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Statistical Approach to Nonlinear Schrödinger Equation. Quantum Case

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Abstract: A new method is proposed to obtain nonlinear Schrödinger equations by the chain of Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) quantum kinetic equations. In that sense, we investigate the dynamics of a quantum system including an infinite number of identical particles which interact via a (special) pair potential in the form of Dirac delta-function.

Keywords: BBGKY hierarchy; quantum kinetic equations; quantum kinetic theory; correlation matrix; non-linear Schrödinger equation.

1 Introduction

One of the well-known universal theories describing non-linear phenomena in nature is the non-linear Schrödinger equation [1,2,3,4,5,6,7,8,9,10,11] In the classical version, it describes the phenomena of light propagation in optical fibers, in the study of gravitational waves on the surface of water under special conditions, in the study of the phenomenon of Bose-Einstein condensation and waves in plasma.

The nonlinear Schrödinger equation in quantum physics is the basis for studying systems of bosons interacting through a potential in the form of a delta function. The study of systems of Bose particles interacting through a potential in the form of a delta function was carried out in the classic work of Lieb-Liniger in 1963 in a one-dimensional system [12].

Both in the classical and in the quantum case, the nonlinear equation describes the evolution of an individual particle, without taking into account its correlation with the rest of the particles of the system. In the real world, all phenomena are described by moving systems of interacting particles. Therefore, the generalization of nonlinear phenomena describing the nonlinear Schrödinger equation, taking into account all correlations between particles, is an urgent task of our time.

The present work is devoted to solving this problem for quantum systems of particles. For this purpose, based on the chain of quantum kinetic equations of Bogolyubov-Born-Green-Kirkwood-Yvon for the one-dimensional case, a chain of quantum equations for correlation matrices is derived. Based on the perturbation theory of series based on the Ishimaru method and considering the interaction between particles in the form of a delta function, the problem is reduced to solving the homogeneous von Neumann equation and inhomogeneous equations for wave functions. The solutions of these equations are determined under initial conditions and it is indicated how to use these solutions to determine the solution of the chain of BBGKY quantum kinetic equations for density matrices.

2 Solution of the chain of BBGKY quantum kinetic equations for dense systems of interacting particles

Let a quantum system of particles interacting through the pair potential Φ be given. Consider, in the framework of quantum statistical physics, a chain of quantum kinetic equations describing the evolution of a given system of particles [13,14,15,16,17,18,19]

$$i\frac{\partial f_n(t,x_1,\dots,x_n;x'_1,\dots,x'_n)}{\partial t} =$$

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$$[H_n, f_n)](t, x_1, \ldots, x_n; x_1', \ldots, x_n')+$$
$$\frac{1}{v} Sp_x \sum_{1\le i\le n} (\Phi(|x_i - x|) - \Phi(|x_i' - x|)\times$$
$$f_{n+1}(t, x_1, \ldots, x_n, x; x_1', \ldots, x_n', x), \qquad (1)$$

where $f_n(t)$ is the density matrix, x_i denotes the position of the ith particle in the three-dimensional Euclidean space R^3, [,] is the Poisson bracket, m is mass of particle and $2m = 1, \hbar = 1$,$0 \le t$ is the time. V is the volume and N is the number of particles of the system; v is the volume per particle, and H is the Hamiltonian of the system:

$$H_n = T_n + U_n = -\sum_{1\le i\le n} \frac{1}{2}\triangle_{x_i} + \sum_{1\le i<j\le n} \Phi(|x_i - x_j|),$$

where

$$T_n = -\sum_{1\le i\le n} \frac{1}{2}\triangle_{x_i}, \quad U_n = \sum_{1\le i<j\le n} \Phi(|x_i - x_j|)$$

and $n \in N$. Let us introduce the following notation:

