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On a Ψ-Caputo-type fractional Stochastic Differential Equation

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Abstract: Consider a Ψ–Caputo fractional stochastic differential equation of order $0<\nu<1$ given by

$$^{C}\mathscr{D}_0^{\nu,\Psi}\varphi(x,t)=\gamma\int_{B(0,t^\nu)}\vartheta(\varphi(y,t))\dot{w}(y,t)dy,\, t>0.$$

Assume a non-negative and bounded function $\varphi(x,0)=\varphi_0(x)$, $x\in B(0,t^\nu)\subset\mathbb{R}^2$, $^{C}\mathscr{D}_0^{\nu,\Psi}$ is a generalized Ψ–Caputo fractional derivative operator, $\vartheta: B(0,t^\nu)\to\mathbb{R}$ is Lipschitz continuous, $\dot{w}(y,t)$ a space-time white noise and $\gamma>0$ the noise level. Under some precise conditions, we present the existence and uniqueness of solution to the class of equation and give upper moment growth bound and the long-term behaviour of the mild solution for the parameter ν such that $\frac{1}{2}<\nu<1$. The result shows that the second moment of the solution to the Ψ–Caputo-type fractional stochastic differential equation exhibits an exponential growth in time at most $c_5\exp\left(c_6\gamma^{\frac{2}{2\nu-1}}\Psi(t)\right)$, $\forall t>0$; and at a rate of $\frac{2}{2\nu-1}$ as the noise level grows large.

Keywords: Asymptotic behaviour, Ψ–fractional calculus, moment growth bound, Ψ-fractional Integral solution, Ψ–Caputo fractional derivative, stochastic Volterra type equation.

1 Introduction, motivation and preliminaries

The Ψ–fractional calculus is a generalized class of fractional operators, given by fractional integration and differentiation in regard to other function. In general, fractional calculus over the years is increasingly acceptable and crucial because of its use in modeling the anomalous diffusion characteristics of real-world processes. Systems with long-time memory and long-range interactivity are known to be most represented by the fractional calculus.

Various generalized fractional integrals and derivatives have been extensively studied. A new and recent direction of research is the study of the existing different generalized fractional integrals and derivatives about other function. Kilbas et al. in [1] have studied the notion of Riemann–Liouville (R–L) fractional calculus as regards some other function Ψ. See the following articles [2–8] and also [9–12] and their references for more on Ψ-fractional calculus.

Motivated by the R–L fractional calculus in relation to some other function Ψ, Almeida in [13] studied Ψ–Caputo derivative with reference to Ψ. Caputo fractional derivatives with regard to other function have been applied in solving boundary value problems, see [14, 15]. Now, we study a class of Ψ–Caputo fractional stochastic differential equation of order $0<\nu<1$

$$\begin{cases} ^{C}\mathscr{D}_0^{\nu,\Psi}\varphi(x,t)=\gamma\int_{B(0,t^\nu)}\vartheta(\varphi(y,t))\dot{w}(y,t)dy,\ 0<t<\infty, \\ \varphi(x,0)=\varphi_0(x),\ x\in B(0,t^\nu), \end{cases} \tag{1}$$

with $\dot{w}(y,t)$ a space-time white noise, $\gamma>0$ a noise level, $\vartheta: B(0,t^\nu)\to\mathbb{R}$ Lipschitz continuous, and $^{C}\mathscr{D}_0^{\nu,\Psi}$ the Caputo-type fractional differential operator of order $0<\nu<1$.

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For all we know, the above model does not exist in literature and we therefore seek to make sense of its solution. Next, we give a mild solution to (1) in $L^2(\mathbf{P})$ sense:

Definition 11 *The function $\{\varphi(x,t), x \in B(0,t^\nu), 0 \le t \le T\}$ is said to be a mild solution of (1) if almost surely,*

$$\begin{aligned}\varphi(x,t) &= \varphi_0(x) + \frac{\gamma}{\Gamma(\nu)}\int_0^t \int_{B(0,t^\nu)} (\Psi(t)-\Psi(s))^{\nu-1}\Psi'(s)\vartheta(\varphi(y,s))\dot{w}(y,s)dyds \\ &= \varphi_0(x) + \frac{\gamma}{\Gamma(\nu)}\int_0^t \int_{B(0,t^\nu)} (\Psi(t)-\Psi(s))^{\nu-1}\Psi'(s)\vartheta(\varphi(y,s))w(dy,ds).\end{aligned}$$

