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The Cordiality for the Join of Pairs of the Third Power of Paths

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Abstract: A graph is said to be cordial if it has a $0-1$ labeling that satisfies certain properties. The third power of path P_n^3, is the graph obtained from the path P_n by adding edges that join all vertices and with $d \leq 3$. In this paper, we show that P_n^3 is cordial if and only if $n \neq 4$. Moreover, we study the cordiality for the sum of pairs of the third power of paths.

Keywords: Prey-predator. Stability analysis. Global Stability. Leslie-Gower Systems, Lyapunov functions.

1 Introduction

It is well known that graph theory has applications in many other fields of study, including physics, chemistry, biology, communication, psychology, sociology, economics, engineering, operator research, and especially computer science [1,2].

One area of graph theory of considerable recent research is that of graph labeling. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, X-ray crystallography, radar, circuit design, communication network addressing and data base management [3].

In a labeling of particular types of graph, the vertices are assigned values from a given set, the edges have a prescribed induced labeling, and the labeling must satisfy certain properties. An excellent reference on this subject in the survey by Gallian [4]. Two of the most important types of labelings are called graceful and harmonious. Graceful labelings were introduced independently by Rosa [5] in 1966 and Golomb [6] in 1972, while harmonious labelings were first studied by Graham and Sloane [7] in 1980. A third important type of labeling, which contains aspects of both of the other two, is called cordial and was introduced by Cahit [8] in 1990. Whereas the label of an edge vw for graceful and harmonious labelings is given respectively by $|f(v)-f(w)|$ and $f(v)+f(w)$ (modulo the number of edges), cordial labelings use only labels 0 and 1 and the induced label $f(v)+f(w)$ (mod 2), which of course equal $|f(v)-f(w)|$. Because arithmetic modulo 2 is an integral part of computer science, cordial labelings have close connections with that field.

More precisely, cordial graphs we defined as folloes.

Let $G=(V,E)$ be a graph, let $f:V\rightarrow\{0,1\}$ labeling of the vertices, and let $f^*:E\rightarrow\{0,1\}$ be the extension of f to the edges of G by the formula $f^*(vw)=f(v)+f(w)$ (mod 2). (Thus for any edges $e=vw, f^*(e)=0$ if its two vertices have the same label and $f^*(e)=1$ if they have different labels). Let v_0 and v_1 be the numbers of vertices labeled 0 and 1 respectively, and let e_0 and e_1 be the corresponding numbers of edges. Such a labeling is called cordial if both $|v_0-v_1|\leq 1$ and $|e_0-e_1|\leq 1$. A graph is called cordial if it has a cordial labeling.

Given two disjoint graphs G and H, their union $G\cup H$ is simply the unions of their sets of vertices and edges, while their join $G+H$ is obtained from $G\cup H$ by adding all edges that join the vertices of G to the vertices of H.

The third power of paths P_n^3, is the graph obtained from the path P_n by adding edges that join all vertices u and v with $d(u,v)\leq 3$. So, the order of the third power of paths P_n^3 is n, and the size of the third power of paths P_n^3 is $3n-6$, in particular $P_1^3=P_1, P_2^3=P_2, P_3^3=C_3$ and $P_4^3=K_4$. The main object of this paper is to extend some important result on paths P_n and P_n^2 to third power of paths P_n^3. Specifically, in [9, 10, 11], we determined that the join of two paths P_n and P_m is cordial for all n and m except for P_2+P_2, and the join of the path P_n and the cycle C_m is cordial for all n and m except for $(n,m)=(1,3),(2,3)$.

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The paper consits of five sections and arranged as follows:
A brief literary of the subject of this work is contined in section one while section two deals with the used terminologies throughout. Section three explores the cordiality of the third power of paths, while section four is devoted to study the cordiality of joins third power of paths, Finally, the last section contains the conclusion.