$$f(t) = \{f_1(t, x_1; x_1'), f_2(t, x_1, x_2; x_1, x_2), \ldots$$
$$\ldots, f_n(t, x_1, x_2, \ldots, x_n; x_1', x_2', \ldots, x_n'), \ldots\}, \qquad n = 1, 2, \ldots;$$
$$(\mathscr{H} f(t))_n = [H_n, f(t)_n]; \qquad (\mathscr{T} f(t)_n = [T_n f(t)_n];$$
$$(\mathscr{D}_x f)_n(t, x_1, x_2, \ldots, x_n; x_1', x_2', \ldots, x_n') =$$
$$f_{n+1}(t, x_1, x_2, \ldots, x_n, x; x_1', x_2', \ldots, x_n', x);$$
$$(\mathscr{A}_x f(t))_n = \frac{1}{v}\sum_{1\le i\le n} [\Phi(|x_i - x|), f(t)_n].$$

Then equation (1) takes the form

$$i\frac{\partial f_n(t)}{\partial t} = \mathscr{H} f_n(t) + Sp_x \mathscr{A}_x f(t). \qquad (2)$$

Statement: The chain of quantum kinetic equations for correlation matrices has the form [20]

$$i\frac{\partial \varphi(t)}{\partial t} = \mathscr{H}\varphi_n(t) + \frac{1}{2}\mathscr{W}(\varphi(t), \varphi(t))+$$
$$+Sp_x(\mathscr{A}_x\mathscr{D}_x\varphi(t)) + Sp_x(\mathscr{A}_x\varphi(t) * \mathscr{D}_x\varphi(t)), \qquad (3)$$

where [21,22]

$$f(t) = \Gamma\varphi(t) = I + \varphi(t) + \frac{\varphi(t)*\varphi(t)}{2} + \ldots + \frac{(*\varphi(t))^n}{n!} + \ldots, \qquad (4)$$
$$(\varphi * \varphi)(X, X') = \sum_{Y\in X; Y'\in X'} \varphi(Y; Y')\varphi(X\setminus Y; X'\setminus Y'),$$

where

$$X = (x_1, x_2, \ldots, x_n), \quad X' = (x_1', x_2', \ldots, x_n'), \quad Y = (x_1, x_2, \ldots, x_{n'}),$$
$$Y' = (x_1', x_2', \ldots, x_{n'}'), \quad n' \in n, \quad n' = 1, 2, .., \quad I * f = f,$$
$$(*f)^n = f * f * \ldots * f \quad n \quad time,$$

$$(\mathscr{U}\varphi(t))_n = \sum_{1\le i<j\le n} (\Phi(|x_i - x_j|) - \Phi(|x_i' - x_j'|))\varphi(t)_n,$$
$$(\mathscr{W}(\varphi(t), \varphi)_n = \sum_{Y\in X; Y'\in X'} (\mathscr{U}(Y, Y'; X\setminus Y, X'\setminus Y')\times$$
$$\varphi(t, Y; Y')\varphi(t, X\setminus Y; X'\setminus Y').$$

Proof: To obtain (3), we substitute (4) in (2):

$$i\frac{\partial \Gamma\varphi(t)}{\partial t} = \mathscr{H}\Gamma\varphi(t) + Sp_x\mathscr{A}_x\Gamma\varphi(t). \qquad (5)$$

On the basis

$$i\frac{\partial}{\partial t}\Gamma\varphi(t) = i\frac{\partial}{\partial t}\varphi(t) * \Gamma\varphi(t), \qquad (6)$$

$$\mathscr{D}_x\Gamma\varphi(t) = \mathscr{D}_x\varphi * \Gamma\varphi(t), \qquad (7)$$

$$\mathscr{A}_x\Gamma\varphi(t) = \mathscr{A}_x\varphi * \Gamma\varphi(t), \qquad (8)$$

$$\mathscr{A}_x\mathscr{D}_x\Gamma\varphi(t) = \mathscr{A}_x\mathscr{D}_x\varphi * \Gamma\varphi(t)+$$
$$\mathscr{A}_x\varphi(t) * \mathscr{D}_x\varphi(t) * \Gamma\varphi(t), \qquad (9)$$

$$\mathscr{T}\Gamma\varphi(t) = \mathscr{T}\varphi(t) * \Gamma\varphi(t), \qquad (10)$$

$$\mathscr{U}\Gamma\varphi(t) = \mathscr{U}\varphi(t) * \Gamma\varphi(t)+$$
$$\frac{1}{2}\mathscr{W}(\varphi(t), \varphi(t)) * \Gamma\varphi(t). \qquad (11)$$