If in addition, $\{\varphi(x,t), x \in B(0,t^\nu), 0 \le t \le T\}$ satisfies

$$\sup_{t\in[0,T]}\sup_{x\in B(0,t^\nu)} \mathbf{E}|\varphi(x,t)|^2 < \infty,$$

then $\{\varphi(x,t), x \in B(0,t^\nu), 0 \le t \le T\}$ is a random field solution to (1).

The following proposition motivated our problem:

Proposition 1([16]). *Let $0<\nu<1$, and consider the generalized fractional differential equation*

$$\begin{cases} {}^C\mathscr{D}_0^{\nu,\Psi} f(t) = Bf(t) + h(t),\ t \ge 0 \\ f(0) = \lambda, \end{cases}$$

with ${}^C\mathscr{D}_0^{\nu,\Psi}$ a Caputo-type fractional differential operator, B a $n\times n$ constant matrix and $h(t)$ a n-dimensional continuous function. Then the solution is identical to the Volterra equation

$$f(t) = \lambda + \frac{1}{\Gamma(\nu)}\int_0^t (\Psi(t)-\Psi(s))^{\nu-1}\Psi'(s)\left[Bf(s)+h(s)\right]ds,\ 0 \le t < \infty.$$

Definition 12 ([11,17]) *Given that $(a,b)(-\infty \le a < b \le \infty)$ and $\alpha > 0$. Suppose $\Psi(t)$ is increasing and positive monotone on $(a,b]$, with a continuous derivative $\Psi'(t)$ on (a,b). The left- and right-sided Ψ-R–L fractional integrals of a function g with regard to some other function Ψ on $[a,b]$ are*

$$I_{a^+}^{\alpha,\Psi} g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \Psi'(x)(\Psi(t)-\Psi(x))^{\alpha-1}g(x)dx,$$

$$I_{b^-}^{\alpha,\Psi} g(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \Psi'(x)(\Psi(x)-\Psi(t))^{\alpha-1}g(x)dx.$$

Alternative definition is as follows:

Definition 13 ([13]) *If $\alpha > 0$, $I=[a,b]$, $h: I \to \mathbb{R}$ an integrable function and $\Psi' \in C'(I)$ an increasing function such that $\Psi'(x) \neq 0\ \forall x \in I$. Then, fractional integrals and fractional derivatives of a function h about other function Ψ are*

$$I_{a^+}^{\alpha,\Psi} h(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \Psi'(t)\big(\Psi(x)-\Psi(t)\big)^{\alpha-1}h(t)dt,$$

and

$$D_{a^+}^{\alpha,\Psi} h(x) := \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^n I_{a^+}^{n-\alpha,\Psi} h(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^n \int_a^x \Psi'(t)\big(\Psi(x)-\Psi(t)\big)^{n-\alpha-1}h(t)dt,$$

where $n = [\alpha]+1$.

Remark. For $\Psi(t) = t$, then $I_{a^+}^{\alpha,\Psi} = I_{a^+}^{\alpha}$ is the usual R–L integral and for $\Psi(t) = \ln t$, then $I_{a^+}^{\alpha,\Psi}$ becomes the Hadamard fractional integral operator.

Definition 14 ([13]) *Suppose $\alpha > 0$, $n \in \mathbb{N}$ and $-\infty \le a < b \le \infty$ be an interval. Let $\phi, \Psi \in C^n(I)$ be two functions such that Ψ is increasing and $\Psi'(x) \neq 0 \; \forall \, x \in I$. The left $\Psi-$ Caputo fractional derivative of g of order α is*

$$^{C}\mathscr{D}_{a^+}^{\alpha,\Psi}\phi(x) := I_{a^+}^{n-\alpha,\Psi}\left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^n \phi(x)$$

and the right $\Psi-$ Caputo fractional derivative of g of order α is

$$^{C}\mathscr{D}_{b^-}^{\alpha,\Psi}\phi(x) := I_{b^-}^{n-\alpha,\Psi}\left(-\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^n \phi(x),$$

where $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$.

