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Fractional Stochastic Modelling of an Investment Model

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Abstract: Financial models have become very crucial in many economics because of the uncertainties within the environment. Investors are always looking for better returns on their investments and therefore, a good financial model is important to any society. In this study, a financial mathematical model is examined in the context of fractional stochastic. The existence and uniqueness of solution of the financial model is studied. A new numerical approach based on Newton polynomial interpolation is utilised to numerically investigate the financial model's dynamics. It is observed that fractional order derivative affects the dynamics of the financial model. The Atangana -Baleanu operator shows better prediction comparing to the Caputo and Caputo-Fabrizio operators. It is suggested that other complex dynamics can be investigated using the newly developed numerical scheme based on the Atangana-Baleanu operator.

Keywords: Caputo-Fabrizio, Atangana-Baleanu operator, Newton polynomial interpolation, Caputo, financial model.

1 Introduction

Many researchers have identified financial models as one the most interesting subject matter, particularly if it exhibits chaotic behaviours. The dynamical characteristics of many financial models present crucial information for decision-making in planning for the future. The world economy has been experiencing some fluctuations in many recent years, and it is difficult to provide a better outlook for the future [1]. The world is becoming more complex than before, and therefore, appropriate financial models could help understand the current situation and adopt the appropriate strategies in managing the system. Understanding such complexity with appropriate mathematical methodologies is crucial than ever been before [2,3,4].

Human activities have been noticed to be associated with the rate of change in most cases. The concepts of differential and integral calculus had been the bedrock of recent development in most of the various aspects of applied mathematics in areas including; financial sector, economics, engineering, space science, and many more [5,6]. Researchers have developed different aspects of fractional derivatives such as fractional fractal order, fractional variable-order derivatives, fractional stochastic, etc., with varying operators. These operators go a long way to provide avenues for accurate prediction [1]. Mankind can analyse and predict accurately using these operators.

The complexity of the modern world and its associated problems keep pushing researchers to make advances in knowledge. Consequently, the current version of the fractional stochastic derivative has been identified as a function that possesses the characteristics to deal with such complexities [7]. The new version of the fractional stochastic derivative is positive, increasing, and also non-zero. Whenever the two functions is considered to be differentiable in non-integer dimension, the corresponding integral is observed to be the popular Riemann-Liouville integral [1,5]. The recently developed derivatives and the associated integral is known to be superior to the existing classical ones [1,3,6,7]. There is, therefore, some good news for researchers ranging from deterministic, integral, partial, differential, stochastic, etc., to expand the frontier of knowledge in applied mathematics.

The financial sector is one of the areas in the economy that does not have long-term stability due to the unstable nature of the industry. In this COVID-19 pandemic, the world financial outlook has not been the best. Some countries are

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struggling to make financial resources available to run their economies. The economy could require an excellent economic mathematical model with appropriate mathematical tools to provide accurate qualitative information for decision-making processes.

In [8], author stressed the relationship that exists between the quantity of investment and the size of employment. In [9], the author studied the mathematical growth model focusing on the fact that to provide full employment demands, a substantial investment growth should be maintained all the time. Authors in [2] constructed a novel financial mathematical model and present the dynamical stability analysis of the model using the Lyapunov global stability approach. In [10], authors considered the synchronization of a financial model by employing the concept of the active control approach. In [11], authors examined an economic model using differential fractional-order operators and comparing the results of the operators. Authors in [12] utilised the concept of fractional calculus to investigate 3D IS-LM macroeconomic model. Authors in [13] constructed a 4D financial model and used the Atangana-Baleanu operator in Caputo sense to examine the dynamics of the model.

Authors in [14] developed a financial model and investigated the model using the Mittag-Leffler function in the context of stochastic. Authors in [15] constructed a complex chaotic financial with a delay and study the dynamics of the model using fractional-order derivatives.

The random effects in the fractional stochastic models will be utilised in this work to provide a better qualitative information about the financial model

2 Connected Definition

In this section the appropriate definitions that are based on the differential and integral operators are presented [16].

$$ {}_0^{C}D_t^w u(t) = \frac{1}{\Gamma(1-w)} \int_0^t \frac{d}{d\tau} u(\tau)(t-\tau)^{-w} d\tau \tag{1} $$

$$ {}_0^{RL}D_t^w u(t) = \frac{1}{\Gamma(1-w)} \frac{d}{dt} \int_0^t u(\tau)(t-\tau)^{-w} d\tau \tag{2} $$

$$ {}_0^{CF}D_t^w u(t) = \frac{M(w)}{1-w} \int_0^t \frac{d}{d\tau} u(\tau) \exp\left[-\frac{w}{1-w}(t-\tau)\right] d\tau \tag{3} $$

$$ {}_0^{AB}D_t^w u(t) = \frac{AB(w)}{1-w} \int_0^t \frac{d}{d\tau} u(\tau) E_w \left[-\frac{w}{1-w}(t-\tau)^w\right] d\tau \tag{4} $$

$$ {}_0^{ABR}D_t^w u(t) = \frac{AB(w)}{1-w} \frac{d}{dt} \int_0^t u(\tau) E_w \left[-\frac{w}{1-w}(t-\tau)^w\right] d\tau \tag{5} $$

where we notice that

$$ E_w(-t^w) = \sum_{k=0}^{\infty} \frac{(-t)^{wk}}{\Gamma(wk+1)} $$

3 Model Formulation

This study examines a financial model constructed by Xu et al.,[1] in which the state variables has to do with interest rate $v_1(t)$,the investment demand $v_2(t)$, the price index $v_3(t)$ and saving amount an individual possesses $v_4(t)$ and as a result of the various interactions among the compartments. The following fractional order derivative in Atangana- Baleanu in

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Caputo sense is obtained. The parameter meaning are the same as in Xu et al.,[1].

$$\begin{aligned}
\frac{dv_1}{dt} &= pv_3(t) + (v_2(t) - v_4(t))v_1(t),\\
\frac{dv_2}{dt} &= 1 - bv_2(t) - v_1^2(t) + dv_2(t)v_4(t),\\
\frac{dv_3}{dt} &= -v_1(t) - cv_3(t),\\
\frac{dv_4}{dt} &= p - qv_4(t) + (mv_1(t) - mv_2(t))v_4(t)
\end{aligned} \tag{6}$$

with the following initial conditions $v_1(t) = 0.1$, $v_2(t) = 0.1$, $v_3(t) = 0.1$, $v_4(t) = 0.1$ In this regard, a stochastic aspect is incorporated into model (6) with respect to Caputo.

$$\begin{aligned}
{}_0^C D_t^w v_1(t) &= pv_3 + (v_2 - v_4)v_1 + \phi_1 v_1(t)dU_1(t),\\
{}_0^C D_t^w v_2(t) &= 1 - bv_2 - v_1^2 + dv_2v_4 + \phi_2 v_2(t)dU_2(t),\\
{}_0^C D_t^w v_3(t) &= -v_1 - cv_3 + \phi_3 v_3(t)dU_3(t),\\
{}_0^C D_t^w v_4(t) &= p - qv_4 + (mv_1 - mv_2)v_4 + \phi_4 v_4(t)dU_4(t)
\end{aligned} \tag{7}$$

In this aspect of the work, Caputo-Fabrizio operator is incorporated in the system equation (6)

$$\begin{aligned}
{}_0^{CF} D_t^w v_1(t) &= pv_3 + (v_2 - v_4)v_1 + \phi_1 v_1(t)dU_1(t),\\
{}_0^{CF} D_t^w v_2(t) &= 1 - bv_2 - v_1^2 + dv_2v_4 + \phi_2 v_2(t)dU_2(t),\\
{}_0^{CF} D_t^w v_3(t) &= -v_1 - cv_3 + \phi_3 v_3(t)dU_3(t),
\end{aligned} \tag{8}$$

$${}_0^{CF} D_t^w v_4(t) = p - qv_4 + (mv_1 - mv_2)v_4 + \phi_4 v_4(t)dU_4(t) \tag{9}$$