Substituting (6)-(11) into (5), multiplying both sides of the equation by $\Gamma(-\varphi, (t))$, we obtain quantum kinetic equations for correlation matrices (3).

This proves the statement.

In order to investigate our system on the basis of reasoning similar to those in [13] , we set:

$$\Phi(|x_i - x_j|) = v\theta(|x_i - x_j|), \qquad (12)$$

and [20,23,24]

$$\varphi_n(t) = v^{n-1}\psi_n(t). \qquad (13)$$

Based on (12),(13), equation (3) for n particles will take the form

$$i\frac{\partial \psi_n(t, X; X')}{\partial t} = (\mathscr{T}\psi)_n(t, X; X') + v(\mathscr{U}\psi)_n(t, X; X')+$$
$$\frac{1}{2}\mathscr{W}(\psi(t), \psi(t))_n(X; X') + vSp_x(\mathscr{A}_x\mathscr{D}_x\psi)_n(t, X; X')+$$
$$Sp_x(\mathscr{A}_x\psi * \mathscr{D}_x\psi)_n(t, X; X'). \qquad (14)$$

3 Solution of equation (14)

To solve equation (14), we use the perturbation theory method and look for a solution to the equation in the form of a series [20]:

$$(\psi)_n(t,X;X') = \sum_{\mu} v^{\mu} \psi_n^{\mu}(t,X;X'), \quad (15)$$

$$n = 1,2,3,...., \quad \mu = 0,1,2,,....$$

Substituting the series (15) in 1-dimensional equation (14) and equating the coefficients of equal powers of v, we obtain the set of homogeneous and inhomogeneous equations.

$$(i\frac{\partial}{\partial t} + \mathscr{L}_1)(\psi)_1^0(t) = 0, \quad (16)$$

$$(i\frac{\partial}{\partial t} + \mathscr{L}_1 + \mathscr{L}_2)(\psi)_2^0(t) = S_2^0(t), \quad (17)$$

....

$$(i\frac{\partial}{\partial t} + \sum_{i=1}^{s} \mathscr{L}_i)(\psi)_n^{\mu}(t) = S_s^{\mu}(t) \quad (18)$$

where

$$\mathscr{L}_1(\psi)_1^0(t,x_1;x_1') = \frac{\triangle_{x_1'} - \triangle_{x_1}}{2m} \psi_1^0(t,x_1;x_1') -$$
$$Sp_x(\theta(|x_1 - x|) - \theta(|x_1' - x|))\psi_1^0(t,x_1;x_1')\psi_1^0(t,x;x),$$
$$(\mathscr{L}_i \psi_1^{\mu}(t,X;X') = \frac{\triangle_{x_i'} - \triangle_{x_i}}{2m} \psi_n^{\mu}(t,X_1;X_1') -$$
$$Sp_x(\tilde{\mathscr{A}_x}\psi^0)(t,x;x_i')(\mathscr{D}_x\psi^{\mu})(t,X \setminus x_i;X' \setminus x_i'),$$
$$S_n^{\mu}(t,X;X') = (\mathscr{U}\psi^{\mu-1})(t,X;X') +$$
$$\frac{1}{2}\sum_{v_1+v_2=\mu}(\mathscr{W}(\psi(t)^{v_1},\psi(t)^{v_2}))(X;X') +$$
$$Sp_x(\tilde{\mathscr{A}_x}\mathscr{D}_x\psi^{\mu-1}(t,X;X')$$
$$+Sp_x \sum_{v_1+v_2=\mu-1}\sum_{Y\in X;Y'\in X'}(\tilde{\mathscr{A}_x}\psi^{v_1}(t,Y;Y') \times$$
$$\mathscr{D}_x\psi^{v_2}(t,X \setminus Y;X' \setminus Y')),$$

where

$$\tilde{\mathscr{A}_x} = \sum_{1\le i\le n}[\theta(|x_1 - x|),\psi_n(t,X;X')].$$