Remark. In [3], given $\beta \in \mathbb{R}$ with $\beta > n$, the Ψ–Caputo fractional derivative of the power function $\phi(x) = \big(\Psi(x) - \Psi(a)\big)^{\beta-1}$ is

$$^{C}\mathscr{D}_{a^+}^{\alpha,\Psi}\phi(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\big(\Psi(x)-\Psi(a)\big)^{\beta-\alpha-1}.$$

Theorem 1([13]). *If $\phi \in C'[a,b]$ and $\alpha > 0$, then $^{C}\mathscr{D}_{a^+}^{\alpha,\Psi} I_{a^+}^{\alpha,\Psi}\phi(x) = \phi(x)$ and $^{C}\mathscr{D}_{b^-}^{\alpha,\Psi} I_{b^-}^{\alpha,\Psi}\phi(x) = \phi(x)$.*

Lemma 1([13]). *Let $\lambda \in \mathbb{R}$, $\alpha > 0$, and $\phi(x) = E_\alpha(\gamma(\Psi(x)-\Psi(a))^\alpha)$ and $\varphi(x) = E_\alpha(\gamma(\Psi(b)-\Psi(x))^\alpha)$, with E_α a Mittag–Leffler function. It follows that*

$$^{C}\mathscr{D}_{a^+}^{\alpha,\Psi}\phi(x) = \gamma\phi(x)$$

and

$$^{C}\mathscr{D}_{b^-}^{\alpha,\Psi}\varphi(x) = \gamma\varphi(x).$$

This is the outline of the paper. Section 2 contains the main results and their proofs. In section 3, a brief summary of the result is given.

2 Main results

Assume a global Lipschitz continuity on ϑ as follows:

Condition 21 *Suppose $0 < \mathrm{Lip}_\vartheta < \infty$. Then assume*

$$|\vartheta(x)-\vartheta(y)| \le \mathrm{Lip}_\vartheta |x-y|, \; \forall x, y \in B(0,t^\nu) \subset \mathbb{R}^2,$$

and let $\vartheta(0) = 0$ for convenience. The constant Lip_ϑ satisfies some linear growth condition

$$\mathrm{Lip}_\vartheta := \sup_{z \in B(0,t^\nu)} \frac{|\vartheta(z)|}{|z|} < \infty.$$

Condition 22 *We assume Ψ to be exponentially bounded: For some positive numbers M, c, we have $\Psi(t) \le Me^{ct}$.*

Now, we give $L^2(\mathbf{P})$ norm of the solution as follows:

$$\|\varphi\|_{2,\beta,\nu}^2 := \sup_{0\le t\le T}\sup_{x\in B(0,t^\nu)} e^{-\beta\Psi(t)}\mathbf{E}|\varphi(x,t)|^2, \; T < \infty,$$

and the result follows

Theorem 2. *Let $c_3 < \frac{1}{(\gamma \mathrm{Lip}_\vartheta)^2}$ for $0 < \mathrm{Lip}_\vartheta < \infty$. Suppose Condition 21 and Condition 22 hold, then there exists a unique solution to equation (1), with $c_3 := \frac{M_1 \pi T^{2\nu} e^{cT}}{\Gamma^2(\nu)} \frac{\Gamma(2\nu-1)}{\beta^{2\nu-1}}$, $\nu > \frac{1}{2}$.*

We will give the proof of the above result by Banach's fixed point theorem. For $0 < \nu < 1$, define the operator

$$\mathscr{A}\varphi(x,t) = \varphi_0(x) + \frac{\gamma}{\Gamma(\nu)}\int_0^t \int_{B(0,t^\nu)} (\Psi(t)-\Psi(s))^{\nu-1}\Psi'(s)\vartheta(\varphi(y,s))w(dy,ds),$$

and the solution of (1) is obtained as a fixed point of the operator $\mathscr{A}$.