Finally, the system equation (6) is reformulated by including Atangana-Baleanu operator in Caputo sense as follows.

$$\begin{aligned}
{}_0^{AB} D_t^w v_1(t) &= pv_3 + (v_2 - v_4)v_1 + \phi_1 v_1(t)dU_1(t),\\
{}_0^{AB} D_t^w v_2(t) &= 1 - bv_2 - v_1^2 + dv_2v_4 + \phi_2 v_2(t)dU_2(t),\\
{}_0^{AB} D_t^w v_3(t) &= -v_1 - cv_3 + \phi_3 v_3(t)dU_3(t),\\
{}_0^{AB} D_t^w v_4(t) &= p - qv_4 + (mv_1 - mv_2)v_4 + \phi_4 v_4(t)dU_4(t)
\end{aligned} \tag{10}$$

where ϕ_i for $i = 1,2,3,4$ represent the magnitude of the stochastic environment and $U_i(t)$ for $i = 1,2,3,4$ is the standard Brownian motion. Many physical systems depict stochastic behaviours such flatualations in financial market, epidemiology and many others. Due to crossover behaviours of many fractional derivative operators and accurate predictions a lot of researchers are adopting this concept for modelling many processes. This work adopted the three common operators namely Caputo, Caputo-Fabrizio and Atangana-Beleanu operators respectively:

4 Existence and Uniqueness

Theorem 1. It is considered that there exists two positive constant A, $\overline{A}$ such that

$$\forall P \in \{1, \dots, 4\}$$

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$$\left|z(y,t)-z(y',t)\right| \leq A\left|y-y'\right|^2 \tag{11}$$

and

$$\forall\ (y,t) \in Q\times[0,T]$$

$$|z(y,t)|^2 \leq \overline{A}\left(1+|y|^2\right) \tag{12}$$

Proof. In this regard, we begin by examining the function $z_1(t,v_1,v_2,v_3,v_4)$. The first condition is verified by the following:

$$\begin{aligned}
|z_1(v_1,t)-z_1(v_1^*,t)|^2 &= |v_2(v_1-v_1^*)-v_4(v_1-v_1^*)|^2 \\
&= |(v_2-v_4)(v_1-v_1^*)|^2 \\
&\leq \left\{2|v_2|^2+2|v_4|^2\right\}|v_1-v_1^*|^2 \\
&\leq \left\{2\sup_{0\leq t\leq T}|v_2|^2+2\sup_{0\leq t\leq T}|v_4|^2\right\}|v_1-v_1^*|^2 \\
&\leq \left\{2\|v_2\|_\infty^2+2\|v_4\|_\infty^2\right\}|v_1-v_1^*|^2 \\
&\leq A_1|v_1-v_1^*|^2
\end{aligned} \tag{13}$$

$$\begin{aligned}
|z_2(v_2,t)-z_2(v_2^*,t)|^2 &= |-b(v_2-v_2^*)+dv_4(v_2-v_2^*)|^2 \\
&= |(-b+dv_4)(v_2-v_2^*)|^2 \\
&\leq \left\{2b^2+2d^2|v_4|^2\right\}|v_2-v_2^*|^2 \\
&\leq \left\{2b^2+2d^2\sup_{0\leq t\leq T}|v_4|^2\right\}|v_2-v_2^*|^2 \\
&\leq \left\{2b^2+2d^2\|v_4\|_\infty^2\right\}|v_2-v_2^*|^2 \\
&\leq A_2|v_2-v_2^*|^2
\end{aligned} \tag{14}$$

$$\begin{aligned}
|z_3(v_3,t)-z_3(v_3^*,t)|^2 &= |-c(v_3-v_3^*)|^2 \\
&\leq c^2|v_3-v_3^*|^2 \\
&\leq \left\{c^2+l_1\right\}|v_3-v_3^*|^2 \\
&\leq A_3|v_3-v_3^*|^2
\end{aligned} \tag{15}$$

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$$
\begin{aligned}
|z_4(v_4,t)-z_4(v_2^*,t)|^2 &= |-q(v_4-v_4^*)+(mv_1-mv_2)(v_4-v_4^*)|^2 \\
&= |(-q+mv_1-mv_2)(v_4-v_4^*)|^2 \\
&\leq \left\{3q^2+3m^2|v_1|^2+3m^2|v_2|^2\right\}|v_4-v_4^*|^2 \\
&\leq \left\{3q^2+3m^2\sup_{0\leq t\leq T}|v_1|^2+3m^2\sup_{0\leq t\leq T}|v_2|^2\right\}|v_4-v_4^*|^2 \\
&\leq \left\{3q^2+3m^2\|v_1\|_\infty^2+3m^2\|v_2\|_\infty^2\right\}|v_4-v_4^*|^2 \\
&\leq A_4|v_4-v_4^*|^2
\end{aligned} \tag{16}
$$

The second condition was examined as follows:

$$
\begin{aligned}
|z_1(v_1,t)|^2 &= |pv_3+(v_2-v_4)v_1|^2 \\
&\leq \left(3p^2|v_3|^2+3|v_2|^2|v_1|^2+3|v_4|^2|v_1|^2\right) \\
&\leq 3\left(p^2\sup_{0\leq t\leq T}|v_3|^2+\sup_{0\leq t\leq T}|v_2|^2|v_1|^2+\sup_{0\leq t\leq T}|v_4|^2|v_1|^2\right) \\
&\leq 3\left(p^2\|v_3\|_\infty^2+\|v_2\|_\infty^2|v_1|^2+\|v_4\|_\infty^2|v_1|^2\right) \\
&\leq 3\left(p^2\|v_3\|_\infty^2\right)\left(1+\frac{\|v_2\|_\infty^2+\|v_4\|_\infty^2}{p^2\|v_3\|_\infty^2}|v_1|^2\right) \\
&\leq \overline{A_1}\left(1+|v_1|^2\right)
\end{aligned} \tag{17}
$$

under the condition $\frac{\|v_2\|_\infty^2+\|v_4\|_\infty^2}{p^2\|v_3\|_\infty^2}<1$

$$
\begin{aligned}
|z_2(v_2,t)|^2 &= \left|1-bv_2-v_1^2+dv_2v_4\right|^2 \\
&\leq \left(4+4b^2|v_2|^2+4\left|v_1^2\right|^2+4d^2|v_2|^2|v_4|^2\right) \\
&\leq 4\left(1+b^2|v_2|^2+\sup_{0\leq t\leq T}\left|v_1^2\right|^2+d^2|v_2|^2\sup_{0\leq t\leq T}|v_4|^2\right) \\
&\leq 4\left(1+b^2|v_2|^2+\left\|v_1^2\right\|_\infty^2+d^2\|v_4\|_\infty^2\right) \\
&\leq 4\left(1+\left\|v_1^2\right\|_\infty^2\right)\left(1+\frac{b^2+d^2|v_2|^2\|v_4\|_\infty^2}{1+\left\|v_1^2\right\|_\infty^2}|v_2|^2\right) \\
&\leq \overline{A_2}\left(1+|v_2|^2\right)
\end{aligned} \tag{18}
$$