2 Terminology and notations

We introduce some notation and terminology for a graph with $4r$ vertices, we let L_{4r} denote the labeling $00110011...0011, R_{4r}$ to denote the labeling $11001100...1100, M_t$ to denote the labeling $0101\ldots r$-times (zero-one repeated r-times), M'_r to denote the labelling $10101\ldots r$-times. O_r denotes the labelling 0000...0000 and 1_r denotes the labelling 111...1111 (one repeated r-times), i.e., 0_5 is 00000, 1_5 is 11111, M_5 is 01010, M_6 is 010101 (This means that if r is odd number, then M_r is 010...01010 and if r is even number, then M_r is 010...010101, M'_5 is 10101 and M'_6 is 101010). In most cases, we modify this by the following symbols of the labeling of the vertices of P_n^3 as for $r>1$, $E_{4r}=0_31_3M'_4M_4M'_4M_4M'_4M_4\ldots(k-1)$ times $\ldots M'_4M_4M'_4M_2$ if $r=2k$, and $D_{4r}=0_31_3M'_4M_4M'_4M_4M'_4M_4\ldots(k-1)$ times $\ldots M'_4M_4M'_4M_2$ if $r=2k+1$. (For example E_{16} is 0001111010010110 and D_{20} is 00011110100101101001), $E_{4r+1}:=0_31_2R_4L_4R_4L_4\ldots(k-1)$ times $\ldots R_4L_4R_4$ if $r=2k$, and $D_{4r+1}:=0_31_2R_4L_4\ldots(k-1)$ times $\ldots R_4L_4$ if $r=2k+1$. (For example E_{17} is 00011110000111100 and D_{21} is 000111100001111000011), $E_{4r+2}:=0_31_3M'_4M_4M'_4M_4M'_4M_4\ldots(k-1)$ times $\ldots M'_4M_4$ if $r=2k+1$. (For example E_{18} is 000111101001011010 and D_{22} is 0001111010010110100101), and $E_{4r+3}:=0_31_2R_4L_4R_4L_4\ldots(k-1)$ times $\ldots R_4L_4R_410$ if $r=2k$, and $D_{4r+1}:=0_31_2R_4L_4R_4L_4\ldots(k-1)$ times $\ldots R_4L_401$ if $r=2k+1$. (For example E_{19} is 0001111000011110010 and D_{22} is 00011110000111100001101). Moreover, we modify the above symbols by replacing ones in the place of zeros, we obtain the new symbols, for example $E'_{4r}:=1_30_3M_4M'_4M_4M'_4M_4M'_4M_4\ldots(k-1)$ times $M_4M'_4M_2$ if $r=2k$, and $D'_{4r}:=1_30_3M_4M'_4M_4M'_4\ldots(k-1)$ times $M_4M'_4M_2M'_2$ if $r=2k+1$. (For example E'_{16} is 1110000101101001 and D'_{20} is 11100001011010010110), and so on. For specific labelings L and M of $G\cup H$ and $G+H$, where G and H are third power of paths, we let $[L;M]$ denote the joint labeling.

Throughout this paper all graphs all graphs are finite and simple, and we also use the following additional notation.

For given labeling of the join $G+H$ or the union $G\cup H$, we let v_i and e_i (for $i=0,1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be Those for H. It follows that $v_0=x_0+y_0$, $v_1=x_1+y_1$, $e_0=a_0+b_0+x_0y_0+x_1y_1$ and $e_1=a_1+b_1+x_0y_1+x_1y_0$, thus $v_0-v_1=(x_0-x_1)+(y_0-y_1)$ and $e_0-e_1=(a_0-a_1)+(b_0-b_1)+(x_0-x_1)(y_0-y_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_0-v_1|\leq 1$ and $|e_0-e_1|\leq 1$.

3 The cordiality of third power of paths

In this section we show that the third power of paths P_n^3 is cordial if and only if $n\neq 4$.

Theorem 3.1. The third power of paths P_n^3 is cordial if and only if $n\neq 4$.

Proof. The necessary condition follows directly from the fact that $P_4^3=K_4$ and the complete graph K_4 is not cordial [12]. Conversely, we assume that $n\neq 4$. If $1\leq n\leq 3$, then the graphs $P_1^3=P_1, P_2^3=P_2$ and $P_3^3=C_3$ are cordial [13]. Now, let $n\geq 5$, then we have the following 4-cases.

Case (1). $n\equiv 0 \pmod 4$.

Let $n=4r$, where $r>1$; then we label the vertices of P_{4r}^3 as E_{4r} if $r=2k$ or D_{4r} if $r=2k+1$. It is easy to verify that $x_0=x_1=2r$ and $a_0=a_1=6r-3$, and consequently $x_0-x_1=0$ and $a_0-a_1=0$.

Case (2). $n\equiv 1 \pmod 4$.