Lemma 2. *Let φ be a random solution such that $\|\varphi\|_{2,\beta,\nu} < \infty$. Suppose Condition 21 and Condition 22 hold. Then for $c_2,\ c_3 > 0$ we have*

$$\|\mathscr{A}\varphi\|^2_{2,\beta,\nu} \le c_2 + c_3\gamma^2\mathrm{Lip}^2_{\vartheta}\|\varphi\|^2_{2,\beta,\nu},$$

where $c_2 := c_1 \sup_{0\le t\le T} e^{-\beta\Psi(t)}$, $c_3 := \dfrac{M_1\pi T^{2\nu}e^{cT}}{\Gamma^2(\nu)}\dfrac{\Gamma(2\nu-1)}{\beta^{2\nu-1}}$, $\nu > \dfrac{1}{2}$.

Proof. Using Itô isometry and suppose that $|\varphi_0(x)|^2 \le c_1$, one gets

$$\begin{aligned}\mathbf{E}|\mathscr{A}\varphi(x,t)|^2 &= |\varphi_0(x)|^2 + \frac{\gamma^2}{\Gamma^2(\nu)}\int_0^t\int_{B(0,t^\nu)}(\Psi(t)-\Psi(s))^{2\alpha-2}\left(\Psi'(s)\right)^2\mathbf{E}|\vartheta(\varphi(y,s))|^2dyds\\ &\le c_1 + \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}\int_0^t\int_{B(0,t^\nu)}(\Psi(t)-\Psi(s))^{2\alpha-2}\left(\Psi'(s)\right)^2\mathbf{E}|\varphi(y,s)|^2dyds\\ &\le c_1 + \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}\pi t^{2\nu}\int_0^t\Psi'(s)\sup_{y\in B(0,t^\nu)}\mathbf{E}|\varphi(y,s)|^2(\Psi(t)-\Psi(s))^{2\alpha-2}\Psi'(s)ds.\end{aligned}$$

Since Ψ is assumed to be exponentially bounded, that is, $\Psi(s) \le Me^{cs},\ s > 0$ and $\Psi'(s) \le M_1e^{cs},\ M_1 = Mc$, then

$$\mathbf{E}|\mathscr{A}\varphi(x,t)|^2 \le c_1 + \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}M_1\pi t^{2\nu}e^{ct}\int_0^t\sup_{y\in B(0,t^\nu)}\mathbf{E}|\varphi(y,s)|^2(\Psi(t)-\Psi(s))^{2\alpha-2}\Psi'(s)ds.$$

Now, multiply both sides by $e^{-\beta\Psi(t)}$ to obtain

$$\begin{aligned}e^{-\beta\Psi(t)}\mathbf{E}|\mathscr{A}\varphi(x,t)|^2 &\le c_1e^{-\beta\Psi(t)}\\ &+ \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}M_1\pi t^{2\nu}e^{ct}\int_0^t e^{-\beta(\Psi(t)-\Psi(s))}(\Psi(t)-\Psi(s))^{2\alpha-2}\Psi'(s)\sup_{y\in B(0,t^\nu)}e^{-\beta\Psi(s)}\mathbf{E}|\varphi(y,s)|^2ds\\ &= c_1e^{-\beta\Psi(t)} + \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}M_1\pi t^{2\nu}e^{ct}\|\varphi\|^2_{2,\beta,\nu}\int_0^t e^{-\beta(\Psi(t)-\Psi(s))}(\Psi(t)-\Psi(s))^{2\alpha-2}\Psi'(s)ds.\end{aligned}$$

Next, we take supremum(s) over $t\in[0,T],\ T<\infty$, and $x\in B(0,t^\nu)$ to obtain

$$\begin{aligned}\|\mathscr{A}\varphi\|^2_{2,\beta,\nu} &\le c_2 + \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}M_1\pi T^{2\nu}e^{cT}\|\varphi\|^2_{2,\beta,\nu}\sup_{0\le t\le T}\int_0^t e^{-\beta(\Psi(t)-\Psi(s))}(\Psi(t)-\Psi(s))^{2\alpha-2}\Psi'(s)ds\\ &\le c_2 + \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}M_1\pi T^{2\nu}e^{cT}\|\varphi\|^2_{2,\beta,\nu}\int_0^\infty e^{-\beta\tau}\tau^{2\nu-2}d\tau,\end{aligned}$$

where the last line follows by substitution. For $\frac{1}{2} < \nu < 1$ and the gamma function Γ,

$$\|\mathscr{A}\varphi\|^2_{2,\beta,\nu} \le c_2 + \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}M_1\pi T^{2\nu}e^{cT}\|\varphi\|^2_{2,\beta,\nu}\frac{\Gamma(2\nu-1)}{\beta^{2\nu-1}}.$$