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such that $\dfrac{b^2+d^2\|v_4\|_\infty^2}{1+\|v_1^2\|_\infty^2}<1$

$$
\begin{aligned}
|z_3(v_3,t)|^2 &= |-v_1-cv_3|^2 \\
&\leq \left(2|v_1|^2+2c^2|v_3|^2\right) \\
&\leq 2\left(\operatorname*{Sup}_{0\leq t\leq T}|v_1|^2+c^2|v_3|^2\right) \\
&\leq 2\left(\|v_1\|_\infty^2+c^2|v_3|^2\right) \\
&\leq 2\|v_1\|_\infty^2\left(1+\frac{c^2}{\|v_1\|_\infty^2}|v_3|^2\right) \\
&\leq \overline{A_3}\left(1+|v_3|^2\right)
\end{aligned}
\tag{19}
$$

under the condition $\dfrac{c^2}{\|v_1\|_\infty^2}<1.$

$$
\begin{aligned}
|z_4(v_4,t)|^2 &= |p-qv_4+(mv_1-mv_2)v_4|^2 \\
&= \left(4p^2+4q^2|v_4|^2+4m^2|v_1|^2|v_4|^2+4m^2|v_2|^2|v_4|^2\right) \\
&\leq 4\left(p^2+q^2|v_4|^2+m^2\operatorname*{Sup}_{0\leq t\leq T}|v_1|^2|v_4|^2+m^2\operatorname*{Sup}_{0\leq t\leq T}|v_2|^2|v_4|^2\right) \\
&\leq 4\left(p^2+q^2|v_4|^2+m^2\|v_1\|_\infty^2|v_4|^2+m^2\|v_2\|_\infty^2|v_4|^2\right) \\
&\leq 4p^2\left(1+\frac{q^2+m^2\|v_1\|_\infty^2+m^2\|v_2\|_\infty^2}{p^2}|v_4|^2\right) \\
&\leq \overline{A_4}\left(1+|v_4|^2\right)
\end{aligned}
\tag{20}
$$

such that $\dfrac{q^2+m^2\|v_1\|_\infty^2+m^2\|v_2\|_\infty^2}{p^2}<1.$

This result to the following;

$$
\begin{aligned}
|z_1(t,v_1)-z_1(t,v_1^*)|^2 &\leq \frac{3}{2}\phi_1^2|v_1-v_1^*|^2\leq \overline{A}_1|v_1-v_1^*|^2, \\
|z_2(t,v_2)-z_2(t,v_2^*)|^2 &\leq \frac{3}{2}\phi_2^2|v_2-v_2^*|^2\leq \overline{A}_2|v_2-v_2^*|^2, \\
|z_3(t,v_3)-z_3(t,v_3^*)|^2 &\leq \frac{3}{2}\phi_3^2|v_3-v_3^*|^2\leq \overline{A}_3|v_3-v_3^*|^2, \\
|z_4(t,v_4)-z_4(t,v_4^*)|^2 &\leq \frac{3}{2}\phi_4^2|v_4-v_4^*|^2\leq \overline{A}_4|v_4-v_4^*|^2.
\end{aligned}
\tag{21}
$$

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The solution of the investment model exists and unique under the condition established below

$$\max = \left\{ \frac{\|v_2\|_\infty^2+\|v_4\|_\infty^2}{p^2\|v_3\|_\infty^2}, \frac{b^2+d^2\|v_4\|_\infty^2}{1+\|v_1^2\|_\infty^2}, \frac{c^2}{\|v_1\|_\infty^2}, \frac{q^2+m^2\|v_1\|_\infty^2+m^2\|v_2\|_\infty^2}{p^2} \right\} < 1$$

5 Numerical Simulation and Discussion

The following initial conditions were used for the purpose of illustration $v_1(0) = 0.1$, $v_2(0) = 0.1$, $v_3(0) = 0.1$ and $v_4(0) = 0.1$ with associated parameter values $p = 0.01$, $b = 0.1$, $c = 0.01$, $q = 0.01$, $m = 0.02$.

$$\begin{aligned}
{}_0^C D_t^w v_1(t) &= (pv_3 + (v_2 - v_4)v_1) + \phi_1 G_1(t, v_1)U_1'(t), \\
{}_0^C D_t^w v_2(t) &= \left(1 - bv_2 - v_1^2 + dv_2 v_4\right) dt + \phi_2 G_2(t, v_2)U_2'(t), \\
{}_0^C D_t^w v_3(t) &= (-v_1 - cv_3)\, dt + \phi_3 G_3(t, v_3)U_3'(t), \\
{}_0^C D_t^w v_4(t) &= (p - qv_4 + (mv_1 - mv_2)v_4) + \phi_4 G_4(t, v_2)U_4'(t)
\end{aligned} \tag{22}$$

The equation (22) is reorganised and simplified as:

$$\begin{aligned}
{}_0^C D_t^w v_1(t) &= v_1(t, v_1, v_2, v_3, v_4) + \phi_1 G_1(t, v_1)U_1'(t), \\
{}_0^C D_t^w v_2(t) &= v_2(t, v_1, v_2, v_3, v_4) + \phi_2 G_2(t, v_2)U_2'(t), \\
{}_0^C D_t^w v_3(t) &= v_3(t, v_1, v_2, v_3, v_4) + \phi_3 G_3(t, v_3)U_3'(t), \\
{}_0^C D_t^w v_4(t) &= v_4(t, v_1, v_2, v_3, v_4) + \phi_4 G_4(t, v_2)U_4'(t)
\end{aligned} \tag{23}$$

The following numerical scheme is obtained based on system equation (22)

$$\begin{aligned}
v_1^{n+1} = {} & \frac{(\Delta t)^w}{\Gamma(w+1)} \sum_{j=2}^{n} v_1\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \times \Theta \\
& + \frac{(\Delta t)^w}{\Gamma(w+1)} \sum_{j=2}^{n} \phi_1 G_1\left(t_{j-2}, v_1^{j-2}\right) \left(U_1(t_{j-1}), -U_1(t_{j-2})\right) \times \Theta \\
& + \frac{(\Delta t)^w}{\Gamma(w+2)} \sum_{j=2}^{n} \begin{bmatrix} \phi_1 G_1\left(t_{j-1}, v_1^{j-1}\right)\left(U_1(t_j) - U_1(t_{j-1})\right) \\ -\phi_1 G_1\left(t_{j-2}, v_1^{j-2}\right)\left(U_1(t_{j-1}) - U_1(t_{j-2})\right) \end{bmatrix} \times \Sigma \\
& + \frac{(\Delta t)^w}{2\Gamma(w+3)} \sum_{j=2}^{n} \begin{bmatrix} \phi_1 G_1\left(t_j, v_1^{j}\right)\left(U_1(t_{j-1}) - U_1(t_j)\right) \\ -2\phi_1 G_1\left(t_{j-1}, v_1^{j-1}\right)\left(U_1(t_j) - U_1(t_{j-1})\right) \\ \phi_1 G_1\left(t_{j-2}, v_1^{j-2}\right)\left(U_1(t_{j-1}) - U_1(t_{j-2})\right) \end{bmatrix} \times \Delta
\end{aligned} \tag{24}$$

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$$
\begin{aligned}
&+\frac{(\Delta t)^w}{\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix} v_1\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right) \\ -v_1\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Sigma \\
&+\frac{w(\Delta t)^w}{2\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix} v_1\left(t_j,v_1^j,v_2^j,v_3^j,v_4^j\right) \\ -2v_1\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right) \\ +v_1\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Delta
\end{aligned}
$$