Equation (16) is the well-known von Neumann equation for the Hartree-Fock system [25,26]

Thus, the solution of equation (14) is reduced to the solution of homogeneous (16) and inhomogeneous (17), (18) von Neumann equations for ψ_1^0 and ψ_n^{μ}, respectively.

As considered in the [20],[23] series $\psi_n(t,X,X') = \sum_{\mu} v^{\mu}\psi_n^{\mu}(t,X,X')$, where ψ_1^0 is defined as a solution to the von Neumann equation and ψ_n^{μ} are determined based on the formula

$$\psi_n^{\mu}(t,X;X') = \int dY \int dY' \int_{-\infty}^{t} dt' S_n^{\mu}(t',Y,Y') \times$$
$$\bigcap_{1\ge i\ge n} \mathscr{G}(t-t',x_i,y_i;x_i',y_i') \quad (19)$$

which is the solution to the equation (14).

Here $\mathscr{G}(t,X,Y;X',Y')$ is the solution to the Cauchy problem [23]

$$i\frac{\partial \mathscr{G}(t-t',x_1,y_1;x_1',y_1')}{\partial t} = -\frac{1}{2}(\triangle_{x_1} - \triangle_{x_1'}) \times$$
$$\mathscr{G}(t-t',x_1,y_1;x_1',y_1') + Sp_x(\theta(|x_1 - x|) -$$
$$\theta(|x_1' - x|))\psi_1^0(t,x_1;x_1')\mathscr{G}(t-t',x,y_1;x,y_1') +$$
$$Sp_x(\theta(|x_1 - x|) - \theta(|x_1' - x|))\psi_1^0(t,x;x)$$
$$\mathscr{G}(t-t',x_1,y_1;x_1',y_1')$$

with the initial condition

$$\mathscr{G}(0,x_1,y_1;x_1',y_1') = \delta(x_1 - y_1)\delta(x_1' - y_i').$$

Consider the 1-dimensional von Neumann equation [4,5, 27,28]:

$$i\frac{\partial \psi_1^0(t,x_1;x_1')}{\partial t} = \frac{\triangle_{x_1'} - \triangle_{x_1}}{2m} \psi_1^0(t,x_1;x_1') +$$
$$Sp_x(\theta(|x_1 - x|) - \theta(|x_1' - x|))\psi_1^0(t,x_1;x_1')\psi_1^0(t,x_1;x_1). \quad (20)$$

We define the density matrix as follows:

$$\psi_1^0(t,x_1;x_1') = \chi(t,x_1)\chi^*(t,x_1'). \quad (21)$$

Substituting (21) into (20) and considering $\theta(|x_i - x_j|)$ in the form of the delta function $\delta(|x_i - x_j|)$, we get the solution of the equations [27,28]:

$$i\frac{\partial \chi(t,x_1)}{\partial t} = -\triangle_{x_1}\chi(t,x_1) + 2c\chi(t,x_1)|\chi(t,x_1)|^2,$$
$$\chi(t,x_1)|_{t=0} = \chi(x_1). \quad (22)$$

$$i\frac{\partial \chi^*(t,x_1')}{\partial t} = -\triangle_{x_1'}\chi^*(t,x_1') + 2c\chi_1^*(t,x_1')|\chi^*(t,x_1')|^2,$$
$$\chi^*(t,x_1')|_{t=0} = \chi^*(x_1'). \quad (23)$$