Following similar steps, we have

Lemma 3. *Let φ and ϕ are random solutions satisfying $\|\varphi\|_{2,\beta,\nu} + \|\phi\|_{2,\beta,\nu} < \infty$. Suppose Condition (21) and Condition 22 hold. Then for a positive number c_3, we have*

$$\|\mathscr{A}\varphi - \mathscr{A}\phi\|^2_{2,\beta,\nu} \le c_3\gamma^2\mathrm{Lip}^2_{\vartheta}\|\varphi-\phi\|^2_{2,\beta,\nu}.$$

Proof(Proof of Theorem 2). We now apply Lemma 2 and Lemma 3. Using Banach fixed point theorem, one obtains $\varphi(x,t) = \mathscr{A}\varphi(x,t)$ and by Lemma 2,

$$\|\varphi\|^2_{2,\beta,\nu} = \|\mathscr{A}\varphi\|^2_{2,\beta,\nu} \le c_2 + c_3\gamma^2\mathrm{Lip}^2_{\vartheta}\|\varphi\|^2_{2,\beta,\nu}.$$

It follows that

$$\|\varphi\|^2_{2,\beta,\nu}[1 - c_3\gamma^2\mathrm{Lip}^2_{\vartheta}] \le c_2 \Rightarrow \|\varphi\|_{2,\beta,\nu} < \infty \Leftrightarrow c_3 < \frac{1}{(\gamma\mathrm{Lip}_{\vartheta})^2}.$$

On the other hand, using Lemma 3, one gets

$$\|\varphi - \phi\|^2_{2,\beta,\nu} = \|\mathscr{A}\varphi - \mathscr{A}\phi\|^2_{2,\beta,\nu} \le c_3\gamma^2 \mathrm{Lip}^2_{\vartheta}\|\varphi - \phi\|^2_{2,\beta,\nu}.$$

Thus, $\|\varphi - \phi\|^2_{2,\beta,\nu}\left[1 - c_3\gamma^2\mathrm{Lip}^2_{\vartheta}\right] \le 0$ and $\|\varphi - \phi\|_{2,\beta,\nu} < 0$ for $c_3 < \frac{1}{(\gamma \mathrm{Lip}_{\vartheta})^2}$. Therefore, using Banach contraction principle, existence and uniqueness of solution follows.

2.1 *Second moment bound*

In order to prove the growth moment bound, we first state the following generalized Gronwall's inequality

Theorem 3([9]). *Suppose φ and ϕ are two integrable functions and h a continuous function with domain $[a,b]$. Let $\Psi \in C[a,b]$ be an increasing function with $\Psi'(t) \neq 0$ for all $t \in [a,b]$. Given that*

1. *φ and ϕ are nonnegative,*
2. *h is nonnegative and nondecreasing.*

If

$$\varphi(t) \le \phi(t) + h(t)\int_a^t \Psi'(\tau)\big(\Psi(t) - \Psi(\tau)\big)^{\alpha-1}u(\tau)d\tau,$$

then

$$\varphi(t) \le \phi(t) + \int_a^t \sum_{k=1}^{\infty}\frac{[h(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)}\Psi'(\tau)[\Psi(t) - \Psi(\tau)]^{\alpha k-1}d\tau.$$

Corollary 1([9]). *Following the assumptions of Theorem 3, suppose ϕ is a nondecreasing function on $[a,b]$. Then*

$$\varphi(t) \le \phi(t)E_{\alpha}\big(h(t)\Gamma(\alpha)[\Psi(t) - \Psi(a)]^{\alpha}\big),\ \forall\, t \in [a,b],$$

with $E_{\alpha}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}$, $\mathrm{Re}\,(\alpha) > 0$, *a Mittag–Leffler function.*

Now, the result:

Theorem 4. *Given that Condition 21 and Condition 22 hold. For all $t > 0$ and $\nu > \frac{1}{2}$, one obtains*

$$\sup_{x\in B(0,t^{\nu})}\mathbf{E}|\varphi(x,t)|^2 \le c_5\exp\big(c_6\gamma^{\frac{2}{2\nu-1}}\psi(t)\big),$$

for some positive constants c_5 and c_6.