Let $n=4r+1$, where $r>1$; then we label the vertices of P_{4r+1}^3 as E_{4r+1} if $r=2k$ or D_{4r+1} if $r=2k+1$. It is easy to verify that $x_0=2r+1, x_1=2r$, $a_0=6r-2$ and $a_1=6r-1$, and consequently $x_0-x_1=1$ and $a_0-a_1=-1$. In case of $r=1$ or P_5^3 we label its vertices as 00011, hence $x_0=3, x_1=2, a_0=4$, and $a_1=5$, and consequently $x_0-x_1=1$ and $a_0-a_1=-1$. Thus P_{4r+1}^3 is cordial.

Case (3). $n\equiv 2 \pmod 4$.

Let $n=4r+2$, where $r>1$; then we label the vertices of P_{4r+2}^3 as E_{4r+2} if $r=2k$ or D_{4r+2} if $r=2k+1$. It is easy to verify that $x_0=x_1=2r$, and $a_0=a_1=6r$, and consequently $x_0-x_1=0$ and $a_0-a_1=0$, $a_0-a_1=-1$. In case of $r=1$ or P_6^3 we label its vertices as 000111, hence $x_0=x_1=3$ and $a_0=a_1=6$, and consequently $x_0-x_1=0$ and $a_0-a_1=0$. Thus P_{4r+2}^3 is cordial.

Case (4). $n\equiv 3 \pmod 4$.

Let $n=4r+3$, where $r>1$; then we label the vertices of P_{4r+3}^3 as E_{4r+3} if $r=2k$ or D_{4r+3} if $r=2k+1$. It is easy to verify that $x_0=2+2r$, $x_1=2r+1$, $a_0=6r+1$, and $a_1=6r+2$, and consequently $x_0-x_1=1$ and $a_0-a_1=-1$,
In case of $r=1$ or P_7^3 we label its vertices as 0001101, hence $x_0=4, x_1=3$, $a_0=7$, and $a_1=8$, and consequently $x_0-x_1=1$ and $a_0-a_1=-1$. Thus P_{4r+3}^3 is cordial. Thus the theorem is proved.

4 Join of pairs of third power of paths

Lemma 4.1. If $n\equiv 0 \pmod 4$ and $n>4$, then the join $P_n^3+P_m^3$ of the third power of paths P_n^3 and P_m^3 is cordial for all $m>4$.

Proof. Let $n = 4r, r > 1$; then we label the vertices of P^3_{4r} as E_{4r} if $r = 2k$ or D_{4r} if $r = 2k+1$, i.e., $x_0 = x_1 = 2r$ and $a_0 = a_1 = 6r-3$. for the labelling of vertices of P^3_m we have the following 4-cases.

Case (1). $n \equiv 0(\bmod 4)$.

Let $m = 4s$, where $s > 1$; then we label the vertices of P^3_{4s} as E_{4s} if $s = 2L$ or D_{4s} if $S = 2L+1$, i.e., $y_0 = y_1 = 2s$ and $b_0 = b_1 = 6s-3$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$.

Case (2). $n \equiv 1(\bmod 4)$.

Let $m = 4s+1$, where $s > 1$; then we label the vertices of P^3_{4s+1} as E_{4s+1} if $s = 2L$ or D_{4s+1} if $r = 2L+1$, i.e., $y_0 = 2s+1$, $y_1 = 2s$, $b_0 = 6s-2$ and $b_1 = 6s-1$. Hence $v_0 - v_1 = 1$ and $e_0 - e_1 = -1$. For $s = 1$ or P^3_5 we label its vertices as 00011, hence $y_0 = 3$, $y_1 = 2$, $b_0 = 4$ and $b_1 = 5$, and consequently $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$.

Case (3). $n \equiv 2(\bmod 4)$.

Let $m = 4s+2$, where $s > 1$; then we label the vertices of P^3_{4s+2} as E_{4s+2} if $r = 2L$ or D_{4s+2} if $r = 2L+1$, i.e., $y_0 = y_1 = 2s+1$ and $b_0 = b_1 = 6s$.

Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$. In case of $s = 1$ or P^3_6 we its vertices as 000111. hence $y_0 = y_1 = 3$ and $b_0 = b_1 = 6$, and consequently $v_0 = v_1 = 0$ and $e_0 - e_1 = 0$.

Case (4). $n \equiv 3(\bmod 4)$.