$$
\begin{aligned}
v_2^{n+1} &= \frac{(\Delta t)^w}{\Gamma(w+1)}\sum_{j=2}^{n} v_2\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\times\Theta \\
&+\frac{(\Delta t)^w}{\Gamma(w+1)}\sum_{j=2}^{n}\phi_2 G_2\left(t_{j-2},v_2^{j-2}\right)\left(U_2(t_{j-1}),-U_2(t_{j-2})\right)\times\Theta \\
&+\frac{(\Delta t)^w}{\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix}\phi_2 G_2\left(t_{j-1},v_2^{j-1}\right)\left(U_2(t_j)-U_2(t_{j-1})\right) \\ -\phi_2 G_2\left(t_{j-2},v_2^{j-2}\right)\left(U_2(t_{j-1})-U_2(t_{j-2})\right)\end{bmatrix}\times\Sigma \\
&+\frac{(\Delta t)^w}{2\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix}\phi_2 G_2\left(t_j,v_2^j\right)\left(U_2(t_{j-1})-U_2(t_j)\right) \\ -2\phi_2 G_2\left(t_{j-1},v_2^{j-1}\right)\left(U_2(t_j)-U_2(t_{j-1})\right) \\ +\phi_2 G_2\left(t_{j-2},v_2^{j-2}\right)\left(U_2(t_{j-1})-U_2(t_{j-2})\right)\end{bmatrix}\times\Delta \qquad (25) \\
&+\frac{(\Delta t)^w}{\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix} v_2\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right) \\ -v_2\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Sigma \\
&+\frac{w(\Delta t)^w}{2\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix} v_2\left(t_j,v_1^j,v_2^j,v_3^j,v_4^j\right) \\ -2v_2\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right) \\ +v_2\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Delta \\
v_3^{n+1} &= \frac{(\Delta t)^w}{\Gamma(w+1)}\sum_{j=2}^{n} v_3\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\times\Theta \\
&+\frac{(\Delta t)^w}{\Gamma(w+1)}\sum_{j=2}^{n}\phi_3 G_3\left(t_{j-2},v_3^{j-2}\right)\left(U_3(t_{j-1}),-U_3(t_{j-2})\right)\times\Theta
\end{aligned}
$$

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$$+\frac{(\Delta t)^w}{\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix}\phi_3 G_3\left(t_{j-1},v_3^{j-1}\right)\left(U_3(t_j)-U_3(t_{j-1})\right)\\ -\phi_3 G_3\left(t_{j-2},v_3^{j-2}\right)\left(U_3(t_{j-1})-U_3(t_{j-2})\right)\end{bmatrix}\times\Sigma$$

$$+\frac{(\Delta t)^w}{2\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix}\phi_3 G_3\left(t_j,v_3^{j}\right)\left(U_3(t_{j-1})-U_3(t_j)\right)\\ -2\phi_3 G_3\left(t_{j-1},v_3^{j-1}\right)\left(U_3(t_j)-U_3(t_{j-1})\right)\\ +\phi_3 G_3\left(t_{j-2},v_3^{j-2}\right)\left(U_3(t_{j-1})-U_3(t_{j-2})\right)\end{bmatrix}\times\Delta \tag{26}$$

$$+\frac{w(\Delta t)^w}{\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix}v_3\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right)\\ -v_3\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Sigma$$

$$+\frac{w(\Delta t)^w}{2\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix}v_3\left(t_j,v_1^{j},v_2^{j},v_3^{j},v_4^{j}\right)\\ -2v_3\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right)\\ +v_3\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Delta$$

$$v_4^{n+1}=\frac{(\Delta t)^w}{\Gamma(w+1)}\sum_{j=2}^{n}v_4\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\times\Theta$$

$$+\frac{(\Delta t)^w}{\Gamma(w+1)}\sum_{j=2}^{n}\phi_4 G_4\left(t_{j-2},v_4^{j-2}\right)\left(U_4(t_{j-1}),-U_4(t_{j-2})\right)\times\Theta$$

$$+\frac{(\Delta t)^w}{\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix}\phi_4 G_4\left(t_{j-1},v_4^{j-1}\right)\left(U_4(t_j)-U_4(t_{j-1})\right)\\ -\phi_4 G_4\left(t_{j-2},v_4^{j-2}\right)\left(U_4(t_{j-1})-U_4(t_{j-2})\right)\end{bmatrix}\times\Sigma$$

$$+\frac{(\Delta t)^w}{2\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix}\phi_4 G_4\left(t_j,v_4^{j}\right)\left(U_4(t_{j-1})-U_4(t_j)\right)\\ -2\phi_4 G_4\left(t_{j-1},v_4^{j-1}\right)\left(U_4(t_j)-U_4(t_{j-1})\right)\\ +\phi_4 G_4\left(t_{j-2},v_4^{j-2}\right)\left(U_4(t_{j-1})-U_4(t_{j-2})\right)\end{bmatrix}\times\Delta \tag{27}$$

$$+\frac{(\Delta t)^w}{\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix}v_4\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right)\\ -v_4\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Sigma$$

$$+\frac{w(\Delta t)^{w}}{2\Gamma(w+3)}\sum_{j=2}^{n}\left[\begin{array}{l} v_4\left(t_j, v_1^j, v_2^j, v_3^j, v_4^j\right) \\ -2v_4\left(t_{j-1}, v_1^{j-1}, v_2^{j-1}, v_3^{j-1}, v_4^{j-1}\right) \\ +v_4\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \end{array}\right]\times\Delta$$

The following numerical scheme is obtained with respect to Caputo-Fabrizio operator (8):

$$\begin{aligned} v_1^{n+1} &= v_1^n + \frac{1-w}{M(w)}\left[\begin{array}{l} v_1\left(t_n, v_1^n, v_2^n, v_3^n, v_4^n\right) \\ -v_1\left(t_{n-1}, v_1^{n-1}, v_2^{n-1}, v_3^{n-1}, v_4^{n-1}\right) \end{array}\right] \\ &+ \frac{w}{M(w)}\left\{\begin{array}{l} \frac{23}{12}v_1\left(t_n, v_1^n, v_2^n, v_3^n, v_4^n\right)\Delta t \\ -\frac{4}{3}v_1\left(t_{n-1}, v_1^{n-1}, v_2^{n-1}, v_3^{n-1}, v_4^{n-1}\right)\Delta t \\ +\frac{5}{12}v_1\left(t_{n-2}, v_1^{n-2}, v_2^{n-2}, v_3^{n-2}, v_4^{n-2}\right)\Delta t \end{array}\right\} \\ &+ \frac{1-w}{M(w)}\phi_1\left[\begin{array}{l} \left(U_1(t_{n+1}) - U_1(t_n)\right)G_1\left(t_{n+1}, v_1^{n+1}\right) \\ -\left(U_1(t_n) - U_1(t_{n-1})\right)G_1\left(t_n, v_1^n\right) \end{array}\right] \\ &+ \frac{w}{M(w)}\phi_1\left\{\begin{array}{l} \frac{5}{12}G_1\left(t_{n-2}, v_1^{n-2}\right)\left(U_1(t_{n-1}) - U_1(t_{n-2})\right) \\ -\frac{4}{3}G_1\left(t_{n-1}, v_1^{n-1}\right)\left(U_1(t_n) - U_1(t_{n-1})\right) \\ +\frac{23}{12}G_1\left(t_n, v_1^n\right)\left(U_1(t_{n+1}) - U_1(t_n)\right) \end{array}\right\} \end{aligned} \tag{28}$$

$$\begin{aligned} v_2^{n+1} &= v_2^n + \frac{1-w}{M(w)}\left[\begin{array}{l} v_2\left(t_n, v_1^n, v_2^n, v_3^n, v_4^n\right) \\ -v_2\left(t_{n-1}, v_1^{n-1}, v_2^{n-1}, v_3^{n-1}, v_4^{n-1}\right) \end{array}\right] \\ &+ \frac{w}{M(w)}\left\{\begin{array}{l} \frac{23}{12}v_2\left(t_n, v_1^n, v_2^n, v_3^n, v_4^n\right)\Delta t \\ -\frac{4}{3}v_2\left(t_{n-1}, v_1^{n-1}, v_2^{n-1}, v_3^{n-1}, v_4^{n-1}\right)\Delta t \\ +\frac{5}{12}v_2\left(t_{n-2}, v_1^{n-2}, v_2^{n-2}, v_3^{n-2}, v_4^{n-2}\right)\Delta t \end{array}\right\} \\ &+ \frac{1-w}{M(w)}\phi_2\left[\begin{array}{l} \left(U_2(t_{n+1}) - U_2(t_n)\right)G_2\left(t_{n+1}, v_2^{n-1}\right) \\ -\left(U_2(t_n) - U_2(t_{n-1})\right)G_2\left(t_n, v_1^n\right) \end{array}\right] \end{aligned} \tag{29}$$