Proof. Since $|\varphi_0(x)|^2 \le c_1$ and Ψ is exponentially bounded, it follows from Lemma 2 that

$$\sup_{x\in B(0,t^{\nu})}\mathbf{E}|\varphi(x,t)|^2 \le c_1 + \frac{\gamma^2\mathrm{Lip}^2_{\vartheta}}{\Gamma^2(\nu)}M_1\pi T^{2\nu}e^{cT}\int_0^t(\Psi(t) - \Psi(s))^{2\nu-2}\Psi'(s)\sup_{y\in B(0,t^{\nu})}\mathbf{E}|\varphi(y,s)|^2ds.$$

Let $f_{\nu}(t) := \sup_{x\in B(0,t^{\nu})}\mathbf{E}|\varphi(x,t)|^2$ to get

$$f_{\nu}(t) \le c_1 + c_4\gamma^2\int_0^t(\Psi(t) - \Psi(s))^{2\nu-2}\Psi'(s)f_{\nu}(s)ds = c_1 + c_4\gamma^2\int_0^t(\Psi(t) - \Psi(s))^{(2\nu-1)-1}\Psi'(s)f_{\nu}(s)ds.$$

Thus, by applying Corollary 1 for $\nu > \frac{1}{2}$, we obtain

$$f_{\nu}(t) \le c_1E_{2\nu-1}\Big(c_4\gamma^2\Gamma(2\nu-1)(\Psi(t) - \Psi(0))^{2\nu-1}\Big) \le c_1E_{2\nu-1}\Big(c_4\gamma^2\Gamma(2\nu-1)(\Psi(t))^{2\nu-1}\Big).$$

Next, using a known inequality, that for $0 < \nu < 1$, $b > 0, t \ge 0$, one obtains

$$E_{\nu}(b(\psi(t))^{\nu}) \le Ce^{b^{\frac{1}{\nu}}\psi(t)},\ C > 0.$$

Thus, for all $\frac{1}{2} < \nu < 1$, we have

$$f_\nu(t) \leq c_1.C\exp\left(c_4^{\frac{1}{2\nu-1}}\gamma^{\frac{2}{2\nu-1}}(\Gamma(2\nu-1))^{\frac{1}{2\nu-1}}\Psi(t)\right)$$

and therefore

$$\sup_{x\in B(0,t^\nu)} \mathbf{E}|\varphi(x,t)|^2 \leq c_5\exp\left(c_6\gamma^{\frac{2}{2\nu-1}}\Psi(t)\right),$$

where $c_5 = c_1.C$ and $c_6 = \left(c_4\Gamma(2\nu-1)\right)^{\frac{1}{2\nu-1}}$.

Moreso, we give an immediate consequence of the above Theorem 4. The result states the rate at which the moment of the solution grows with regards to the noise parameter γ:

Corollary 2. *Let $\nu > \frac{1}{2}$ and conditions of Theorem 4 hold. For $x \in B(0,t^\nu)$ and some $t > 0$,*

$$\limsup_{\gamma\to\infty} \frac{\log\log\mathbf{E}|\varphi(x,t)|^2}{\log\gamma} \leq \frac{2}{2\nu-1}.$$

3 Conclusion

The long term behaviour of solution to a Ψ–Caputo time-fractional stochastic differential equation with respect to a noise level γ was studied. We proved the existence and uniqueness of solution under some linearity condition on ϑ using Banach fixed point theorem, and also estimated an upper second moment bound of the solution. It was observed that the second moment of the mild solution exhibits an exponential growth rate in time as a function of Ψ.

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