Let $m = 4s+3$, where $s > 1$; then we label the vertices of P^3_{4s+3} as E_{4s+3} if $r = 2L$ or D_{4s+3} if $r = 2L+1$, i.e., $y_0 = 2s+2$, $y_1 = 2s+1$, $b_0 = 6s+1$. and $b_1 = 6s+2$

Hence $v_0 - v_1 = 1$ and $e_0 - e_1 = -1$. In case of $r = 1$ or P^3_7 we label its vertices as 0001101. hence $y_0 = 4$, $y_1 = 3$, $b_0 = 7$ and $b_1 = 8$, and consequently $v_0 = v_1 = 1$ and $e_0 - e_1 = -1$. Thus the lemma follows.

Lemma 4.2. If $n \equiv 2(\bmod 4)$, then the join $P^3_n + P^3_m$ of the third power of paths P^3_n and P^3_m is cordial for all $m > 4$.

Proof. Let $n = 4r+2, r > 1$; then we label the vertices of P^3_{4r+2} as E_{4r+2} if $r = 2k$ or D_{4r+2} if $r = 2k+1$, i.e., $x_0 = x_1 = 2r+1$ and $a_0 = a_1 = 6r$. For $r = 1$ or P^3_6, we label its vertices as 000111, i.e., $x_0 = x_1 = 3$ and $a_0 = a_1 = 6$. For the labelling of vertices of P^3_m we have the following 4-cases.

Case (1). $m \equiv 0(\bmod 4)$.

Let $m = 4s$, where $s > 1$; then we label the vertices of P^3_{4s} as E_{4s} if $s = 2L$ or D_{4s} if $S = 2L+1$, i.e., $y_0 = y_1 = 2s$ and $b_0 = b_1 = 6s-3$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$.

Case (2). $m \equiv 1(\bmod 4)$.

Let $m = 4s+1$, where $s > 1$; then we label the vertices of P^3_{4s+1} as E_{4s+1} if $s = 2L$ or D_{4s+1} if $r = 2L+1$, i.e., $y_0 = 2s+1$, $y_1 = 2s$, $b_0 = 6s-2$ and $b_1 = 6s-1$. Hence $v_0 - v_1 = 1$ and $e_0 - e_1 = -1$. For $s = 1$ or P^3_5 we label its vertices as 00011, hence $y_0 = 3$, $y_1 = 2$, $b_0 = 4$ and $b_1 = 5$, and consequently $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$.

Case (3). $m \equiv 2(\bmod 4)$.

Let $m = 4s+2$, where $s > 1$; then we label the vertices of P^3_{4s+2} as E_{4s+2} if $r = 2L$ or D_{4s+2} if $r = 2L+1$, i.e., $y_0 = y_1 = 2s+1$ and $b_0 = b_1 = 6s$.

Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$. In case of $s = 1$ or P^3_6 we its vertices as 000111. hence $y_0 = y_1 = 3$ and $b_0 = b_1 = 6$, and consequently $v_0 = v_1 = 0$ and $e_0 - e_1 = 0$.

Case (4). $m \equiv 3(\bmod 4)$.

Let $m = 4s+3$, where $s > 1$; then we label the vertices of P^3_{4s+3} as E_{4s+3} if $r = 2L$ or D_{4s+3} if $r = 2L+1$, i.e., $y_0 = 2s+2$, $y_1 = 2s+1$, $b_0 = 6s+1$. and $b_1 = 6s+2$

Hence $v_0 - v_1 = 1$ and $e_0 - e_1 = -1$. In case of $r = 1$ or P^3_7 we label its vertices as 0001101, hence $y_0 = 4$, $y_1 = 3$, $b_0 = 7$ and $b_1 = 8$, and consequently $v_0 = v_1 = 1$ and $e_0 - e_1 = -1$. Thus the lemma follows.

Lemma 4.3. If $n \equiv 1(\bmod 4)$ and $n > 5$, then the join $P^3_n + P^3_m$ of the third power of paths P^3_n and P^3_m is cordial for all $m > 4$.