Equations (22), (23) are non-linear Schrödinger equations. If we know the solutions to these equations, we can use them to determine the solution to the von Neumann equation. As is known [4,5] at $c > 0$, the solution of Eq. (22) has the form

$$\chi(t,x_1) = \sqrt{\frac{2}{c}} \frac{(\lambda + i\nu)^2 + exp[2\nu(x_1 - x_0 - 2\lambda t)]}{1 + exp[2\nu(x_1 - x_0 - 2\lambda t)]}, \quad (24)$$

where v is the speed, the parameter λ characterizes the amplitude. The speed v is expressed through the parameter λ as $v = \sqrt{1-\lambda^2}$.

It should be noted that the following two relationships take place:

$$\frac{2}{c}|\chi(t,x_1)|^2 = 1 - \frac{v^2}{ch^2 v(x_1 - x_0 - 2\lambda t)}$$

and

$$\int |\chi(t,x_1)|^2 dx_1 = n,$$

where n is the number of particles in the system [27].

Thus, the equations (22), (23) are non-linear Schrödinger equations. If we know the solutions of the equations (22), (23), we can determine the solution of the von Neumann equation through them. Further, based on the von Neumann solution, using and (13) and (4) we can determine the density matrix, which is the solution of the BBGKY chain of quantum kinetic equations for the one-dimensional case.

4 Conclusion

Thus, the non-linear Schrödinger equation and its solution are derived from the chain of BBGKY quantum kinetic equations. This allows, in contrast to the nonlinear Schrödinger equation, on the basis of the chain of BBGKY to investigate nonlinear processes in large systems consisting of an arbitrary number of interacting particles.

It should be noted that the derivation of the nonlinear Schrödinger equation could be easily derived following the derivation of the Gross-Pitaevskii equation from the BBGKY chain of quantum kinetic equations, without taking into account the contribution of the external field [29].

Conflict of Interest

The authors declare that they have no conflict of interest

References

[1] C. Gardner, G. Green, M. Kruskal, R. Miura. Method for solving the Korteveg-de Vries equation. *Phys. Rev. Lett.*, **19**, 1095-1097 (1967).

[2] F. Calogero. Solution of the One-Dimensional N-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials. *J. Math. Phys.*, **12**, 419 (1971); https://doi.org/10.1063/1.1665604.

[3] V. E. Zakharov, L. D. Faddeev. Korteveg-de Vries equation, a completely integrable Hamiltonian system. *Funct. Anal. Appl.*, **5**, 280-287 (1971).

[4] V. E. Zaharov, A. B. Shabat. Exact Theory of two-dimensional self-focusing and onedimensional self modulation of waves in nonlinear media. *Zh. Eksp. Teor. Fiz.*, **61**, 118-134 (1971).

[5] V. E. Zakharov, A. B. Shabat. Interaction between solitons in a stable medium. *Zh. Eksp. Tear. Fiz.*, **64**,1627-1639 (1973).

[6] M. D. Kruskal. The Korteweg-de Vries equation and related evolution equations. *Lect. Appl. Math.*. **15**, 61-83, (1974).

[7] J. Ginibre, G. Velo. On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *Journal of Functional Analysis*, **32**, Issue 1, 1-32 (1979).

[8] Tosio Kato. On nonlinear Schrödinger equations. *Annales de l'I.H.P. Physique théorique*, **46**, Issue 1, 113-129 (1987).

[9] Marco Secondini, Enrico Forestieri. The nonlinear Schrödinger equation in fiber-optic systems. *Rivista di Matematica della Universita di Parma*, **8**, 69-98 (2008).

[10] Xiaotian Jiang, Danshi Wang, Qirui Fan, Min Zhang, Chao Lu, and Alan Pak Tao Lau. *Solving the Nonlinear Schrödinger Equation in Optical Fibers Using Physics-informed Neural Network*. Optical Fiber Communication Conference (OFC) (2021),in Proc.OSA Technical Digest, Optica Publishing Group, (2021), https://doi.org/10.1364/OFC.2021.M3H. 8.