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$$+\frac{w}{M(w)}\phi_2\begin{Bmatrix}\frac{5}{12}G_2(t_{n-2},v_2^{n-2})\left(U_2(t_{n-1})-U_2(t_{n-2})\right)\\-\frac{4}{3}G_2(t_{n-1},v_2^{n-1})\left(U_2(t_n)-U_2(t_{n-1})\right)\\+\frac{23}{12}G_2(t_n,v_2^n)\left(U_2(t_{n+1})-U_2(t_n)\right)\end{Bmatrix}$$

$$v_3^{n+1}=v_3^n+\frac{1-w}{M(w)}\begin{bmatrix}v_3\left(t_n,v_1^n,v_2^n,v_3^n,v_4^n\right)\\-v_3\left(t_{n-1},v_1^{n-1},v_2^{n-1},v_3^{n-1},v_4^{n-1}\right)\end{bmatrix}$$

$$+\frac{w}{M(w)}\begin{Bmatrix}\frac{23}{12}v_3\left(t_n,v_1^n,v_2^n,v_3^n,v_4^n\right)\Delta t\\-\frac{4}{3}v_3\left(t_{n-1},v_1^{n-1},v_2^{n-1},v_3^{n-1},v_4^{n-1}\right)\Delta t\\+\frac{5}{12}v_3\left(t_{n-2},v_1^{n-2},v_2^{n-2},v_3^{n-2},v_4^{n-2}\right)\Delta t\end{Bmatrix}$$

$$+\frac{1-w}{M(w)}\phi_3\begin{bmatrix}\left(U_3(t_{n+1})-U_3(t_n)\right)G_2(t_{n+1},v_3^{n+1})\\-\left(U_3(t_n)-U_3(t_{n-1})\right)G_3(t_n,v_3^n)\end{bmatrix} \tag{30}$$

$$+\frac{w}{M(w)}\phi_3\begin{Bmatrix}\frac{5}{12}G_3(t_{n-2},v_3^{n-2})\left(U_3(t_{n-1})-U_3(t_{n-2})\right)\\-\frac{4}{3}G_3(t_{n-1},v_3^{n-1})\left(U_3(t_n)-U_3(t_{n-1})\right)\\+\frac{23}{12}G_3(t_n,v_3^n)\left(U_3(t_{n+1})-U_3(t_n)\right)\end{Bmatrix}$$

$$v_4^{n+1}=v_4^n+\frac{1-w}{M(w)}\begin{bmatrix}v_4\left(t_n,v_1^n,v_2^n,v_3^n,v_4^n\right)\\-v_4\left(t_{n-1},v_1^{n-1},v_2^{n-1},v_3^{n-1},v_4^{n-1}\right)\end{bmatrix}$$

$$+\frac{w}{M(w)}\begin{Bmatrix}\frac{23}{12}v_4\left(t_n,v_1^n,v_2^n,v_3^n,v_4^n\right)\Delta t\\-\frac{4}{3}v_4\left(t_{n-1},v_1^{n-1},v_2^{n-1},v_3^{n-1},v_4^{n-1}\right)\Delta t\\+\frac{5}{12}v_4\left(t_{n-2},v_1^{n-2},v_2^{n-2},v_3^{n-2},v_4^{n-2}\right)\Delta t\end{Bmatrix} \tag{31}$$

$$+\frac{1-w}{M(w)}\phi_4\begin{bmatrix}\left(U_4(t_{n+1})-U_4(t_n)\right)G_4(t_{n+1},v_4^{n+1})\\-\left(U_4(t_n)-U_4(t_{n-1})\right)G_4(t_n,v_4^n)\end{bmatrix}$$

$$+\frac{w}{M(w)}\phi_4\begin{Bmatrix}+\frac{5}{12}G_4(t_{n-2},v_4^{n-2})\left(U_4(t_{n-1})-U_4(t_{n-2})\right)\\-\frac{4}{3}G_4(t_{n-1},v_4^{n-1})\left(U_4(t_n)-U_4(t_{n-1})\right)\\+\frac{23}{12}G_4(t_n,v_4^n)\left(U_4(t_{n+1})-U_4(t_n)\right)\end{Bmatrix}$$

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The following (AB) numerical scheme is obtained with respect to (10):

$$\begin{aligned}
v_1^{n+1} &= \frac{1-w}{AB(w)} v_1\left(t_n, v_1^n, v_2^n, v_3^n, v_4^n\right) \\
&+ \frac{w(\Delta t)^w}{AB(w)\Gamma(w+1)} \sum_{j=2}^{n} v_1\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \times \Theta \\
&+ \frac{w(\Delta t)^w}{AB(w)\Gamma(w+1)} \sum_{j=2}^{n} \phi_1 G_1\left(t_{j-2}, v_1^{j-2}\right)\left(U_1(t_{j-1}), -U_1(t_{j-2})\right) \times \Theta \\
&+ \frac{w(\Delta t)^w}{AB(w)\Gamma(w+2)} \sum_{j=2}^{n} \begin{bmatrix} \phi_1 G_1\left(t_{j-1}, v_1^{j-1}\right)\left(U_1(t_j) - U_1(t_{j-1})\right) \\ -\phi_1 G_1\left(t_{j-2}, v_1^{j-2}\right)\left(U_1(t_{j-1}) - U_1(t_{j-2})\right) \end{bmatrix} \times \Sigma \\
&+ \frac{w(\Delta t)^w}{2AB(w)\Gamma(w+3)} \sum_{j=2}^{n} \begin{bmatrix} \phi_1 G_1\left(t_j, v_1^j\right)\left(U_1(t_{j-1}) - U_1(t_j)\right) \\ -2\phi_1 G_1\left(t_{j-1}, v_1^{j-1}\right)\left(U_1(t_j) - U_1(t_{j-1})\right) \\ +\phi_1 G_1\left(t_{j-2}, v_1^{j-2}\right)\left(U_1(t_{j-1}) - U_1(t_{j-2})\right) \end{bmatrix} \times \Delta \\
&+ \frac{w(\Delta t)^w}{AB(w)\Gamma(w+2)} \sum_{j=2}^{n} \begin{bmatrix} v_1\left(t_{j-1}, v_1^{j-1}, v_2^{j-1}, v_3^{j-1}, v_4^{j-1}\right) \\ -v_1\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \end{bmatrix} \times \Sigma \\
&+ \frac{w(\Delta t)^w}{2AB(w)\Gamma(w+3)} \sum_{j=2}^{n} \begin{bmatrix} v_1\left(t_j, v_1^j, v_2^j, v_3^j, v_4^j\right) \\ -2v_1\left(t_{j-1}, v_1^{j-1}, v_2^{j-1}, v_3^{j-1}, v_4^{j-1}\right) \\ +v_1\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \end{bmatrix} \times \Delta
\end{aligned} \tag{32}$$

$$\begin{aligned}
v_2^{n+1} &= \frac{1-w}{AB(w)} v_2\left(t_n, v_1^n, v_2^n, v_3^n, v_4^n\right) \\
&+ \frac{w(\Delta t)^w}{AB(w)\Gamma(w+1)} \sum_{j=2}^{n} v_2\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \times \Theta \\
&+ \frac{w(\Delta t)^w}{AB(w)\Gamma(w+1)} \sum_{j=2}^{n} \phi_2 G_2\left(t_{j-2}, v_2^{j-2}\right)\left(U_2(t_{j-1}), -U_2(t_{j-2})\right) \times \Theta \\
&+ \frac{w(\Delta t)^w}{AB(w)\Gamma(w+2)} \sum_{j=2}^{n} \begin{bmatrix} \phi_2 G_2\left(t_{j-1}, v_2^{j-1}\right)\left(U_2(t_j) - U_2(t_{j-1})\right) \\ -\phi_2 G_2\left(t_{j-2}, v_2^{j-2}\right)\left(U_2(t_{j-1}) - U_2(t_{j-2})\right) \end{bmatrix} \times \Sigma
\end{aligned} \tag{33}$$