Proof. Let $n = 4r+2, r > 2$; then we label the vertices of P^3_{4r+1} as $E''_{4r+1} := 1_3 0_3 L_4 R_4 \ldots (k-1)\text{-times} \ldots L - 4R_4$ 101 if $r = 2k$, and $D''_{4r+1} := 1_3 0_3 L_4 R_4 \ldots (k-1)\text{-times} \ldots L_4 R_4 L_4$ 101 if $r = 2k+1$. It is easy to verify that $x_0 = 2r$, $x_1 = 24+1$ and $a_0 = 6r-1$ and $a_1 = 64-2$. For $r = 2$ or P^3_9, we label its the vertices as 111100010, i.e., $x_0 = 4$, $x_1 = 5$ and $a_0 = 11$ and $a_1 = 10$. For the labelling of vertices of P^3_m we have the following 4-cases.

Case (1). $m \equiv 0(\bmod 4)$.

Let $m = 4s$, where $s > 1$; then we label the vertices of P^3_{4s} as E_{4s} if $s = 2L$ or D_{4s} if $S = 2L+1$, i.e., $y_0 = y_1 = 2s$ and $b_0 = b_1 = 6s-3$. Hence $v_0 - v_1 = -1$ and $e_0 - e_1 = -1$.

Case (2). $m \equiv 1(\bmod 4)$.

Let $m = 4s+1$, where $s > 1$; then we label the vertices of P^3_{4s+1} as E_{4s+1} if $s = 2L$ or D_{4s+1} if $r = 2L+1$, i.e., $y_0 = 2s+1$, $y_1 = 2s$, $b_0 = 6s-2$ and $b_1 = 6s-1$. Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$. For $s = 1$ or P^3_5 we label its vertices as 00011, hence $y_0 = 3$, $y_1 = 2$, $b_0 = 4$ and $b_1 = 5$, and consequently $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$.

Case (3). $m \equiv 2(\bmod 4)$.

Let $m = 4s+2$, where $s > 1$; then we label the vertices of P^3_{4s+2} as E_{4s+2} if $r = 2L$ or D_{4s+2} if $r = 2L+1$, i.e., $y_0 = y_1 = 2s+1$ and $b_0 = b_1 = 6s$.

Hence $v_0 - v_1 = -1$ and $e_0 - e_1 = 1$. In case of $s = 1$ or P^3_6 we label its vertices as 000111. hence $y_0 = y_1 = 3$ and $b_0 = b_1 = 6$, and consequently $v_0 = v_1 = -1$ and $e_0 - e_1 = 1$.

Case (4). $m \equiv 3(\bmod 4)$.

Let $m = 4s+3$, where $s > 1$; then we label the vertices of P^3_{4s+3} as E_{4s+3} if $r = 2L$ or D_{4s+3} if $r = 2L+1$, i.e., $y_0 = 2s+2$, $y_1 = 2s+1$, $b_0 = 6s+1$. and $b_1 = 6s+2$

Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$. In case of $r = 1$ or P^3_7 we label its vertices as 0001101, hence $y_0 = 4$, $y_1 = 3$, $b_0 = 7$ and $b_1 = 8$, and consequently $v_0 = v_1 = 0$ and $e_0 - e_1 = -1$. Thus the lemma follows.

Lemma 4.4. If $n \equiv 3(\bmod 4)$, then the join $P^3_n + P^3_m$ of the third power of paths P^3_n and P^3_m is cordial for all $m > 4$.

Proof. Let $n = 4r+3, r > 1$; then we label the vertices of P^3_{4r+1} as $E'''_{4r+1} := 1_3 0_3 L_4 R_4 \ldots (k-1)\text{-times} \ldots L - 4R_4 M_4$ 1 if $r = 2k$, and $D'''_{4r+1} := 1_3 0_3 L_4 R_4 \ldots k\text{-times} \ldots L_4 R_4$ 1 if $r = 2k+1$. It is easy to verify that $x_0 = 2r+1$, $x_1 = 24+2$ and $a_0 = 6r+2$ and $a_1 = 64+1$. For $r = 1$ or P^3_7, we label its the vertices as 0001111, i.e., $x_0 = 3$, $x_1 = 4$, $a_0 = 8$ and $a_1 = 7$. For the labelling of vertices of P_m^3 we have the following 4-cases.

Case (1). $m \equiv 0 (\text{mod } 4)$.

Let $m = 4s$, where $s > 1$; then we label the vertices of P_{4s}^3 as E_{4s} if $s = 2L$ or D_{4s} if $S = 2L+1$, i.e., $y_0 = y_1 = 2s$ and $b_0 = b_1 = 6s-3$. Hence $v_0 - v_1 = -1$ and $e_0 - e_1 = -1$.