[11] Anwar Ja'afar Mohamad Jawad, Mahmood Jawad. Study of Optical Soliton of Nonlinear Optical Fibers by Nonlinear Schrodinger Equation. *NeuroQuantology*, **20**, Issue 4, 417-422 (2022).

[12] E.H.Lieb and W. Liniger.Exact analysis of an interacting Bose gas. I: The general solution and the ground state, *Phys. Rev.*, **130**, 1605-1616 (1963).

[13] N.N.Bogolyubov, *Lectures in quantum statistics. Questions of statistical quantum systems*. Soviet school, Kiev(1949); *Selected Papers, 2*. Naukova Dumka, Kiev, (1970).

[14] N. N. Bogoliubov and N.N.(Jr.) Bogoliubov. *Introduction to Quantum Statistical Mechanics* [in Russian], Nauka, Moscow (1984); [English transl.], World Scientific, Singapore (2009).

[15] M. Born, H Green. A general kinetic theory of liquids I. The molecular distribution functions.*Proc.Roy.Soc.A.*, **188**, Issue 1012, 10-18 (1946).

[16] H Green. A general kinetic theory of liquids. II. Equilibrium properties. *Proc.Roy.Soc.A.*, **189**, Issue 1016, 103-117 (1947).

[17] J. Kirkwood. The Statistical Mechanical Theory of Transport Processes I. General Theory. *The Journal of Chemical Physics*, **14** Issue 3, 180-201 (1946).

[18] J. Kirkwood. The Statistical Mechanical Theory of Transport Processes II. Transport in Gases. *The Journal of Chemical Physics*, **15**, Issue 1, 72-76 (1947).

[19] J.Yvon, Yves Rocard. *La Theoria Statistique des Fluides et L'equation d'etat*. Herrmann and Cie, Paris, (1935).

[20] M.Yu.Rasulova.On evolution of perturbations of stationary solutions of Bogolyubov's kinetic equations. *Journal Theoretical nd Mathematical Physics*, **42**, 124-132 (1980).

[21] D.Ruelle. Cluster Property of the Correlation Functions of Classical Gases. *Rev. Phys.*, **36**, 580-584 (1964); *Statistical Mechanics:Rigorous results*. Benjamin, New York (1969).

[22] M. Yu. Rasulova and A. K. Vidibida. *Kinetic equations for distribution functions and density matrices*. ITP-27P Preprint (1976).

[23] S.Ichimaru. *Basic Principles Of Plasma Physics: A Statistical Approach*. W. A. Benjamin, Inc, (1973).

[24] R.L.Liboff. *Kinetic Theory, Classical, Quantum, and relativistic Descriptions*. Springer, (2003).
[25] A.Arnold. Self-Consistent Relaxation-Time Models in Quantum Mechanics. *Comm.Part.Diff.Eq.*.**21**, Issue 3-4, 473-506 (1996).
[26] A. Arnold, R. Bosi, S.Jeschke, E. Zorn. *On global classical solutions of the time-dependent von Neumann equation for Hartree-Fock systems*. arXiv:math-ph/0305027 (2003).
[27] D. Ya. Petrina and V. Z. Enolskii. Oscillations of one-dimensional systems. *Dopovidi Akad. Nauk Ukrainskoi Rsr Seriya a-Fiziko-Matematichni Ta Technichni Nauki*, **8**, 756 (1976).
[28] M.Yu. Rasulova. Physica A: Statistical Mechanics and its Applications **315**3 Issue 1-2, 72-78 (2002), DOI:10.1016/S0378-4371(02)01244-X.
[29] N.N.Bogolubov, M.Yu.Rasulova, T.Akramov. Generalization of the Gross-Pitaevskii equation for multi-particle cases. *Int.Journal of Modern Phys.B*, **35** Issue 30, 2150311 (2021).

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