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$$+\frac{w(\Delta t)^{w}}{2AB(w)\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix}\phi_2G_2\left(t_j,v_2^j\right)\left(U_2(t_{j-1})-U_2(t_j)\right)\\ -2\phi_2G_2\left(t_{j-1},v_2^{j-1}\right)\left(U_2(t_j)-U_2(t_{j-1})\right)\\ +\phi_2G_2\left(t_{j-2},v_2^{j-2}\right)\left(U_2(t_{j-1})-U_2(t_{j-2})\right)\end{bmatrix}\times\Delta \quad (34)$$

$$+\frac{w(\Delta t)^{w}}{AB(w)\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix}v_2\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right)\\ -v_2\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\sum$$

$$+\frac{w(\Delta t)^{w}}{2AB(w)\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix}v_2\left(t_j,v_1^j,v_2^j,v_3^j,v_4^j\right)\\ -2v_2\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right)\\ +v_2\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Delta$$

$$\begin{aligned}v_3^{n+1}=&\frac{1-w}{AB(w)}v_3\left(t_n,v_1^n,v_2^n,v_3^n,v_4^n\right)\\ &+\frac{w(\Delta t)^{w}}{AB(w)\Gamma(w+1)}\sum_{j=2}^{n}v_3\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\times\Theta\\ &+\frac{w(\Delta t)^{w}}{AB(w)\Gamma(w+1)}\sum_{j=2}^{n}\phi_3G_3\left(t_{j-2},v_3^{j-2}\right)\left(U_3(t_{j-1}),-U_3(t_{j-2})\right)\times\Theta\\ &+\frac{w(\Delta t)^{w}}{AB(w)\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix}\phi_3G_3\left(t_{j-1},v_3^{j-1}\right)\left(U_3(t_j)-U_3(t_{j-1})\right)\\ -\phi_3G_3\left(t_{j-2},v_2^{j-2}\right)\left(U_3(t_{j-1})-U_3(t_{j-2})\right)\end{bmatrix}\times\sum\\ &+\frac{w(\Delta t)^{w}}{2AB(w)\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix}\phi_3G_3\left(t_j,v_3^j\right)\left(U_3(t_{j-1})-U_3(t_j)\right)\\ -2\phi_3G_3\left(t_{j-1},v_3^{j-1}\right)\left(U_3(t_j)-U_3(t_{j-1})\right)\\ +\phi_3G_3\left(t_{j-2},v_3^{j-2}\right)\left(U_3(t_{j-1})-U_3(t_{j-2})\right)\end{bmatrix}\times\Delta \quad (35)\\ &+\frac{w(\Delta t)^{w}}{AB(w)\Gamma(w+2)}\sum_{j=2}^{n}\begin{bmatrix}v_3\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right)\\ -v_3\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\sum\\ &+\frac{w(\Delta t)^{w}}{2AB(w)\Gamma(w+3)}\sum_{j=2}^{n}\begin{bmatrix}v_3\left(t_j,v_1^j,v_2^j,v_3^j,v_4^j\right)\\ -2v_3\left(t_{j-1},v_1^{j-1},v_2^{j-1},v_3^{j-1},v_4^{j-1}\right)\\ +v_3\left(t_{j-2},v_1^{j-2},v_2^{j-2},v_3^{j-2},v_4^{j-2}\right)\end{bmatrix}\times\Delta\end{aligned}$$

$$
\begin{aligned}
v_4^{n+1} = & \frac{1-w}{AB(w)} v_4\left(t_n, v_1^n, v_2^n, v_3^n, v_4^n\right) \\
& + \frac{w(\Delta t)^w}{AB(w)\Gamma(w+1)} \sum_{j=2}^{n} v_4\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \times \Theta \\
& + \frac{w(\Delta t)^w}{AB(w)\Gamma(w+1)} \sum_{j=2}^{n} \phi_4 G_4\left(t_{j-2}, v_4^{j-2}\right)\left(U_4(t_{j-1}), -U_4(t_{j-2})\right) \times \Theta \\
& + \frac{w(\Delta t)^w}{AB(w)\Gamma(w+2)} \sum_{j=2}^{n} \begin{bmatrix} \phi_4 G_4\left(t_{j-1}, v_3^{j-1}\right)\left(U_3(t_j) - U_3(t_{j-1})\right) \\ -\phi_4 G_4\left(t_{j-2}, v_2^{j-2}\right)\left(U_4(t_{j-1}) - U_4(t_{j-2})\right) \end{bmatrix} \times \Sigma \\
& + \frac{w(\Delta t)^w}{2AB(w)\Gamma(w+3)} \sum_{j=2}^{n} \begin{bmatrix} \phi_4 G_4\left(t_j, v_4^j\right)\left(U_4(t_{j-1}) - U_4(t_j)\right) \\ -2\phi_4 G_4\left(t_{j-1}, v_4^{j-1}\right)\left(U_4(t_j) - U_4(t_{j-1})\right) \\ +\phi_4 G_4\left(t_{j-2}, v_4^{j-2}\right)\left(U_4(t_{j-1}) - U_4(t_{j-2})\right) \end{bmatrix} \times \Delta \\
& + \frac{w(\Delta t)^w}{AB(w)\Gamma(w+2)} \sum_{j=2}^{n} \begin{bmatrix} v_4\left(t_{j-1}, v_1^{j-1}, v_2^{j-1}, v_3^{j-1}, v_4^{j-1}\right) \\ -v_4\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \end{bmatrix} \times \Sigma \\
& + \frac{w(\Delta t)^w}{2AB(w)\Gamma(w+3)} \sum_{j=2}^{n} \begin{bmatrix} v_4\left(t_j, v_1^j, v_2^j, v_3^j, v_4^j\right) \\ -2v_4\left(t_{j-1}, v_1^{j-1}, v_2^{j-1}, v_3^{j-1}, v_4^{j-1}\right) \\ +v_4\left(t_{j-2}, v_1^{j-2}, v_2^{j-2}, v_3^{j-2}, v_4^{j-2}\right) \end{bmatrix} \times \Delta
\end{aligned} \tag{36}
$$

where

$$
\begin{aligned}
\Theta &= \left[(n-j+1)^w - (n-j)^w\right], \\
\Sigma &= \begin{bmatrix} (n-j+1)^w - (n-j+3+2w) \\ -(n-j)^w(n-j+3+3w) \end{bmatrix}, \\
\Delta &= \begin{bmatrix} (n-j+1)^w \begin{bmatrix} 2(n-j)^2 + (3w+10)(n-j) \\ +2w^2 + 9w + 12 \end{bmatrix} \\ -(n-j) \begin{bmatrix} 2(n-j)^2 + (5w+10)(n-j) \\ +6w^2 + 18w + 12 \end{bmatrix} \end{bmatrix}
\end{aligned} \tag{37}
$$

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The Figure 1 is the numerical scheme of fractional order stochastic financial model (22) in Caputo operator. Figure 1(a) is the interest rate and it can be observed that the lowest interest rate occurs at day 6th and the highest interest rate occurs on the 15th day as the fractional order derivative get to 1 which is the integer. However, the non-integer aspect indicates volatility in the scheme of interest and one can suggest that planning in such an economy will be difficult. The Figure 1 (b) depicts the investment demand and the lowest investment demand happens on the 7th day and the highest investment that people demanded occurs on the 14th day at 1. The fractional dynamics in this environment is not different from interest rate because one can observe a high fluctuation in the market. The price index is the Figure 1 (c) where the lowest price index occurs on the 15th day and highest index happens on the 6th day concerning the increases of the fractional order at 1. Generally, as the fractional order derivative increases from 0.65 to 0.90, the price index reduces with some volatility in the economy. The Figure 1 (d) is the savings amount the individuals has in the bank and lowest saving time happens on the 6th day and the highest saving day occurs on the 16th day at 1. The fractional order derivative aspect of Figure 1 (d) indicates that individuals' saving amount increases as the fractional order increases towards 0.90 with some amount of fluctuation.