Case (2). $m \equiv 1 (\text{mod } 4)$.

Let $m = 4s+1$, where $s > 1$; then we label the vertices of P_{4s+1}^3 as E_{4s+1} if $s = 2L$ or D_{4s+1} if $r = 2L+1$, i.e., $y_0 = 2s+1$, $y_1 = 2s$, $b_0 = 6s-2$ and $b_1 = 6s-1$. Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$. For $s = 1$ or P_5^3 we label its vertices as 00011, hence $y_0 = 3$, $y_1 = 2$, $b_0 = 4$ and $b_1 = 5$, and consequently $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$.

Case (3). $m \equiv 2 (\text{mod } 4)$.

Let $m = 4s+2$, where $s > 1$; then we label the vertices of P_{4s+2}^3 as E_{4s+2} if $r = 2L$ or D_{4s+2} if $r = 2L+1$, i.e., $y_0 = y_1 = 2s+1$ and $b_0 = b_1 = 6s$.
Hence $v_0 - v_1 = -1$ and $e_0 - e_1 = 1$. In case of $s = 1$ or P_6^3 we label its vertices as 000111. hence $y_0 = y_1 = 3$ and $b_0 = b_1 = 6$, and consequently $v_0 = v_1 = -1$ and $e_0 - e_1 = 1$.

Case (4). $m \equiv 3 (\text{mod } 4)$.

Let $m = 4s+3$, where $s > 1$; then we label the vertices of P_{4s+3}^3 as E_{4s+3} if $r = 2L$ or D_{4s+3} if $r = 2L+1$, i.e., $y_0 = 2s+2$, $y_1 = 2s+1$, $b_0 = 6s+1$. and $b_1 = 6s+2$
Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = -1$. In case of $r = 1$ or P_7^3 we label its vertices as 0001101, hence $y_0 = 4$, $y_1 = 3$, $b_0 = 7$ and $b_1 = 8$, and consequently $v_0 = v_1 = 0$ and $e_0 - e_1 = -1$. Thus the lemma follows.

It is easy to see that $P_1 + P_1 \equiv P_2$, $P_1 + P_2 \equiv C_3$, $P_1 + C_3 \equiv K_4$, $P_1 + K_4 \equiv K_5$, $P_2 + P_2 \equiv K_4$, $P_2 + C_3 \equiv K_5$, $P_2 + K_4 \equiv K_6$, $C_3 + C_3 \equiv K_6$, $C_3 + K_4 \equiv K_7$ and $K_4 + K_4 \equiv K_8$. Then we can establish the following lemma.

Lemma 4.5. If $1 \le n, m \le 4$, then the join $P_n^3 + P_m^3$ of the third power of paths P_n^3 and P_m^3 is cordial if and only if $(n,m) \neq (1,3),(1,4),(2,2),(2,3),(2,4),(3,1), (3,2), (3,3),(3,4),(4,1),(4,2),(4,3),(4,4)$.

Proof. The necessary condition follows from the following facts that $P_1^3 = P_1$, $P_2^3 = P_2$, $P_3^3 = C_3$, $P_4^4 = K_4$ and the complete graph K_n is cordial if and only if $n \le 3$ [10]. The sufficient condition follows directly from the fact that P_2 and C_3 are cordial [10]. Thus the lemma follows.

Lemma 4.6. If $m \equiv 1$ (mod 4), then the join $P_5^3 + P_m^3$ is cordial for all $m > 5$.

Proof. Let $m = 4s+1$ and $s > 1$, then for $s > 2$, the following labeling is suffice $P_5^3 + P_{4s+1}^3 : [01001; E''_{4s+1}$ if $s = 2L]$ or $P_5^3 + P_{4s+1}^3 : [01001; D''_{4s+1}$ if $s = 2L+1]$, where E''_{4s+1} and D''_{4s+1} are defined in Lemma 4.3 above. For $s = 2$ or P_9^3, the following labeling is suffice $P_5^3 + P_9^3 : [01001; E''_{4s+1}]$. Thus the lemma follows.

Example 4.1. The graphs $P_1^3 + P_6^3$, $P_1^3 + P_7^3$, $P_2^3 + P_6^3$, $P_2^3 + P_7^3$, $P_3^3 + P_6^3$, and $P_3^3 + P_7^3$ are cordial.

