estimation of population mean using transformed auxiliary variable and non-response, REVISTA INVESTIGACION OPERACIONAL 41(3), 438–444, 2020.
[26] W. A. Fuller, Measurement Error Models, New York: Wiley.
[27] P. Biemer and S. L. Stokes, Approaches to the modeling of measurement error, In B. et al. (Ed.), Measurement error in Surveys, Hoboken, NJ, pp. 485–516. Wiley, 1991.
[28] Shalabh, Ratio method of estimation in the presence of measurement error, Journal of Indian Society of Agricultural Statistics, **50(2)**, 150-155, 1997.
[29] H. Singh and N. Karpe, Ratio-product estimator for population mean in presence of measurement errors, Journal of Applied Statistical Science, **16(4)**, 437–452, 2008.
[30] M. R. Kumar, R. Singh, N. Sawan, and P. Chauhan, Exponential ratio method of estimators in the presence of measurement errors, International Journal of Agricultural and Statistical Sciences, **7(2)**, 457–461, 2011.
[31] T. G. Gregoire and C. Salas, Ratio estimation with measurement error in the auxiliary variate, Biometrics, **65(2)**, 590–598, 2009.
[32] G. Diane and M. Giordan, Finite population variance estimation in presence of measurement errors, Communications in Statistics - Theory and Methods, **41(23)**, 4302–4314, 2012.
[33] D. Shukla, S. Pathak, and N. Thakur, An estimator for mean estimation in presence of measurement error, Research and Reviews: A Journal of Statistics, **1(1)**, 1–8, 2012.
[34] Shalabh and J. R. Tsai, Ratio and product methods of estimation of population mean in the presence of correlated measurement errors, Communications in Statistics - Simulation and Computation, **46(7)**, 5566–5593, 2017.
[35] K. K. Tiwari, S. Bhougal, S. Kumar, and K. U. I. Rather, Using Randomized Response to Estimate the Population Mean of a Sensitive Variable under the Influence of Measurement Error, Journal of Statistical Theory and Practice, **16(2)**, 1-11, 2022.
[36] K. K. Tiwari, S. Bhougal, S. Kumar, and R. Onyango, Assessing the effect of nonresponse and measurement error using a novel class of efficient estimators, Journal of Mathematics, 2022, Article ID 4946265, 1–9, 2022. doi: 10.1155/2022/4946265.
[37] S. Jackman, Correcting surveys for non-response and measurement error using auxiliary information, Electoral Studies, **18(1)**, 7–27, 1999.
[38] P. P. Biemer, Nonresponse bias and measurement bias in a comparison of face to face and telephone interviewing, Journal of Official Statistics, **17 (2)**, 295–320, 2001.
[39] J. J. Hox, E. D. Leeuw, and H. T. Chang, Nonresponse versus measurement error: Are reluctant respondents worth pursuing? Bulletin of Sociological Methodology, **113(1)**, 5–19, 2012.
[40] S. Kumar, S. Bhougal, and N. Nataraja, Estimation of population mean in the presence of non-response and measurement error, Revista Colombiana de Estadistica, **38(1)**, 145–161, 2015.
[41] R. Singh and P. Sharma, Method of estimation in the presence of non-response and measurement errors simultaneously, Journal of Modern Applied Statistical Methods, **14(1)**, 107–121, 2015.
[42] M. Azeem and M. Hanif, Joint influence of measurement error and non-response on estimation of population mean, Communications in Statistics - Theory and Methods, **46(4)**, 1679–1693, 2016.
[43] S. Kumar, Improved estimation of population mean in presence of non-response and measurement error, Journal of Statistical Theory and Practice, **10(4)**, 707–720, 2016.
[44] S. Kumar and S. Bhougal (2018). Study on non response and measurement error using double sampling scheme, J. Stat. Appl. Pro. Lett. **5(1)**, 43–52, 2018.
[45] S. Kumar, M. Trehan, and J. P. S. Joorel, A simulation study: estimation of population mean using two auxiliary variables in stratified random sampling, Journal of Statistical Computation and Simulation, **88(18)**, 3694–3707, 2018.
[46] G. N. Singh, D. Bhattacharyya, and A. Bandyopadhyay, Formulation of logarithmic type estimators to estimate population mean in successive sampling in presence of random non response and measurement errors, Communications in Statistics - Simulation and Computation, **51(3)**, 901-923, 2019.
[47] D. T. Searls, The utilization of a known coefficient of variation in the estimation procedure, Journal of the American Statistical Association, **59**, 1225–1226, 1964.
[48] M. Murthy, Product method of estimation, Sankhya A, **26(1)**, 69–74, 1964.
[49] T. J. Rao, On certail methods of improving ration and regression estimators, Communications in Statistics - Theory and Methods, **20(10)**, 3325–3340, 1991.
[50] S. Bahl and R. Tuteja, Ratio and product-type exponential estimators, Journal of Information and Optimization Sciences, **12(1)**, 159–164, 1991.
[51] C. Kadilar and H. Cingi, Ratio estimators in simple random sampling, Applied Mathematics and Computation, **151(5)**, 893–902, 2004.
[52] L. K. Grover and P. Kaur, An improved estimator of the finite population mean in simple random sampling, Model Assisted Statistics and Applications, **6(1)**, 47–55, 2011.
[53] W. G. Cochran, The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce, The Journal of Agricultural Science, **30(2)**, 262–275, 1940.