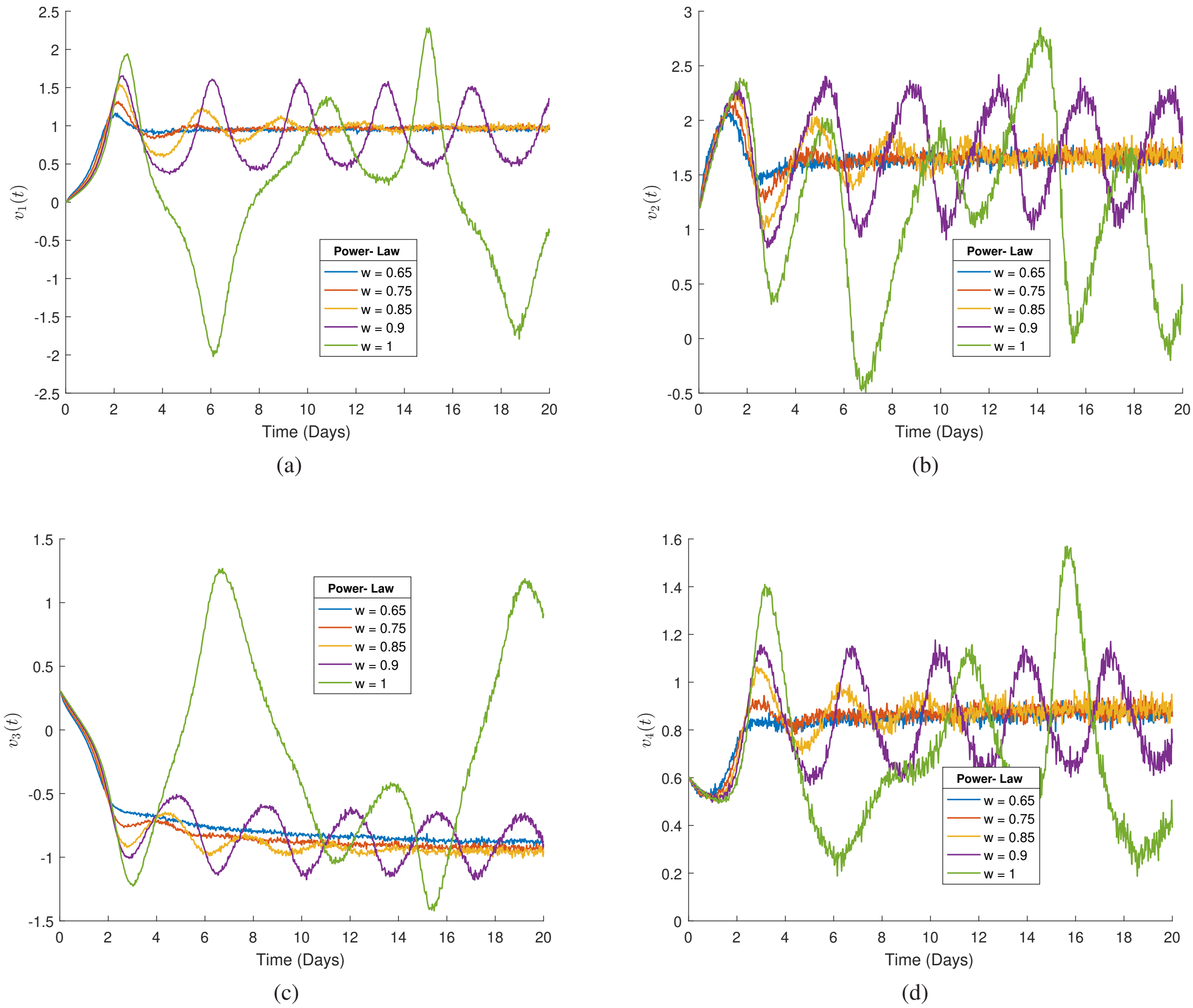

Fig. 1: simulation results for model (22), Caputo at $w = 1, 0.9, 0.80, 0.75, 0.65$ and stochastic constants $\phi = 0.2, 0.4, 0.6, 0.8$

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The Figure 2 depicts the numerical scheme of fractional order stochastic financial model (8) in Caputo-Fabrizio in Caputo sense. Figure 2(a) shows the interest rate and there is high fluctuations in the system and no stable interest rate with the economy as the fractional order derivative increases towards 1. Figure 2(b) shows that the investment demand and similar high fluctuations can be observed, making interest demand difficult because of the high uncertainty. In Figure 2 (c), the price index depicts a similar unstable system. It is difficult to predict what is likely to happen in the next day as the fractional order derivative increases towards 1. The Figure 2 (d) indicates the savings amount the individuals possess in the bank. The volatility is so high such that, it is impossible to predict what an individual is likely to save as the fractional order derivative increases towards 1.