Solution. The following labeling are suffice.
$P_1^3 + P_6^3 \equiv P_1 + P_6^3 : [0; 000, 111]$, $P_1^3 + P_7^3 \equiv P_1 + P_7^3 : [0; 0001111]$,
$P_2^3 + P_6^3 \equiv P_2 + P_6^3 : [01; 000, 111]$, $P_2^3 + P_7^3 \equiv P_2 + P_7^3 : [01; 0001111]$,
$P_3^3 + P_6^3 \equiv C_3 + P_6^3 : [011; 000, 11]$ and $P_3^3 + P_7^3 \equiv C_3 + P_7^3 : [001; 000111]$.

Example 4.2. The graphs $P_1^3 + P_5^3$, $P_2^3 + P_5^3$, $P_3^3 + P_5^3$, $P_4^3 + P_6^3$, $P_4^3 + P_8^3$, and $P_5^3 + P_5^3$ are not cordial.

Solution. The solution follows by investigating all possible labelings of P_1, P_2, C_3, K_4, P_5^5, P_6^3

Lemma 4.7. If $1 \le n \le 3$, then the join $P_n^3 + P_m^3$ of the third power of paths P_n^3 and P_m^3 is cordial for $m > 6$.

Proof. Let $m = 4s + j$ for $0 \le j \le 3$ and $s > 1$; then we have the following cases.

Case (1). $n = 1$,

The appropriate labelings are the following:
$P_1^3 + P_{4s}^3 \equiv P_1 + P_{4s}^3 : [1; E_{4s}$ if $s = 2L]$ or $P_1^3 + P_{4s}^3 \equiv P_1 + P_{4s}^3 : [1; D_{4s}$ if $s = 2L+1]$, $P_1^3 + P_{4s+1}^3 \equiv P_1 + P_{4s+1}^3 : [0; E''_{4s+1}$ if $S = 2L]$ or $P_1^3 + P_{4s+1}^3 \equiv P_1 + P_{4s+1}^3 : [0; D''_{4s+1}$ if $s = 2L+1]$, where $s > 2$ and E''_{4s+1} and D''_{4s+1} are defined in Lemma 4.3, and for $s = 2, P_1 + P_9^3 : [0; 111100010]$, $P_1^3 + P_{4s+2}^3 \equiv P_1 + P_{4s+2}^3 : [1; E_{4s+2}$ if $s = 2L]$ or $P_1^3 + P_{4s+2}^3 \equiv P_1 + P_{4s+2}^3 : [1; D_{4s+2}$ if $s = 2L]$ and $P_1^3 + P_{4s+3}^3 \equiv P_1 + P_{4s+3}^3 : [0; E'''_{4s+3}$ if $s = 2L]$ or $P_1^3 + P_{4s+3}^3 \equiv P_1 + P_{4s+3}^3 : [0; D'''_{4s+3}$ if $s = 2L+1]$, where $s > 1$ and E'''_{4s+3} and D'''_{4s+3} are defined in Lemma 4.3.

Case (2). $n = 2$,

The appropriate labelings are the following:
$P_2^3 + P_{4s}^3 \equiv P_2 + P_{4s}^3 : [01; E_{4s}$ if $s = 2L]$ or $P_2^3 + P_{4s}^3 \equiv P_2 + P_{4s}^3 : [1; D_{4s}$ if $s = 2L+1]$, $P_2^3 + P_{4s+1}^3 \equiv P_2 + P_{4s+1}^3 : [01; E''_{4s+1}$ if $S = 2L]$ or $P_2^3 + P_{4s+1}^3 \equiv P_2 + P_{4s+1}^3 : [01; D''_{4s+1}$ if $s = 2L+1]$, where $s > 2$ and E''_{4s+1} and D''_{4s+1} are defined in Lemma 4.3, and for $s = 2, P_2 + P_9^3 : [01; 111100010]$, $P_2^3 + P_{4s+2}^3 \equiv P_2 + P_{4s+2}^3 : [01; E_{4s+2}$ if $s = 2L]$ or $P_2^3 + P_{4s+2}^3 \equiv P_2 + P_{4s+2}^3 : [01; D_{4s+2}$ if $s = 2L+1]$ where $s > 1$ and $P_2^3 + P_{4s+3}^3 \equiv P_2 + P_{4s+3}^3 : [01; E'''_{4s+3}$ if $s = 2L]$ or $P_2^3 + P_{4s+3}^3 \equiv P_2 + P_{4s+3}^3 : [01; D'''_{4s+3}$ if $s = 2L+1]$, where $s > 1$ and E'''_{4s+3} and D'''_{4s+3} are defined in Lemma 4.3.