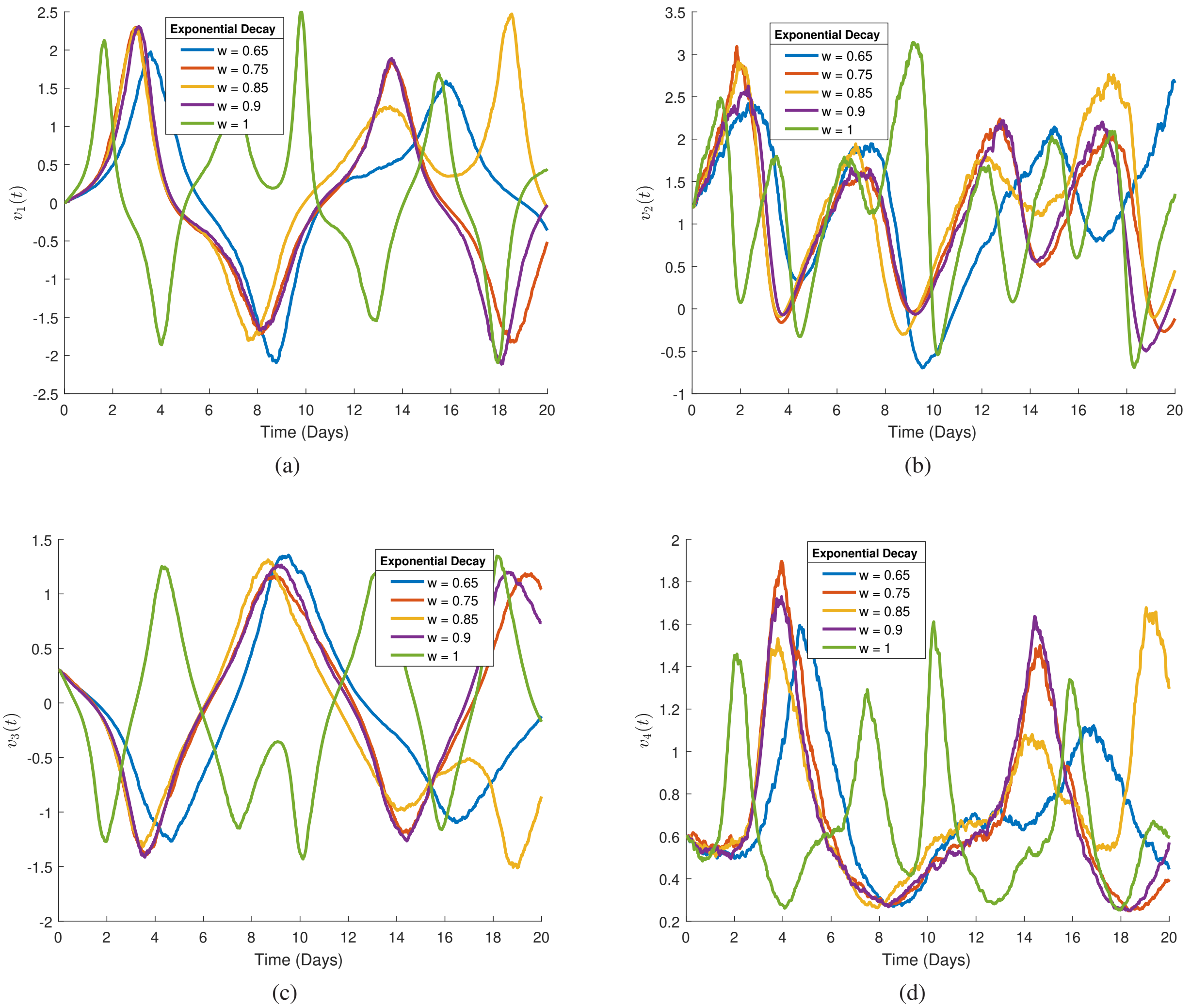

Fig. 2: simulation results for model (8), Caputo-Fabrizio operator at $w = 1, 0.9, 0.80, 0.75, 0.65$ and stochastic constants $\phi = 0.2, 0.4, 0.6, 0.8$

The Figure 3 shows the numerical scheme of the fractional stochastic financial model (10) of Atangana- Baleanu operator in Caputo sense. Figure 3(a) shows the interest rate and one can observe that the lowest interest rate appears on the 6th day and uppermost interest rate transpire on the 16th day at 1. Generally, within the fractional order derivative interval 0.65 to 0.90 one can observe a rise in interest rate. Figure 3 (b) represents the investment demand, and on the 7th day it experiences the lowest investment demand people made and highest investment ensures on the 14th day within the economy at 1. In Figure 3 (b), one can see that the fractional order derivative aspect show some amount of constant

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investment demand and this can afford investment to plan well for the future. The Figure 3 (c) is the price index and the highest price index happens on the 7th day and lowest occurs on the 15th day. Highest index happens on the 6th day concerning the increases of the fractional order derivative towards 1. Generally, as the fractional order increases towards 1. It can be observed from the Figure 3 (c) that from fractional order derivative 0.65 to 0.90 there is a general reduction of price index in the economy. The Figure 3 (d) depicts the savings amount of individual and the least ensues on the 6th day. The uppermost of the savings amount the individual has is on the 16th day at 1 which is the integer order. The non-integer fractional order derivative at 0.65 towards 0.90 shows a rise in the amount of saving.

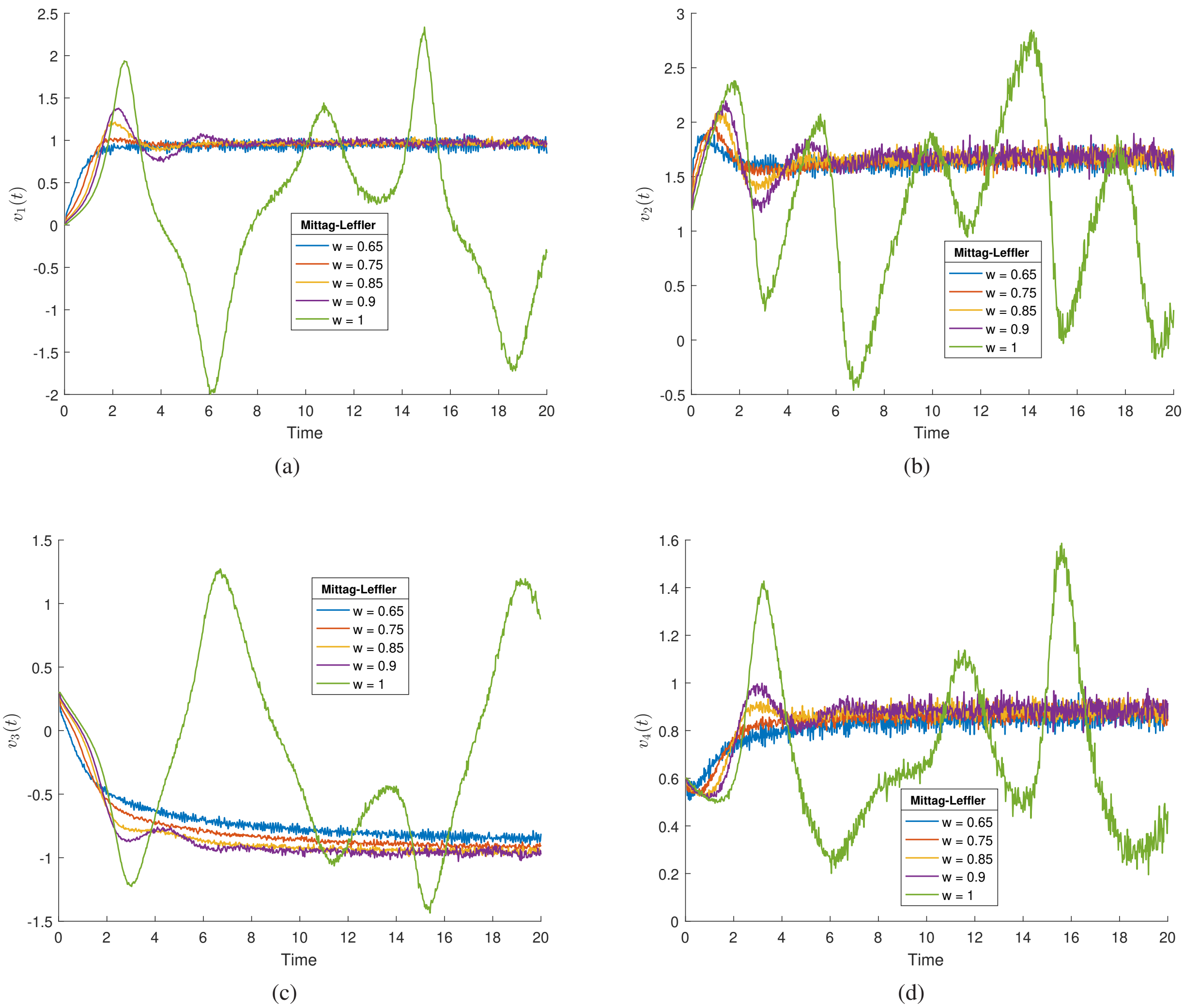

Fig. 3: simulation results for model (10), Atangana-Baleau operator at $w = 1, 0.9, 0.80, 0.75, 0.65$ and stochastic constants $\phi = 0.2, 0.4, 0.6, 0.8$

6 Conclusion

The paper examined the fractional stochastic dynamics of a financial model. The study employed the newly developed numerical scheme based on Newton polynomial interpolation. The existence and uniquenss of the financial model had been shown to exist. The financial model was solved numerically based on Caputo, Caputo-Fabrizio and Atangana-Baleanu operators respectively. It was established that the non- integer order had different dynamics which

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yielded some fluctuations within the financial system. Comparing all the operators, the Atangana-Beleanu predicted better than the rest of the operators. However, the Caputo-Fabrizio had high unstable volatility and could not give better prediction. The numerical results indicated a vast difference in the integer and non-integer dynamics. This provides a clear evidence that memory effect of fractional derivative is important in predicting the future. The AB operator due to its non-local and nonsingular properties as well as the crossover ability provided better prediction from the numerical results compared with the other two operators. Therefore, it can be suggested that other complex financial and economic models can be examined using fractional stochastic in the light of Atangana-Beleanu operator.

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