Case (3). $n = 3$,

The appropriate labelings are the following:
$P_3^3 + P_{4s}^3 \equiv C_3 + P_{4s}^3 : [011; E_{4s}$ if $s = 2L]$ or $P_3^3 + P_{4s}^3 \equiv C_3 + P_{4s}^3 : [011; D_{4s}$ if $s = 2L+1]$, $P_3^3 + P_{4s+1}^3 \equiv C_3 + P_{4s+1}^3 : [001; E''_{4s+1}$ if $S = 2L]$ or $P_3^3 + P_{4s+1}^3 \equiv C_3 + P_{4s+1}^3 : [001; D''_{4s+1}$ if $s = 2L+1]$, where $s > 2$ and E''_{4s+1} and D''_{4s+1} are defined in Lemma 4.3, and for $s = 2$ or $P_9^3, V_3 + P_9^3 : [001; 111100010]$, $P_3^3 + P_{4s+2}^3 \equiv C_3 + P_{4s+2}^3 : [011; E_{4s+2}$ if $s = 2L]$ or $P_3^3 + P_{4s+2}^3 \equiv C_3 + P_{4s+2}^3 : [011; D_{4s+2}$ if $s = 2L+1]$ where $s > 1$ and $P_3^3 + P_{4s+3}^3 \equiv C_3 + P_{4s+3}^3 : [001; E'''_{4s+3}$ if $s = 2L]$ or $P_3^3 + P_{4s+3}^3 \equiv C_3 + P_{4s+3}^3 : [001; D''_{4s+3}$ if $s = 2L+1]$, where $s > 1$ and E''_{4s+3} and D''_{4s+3} are defined in Lemma 4.3, the lemma follows

Lemma 4.8. If $m > 5$ and $m \equiv 1,3 \pmod 4$, then the join $P_4^3 + P_m^3$ is cordial.

Proof. Let $m = 4s + j$ for $j = 1,3$, then we have the following cases.

Case (1). $j = 1$,

The appropriate labelings are the following: $P_4^3 + P_{4s+1}^3 \equiv K_4 + P_{4s+1}^3$: $[0011; E''_{4s+1}$ if $s = 2L]$ or $P_4^3 + P_{4s+1}^3 \equiv K_4 + P_{4s+1}^3$: $[0011; D''_{4s+1}$ if $s = 2L]$, where $s > 2$ and E''_{4s+1} and D''_{4s+1} are defined in Lemma 4.3, and for $s = 2$ or $P_9^3, K_4 + P_9^3$: [0011;111100010].

Case (2). $j = 3$,

The appropriate labelings are the following: $P_4^3 + P_{4s+1}^3 \equiv K_4 + P_{4s+1}^3$: $[0011; E'''_{4s+1}$ if $s = 2L]$ or $P_4^3 + P_{4s+1}^3 \equiv K_4 + P_{4s+1}^3$: $[0011; D''_{4s+1}$ if $s = 2L+1]$, where $s > 1$ and E'''_{4s+1} and D'''_{4s+1} are defined in Lemma 4.3, and for $s = 1$ or $P_7^3, K_4 + P_7^3$: [0011;0001111]. Thus the lemma follows.

5 Applications

Graph theory is used in so many of our daily routine activities. Almost everything in this world is interconnected. In Google maps, cities are represented as vertices while roads are represented as edges and graph theory is used to find the shortest path between two cities .This is also applied to network. The labeling (coloring) of vertices is used to find a proper coloring of the map with only four colors.

The graph labeling is also connected to a wide range of applications such as x-ray crystallography, coding theory, radar, astronomy, circuit design, network, and communication design.

6 Conclusion

In this paper, the cordiality of the third power of paths P_n^3 is examined and show that P_n^3 is cordial iff $n \neq 4$.

We prove that the join $P_n^3 + P_m^3$ of the third power of paths P_n^3 and P_m^3 is cordial if and only if $(n,m) \neq (1,3),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),$ $(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,6),(4,8),(4,1),$ $(5,2),(5,3),(5,4),(5,5),(6,4),(8,4)$.

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Conflict of Interest

The authors declare that they have no conflict of interest

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